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ON TOTAL STABILITY AND ASYMPTOTIC STABILITY

GEORGE SEIFERT

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In some recent work (cf. [3]) on the existence of almost-periodic solutions of systems of differential equations, sufficient conditions have been given in terms of a concept known as total stability [2], or alternately, stability under constant (or continuously-acting) disturbances [1].

It is known [2] that if a solution of a system of differential equations is uniformly asymptotically stable, then it is totally stable. It is also known that the converse is not in general true; for example, the first order autonomous scalar differential equation

$$\dot{x} = f(x),$$

where $f(x) = -x^2 \sin\frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$, has the property that the solution $x = 0$ is totally stable, but clearly not asymptotically stable.

However, for linear homogeneous systems, the zero solution will be uniformly asymptotically stable if it is totally stable; cf. [2], Theorem 28.

It is the purpose of this note to give a somewhat more general sufficient condition which will, in addition to total stability, guarantee that a solution be uniformly asymptotically stable. The proof of our theorem is based on a slight extension of the main idea of the proof of Theorem 28 in [2]. In fact, we obtain this result for linear homogeneous systems as a simple corollary of our theorem.

We consider the system

$$\dot{x} = f(t, x); \tag{1}$$

here the dot denotes differentiation with respect to t, and x and $f(t, x)$ are elements of X, a normed n-dimensional vector space over the complex field. If x is in X, $|x|$ denotes the norm of x. We assume that

(i) f is continuous in (t, x) for $t \geqq 0$, $|x| < \rho$,
and

(ii) $f(t, 0) = 0$ for all $t \geqq 0$.

We use the following stability definitions (cf. [2], [1]):

(iii) The solution $x = 0$ of (1) is totally stable if given ε, $0 < \varepsilon < \rho$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for each $t_0 \geqq 0$, any solution $y(t)$ of

$$\dot{y} = f(t, y) + g(t, y) \tag{2}$$

such that $|y(t_0)| < \delta$ satisfies $|y(t)| < \varepsilon$ for $t \geqq t_0$; here g is any function of (t, y) to X continuous in (t, y) for $t \geqq 0$, $|y| < \rho$, and such that $|g(t, y)| < \delta$ $t \geqq t_0$, $|y| < \varepsilon$.

(iv) The solution $x=0$ of (1) is uniformly asymptotically stable if given ε, $0 < \varepsilon < \rho$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $t_0 \geqq 0$ and then any solution $x(t)$ of (1) with $|x(t_0)| < \delta$ satisfies $|x(t)| < \varepsilon$ for $t \geqq t_0$, and

$$\lim_{t \to +\infty} x(t) = 0$$

uniformly for $t_0 \geqq 0$, $|x_0| < \delta$.

REMARK 1. In the definition of total stability given in (iii) here, the function g may depend on t_0. This seems to be presumed in the definition given in [1], [2], and [3], although never explicitly mentioned.

THEOREM. *Suppose the solution $x = 0$ of (1) is totally stable, and that*

$$\lim_{\lambda \to 0+} e^{\lambda t} f(t, e^{-\lambda t} x) = f(t, x) \tag{3}$$

uniformly in (t, x) for $t \geqq 0$, $|x| < \rho$. Then $x=0$ is uniformly asymptotically stable.

PROOF. Let ε, $0 < \varepsilon < \rho$ be given and $\delta = \delta(\varepsilon)$ be as determined by the total stability of $x = 0$. For each $t_0 \geqq 0$ consider the system

$$\dot{y} = e^{\lambda(t-t_0)} f(t, e^{-\lambda(t-t_0)} y) + \lambda y. \tag{4}$$

Clearly if $x(t)$ is a solution of (1), then $y(t) = e^{\lambda(t-t_0)} x(t)$ is a solution of (4), and conversely.

Fix $\lambda = \lambda(\varepsilon)$ so that $0 < \lambda < \delta/2\varepsilon$, and that

$$|e^{\lambda t} f(t, e^{-\lambda t} y) - f(t, y)| < \delta/2$$

for $t \geqq 0$ and $|y| < \rho$. Then if we define g by

$$g(t, y) = e^{\lambda(t-t_0)} f(t, e^{-\lambda(t-t_0)} y) - f(t, y) + \lambda y\,,$$

it follows that $|g(t, y)| < \delta$ for $t \geqq t_0$ and $|y| < \varepsilon$. Thus since (4) can be written as

$$\dot{y} = f(t, y) + g(t, y)\,, \tag{5}$$

it follows that if $y(t)$ is any solution of (4) with $|y(t_0)| < \delta$, then $|y(\mathrm{t})| < \varepsilon$ for $t \geqq t_0$. But $|y(t_0)| = |x(t_0)| < \delta$, and it follows from the fact that

$$x(t) = e^{-\lambda(t-t_0)} y(t)$$

that $x=0$ is a uniformly asymptotically stable solution of (1).

REMARK 2. We actually have obtained in this theorem sufficient conditions that the $x = 0$ solution of (1) be exponentially asymptotically stable (cf. [3]), a type of stability stronger than uniform asymptotic stability.

REMARK 3. If the function g is to be independent of t_0; i.e., if $|g(t, y)| < \delta$ for $t \geqq 0$, $|y| < \varepsilon$ in (iii), then we may conclude from the hypotheses of our theorem that $x = 0$ is an asymptotically stable solution of (1); again, cf. [2] for a definition of the latter type of stability.

COROLLARY. *If $f(t, x) = A(t)x + h(t, x)$, where $A(t)$ is a $n \times n$ matrix whose elements are continuous for $t \geqq 0$, h is continuous in (t, x) for $t \geqq 0$, $|x| < \rho$, and there exists a scalar function Φ such that $\Phi(t) \to 0$ as $t \to +\infty$, and such that*

$$|h(t, x)| \leqq \Phi(t)|x|$$

for $t \geqq 0$, $|x| < \rho$, then if $x = 0$ is totally stable, it is uniformly asymptotically stable.

From this corollary, the proof of which is straight forward and is therefore omitted, the result for linear homogeneous systems; i.e., systems of the form $\dot{x} = A(t)x$, follows immediately.

REMARK 4. Extensions of our theorem based on more general changes of variables suggest themselves for further study. For example, transformations of the form

$$x = B(t, \lambda)y$$

where $B(t, \lambda)$ is an $n \times n$ matrix, non-singular and sufficiently smooth for $t \geqq 0$, $\lambda > 0$, and such that $B(t, \lambda) \to 0$ as $t \to +\infty$ for $|\lambda|$ sufficiently small, could be considered.

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DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
IOWA, U. S. A.