Tôhoku Math. Journ.
20(1968), 333-345.

ON CONTRAVARIANT C-ANALYTIC 1-FORMS IN A COMPACT SASAKIAN SPACE

YÔSUKE OGAWA

(Received February 5, 1968)

Introduction. Let $F_i{}^j$ be the complex structure tensor in a Kaehlerian space. If a 1-form u leaves invariant the structure tensor $F_i{}^j$, then it is called a contravariant analytic 1-form. It is defined by the relation

$$\nabla_i u_j - F_i{}^a F_j{}^b \nabla_a u_b = 0 .$$

In a compact Kaehler Einstein space, we take a contravariant analytic 1-form u. Then there exist Killing 1-forms v and w such that u can be written as

$$u_i = v_i + F_i{}^j w_j ,$$

which is known as a theorem of Matsushima [3]. We consider the analogy of this theorem in a compact C-Einstein space. Denote the structure tensors of a Sasakian space by $(\varphi_\lambda{}^\mu, \eta_\lambda, g_{\lambda\mu})$. Then it is known that a 1-form u which leaves invariant the tensor $\varphi_\lambda{}^\mu$ is Killing in a compact contact space (S. Tanno [2]). Therefore if we take a $\varphi_\lambda{}^\mu$-preserving 1-form instead of a contravariant analytic 1-form in a Kaehlerian case, the analogy of the theorem of Matsushima is trivial.

In the former paper [6], we introduced the notion of C-Killing 1-forms on a Sasakian space. We call a 1-form u C-Killing if it satisfies $\delta u=0$ and leaves invariant $g_{\lambda\mu} - \eta_\lambda \eta_\mu$. Especially if a C-Killing form u satisfies $u'=i(\eta)u$ $=$constant, it is called special C-Killing. In a compact Sasakian space, it is known that a 1-form u is special C-Killing if and only if it satisfies

$$\nabla_\lambda u_\mu + \nabla_\mu u_\lambda = 2u^\rho(\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda) .$$

Now we study the analogy of the theorem of Matsushima taking the C-Killing forms for Killing forms.

In a Sasakian space, we define a contravariant C-analytic 1-form u by the relation

$$\nabla_\lambda u_\mu - \varphi_\lambda{}^\rho \varphi_\mu{}^\sigma \nabla_\rho u_\sigma = u^\rho(\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda).$$

It is our purpose to show that for any contravariant C-analytic 1-form u on a compact C-Einstein space, there exist special C-Killing 1-forms v, w such that the relation

$$u_\lambda = v_\lambda + \varphi_\lambda{}^\mu w_\mu$$

holds good, which is an analogy of the theorem of Matsushima.

I should like to express my hearty thanks to Prof. S. Tachibana for his kind suggestions and many valuable criticism.

1. Preliminaries. An n-dimensional Riemannian space M^n is called a Sasakian space (or normal contact metric space) if it admits a unit Killing vector field η^λ satisfying

$$\nabla_\lambda \nabla_\mu \eta_\nu = \eta_\mu g_{\lambda\nu} - \eta_\nu g_{\lambda\mu},$$

where $g_{\lambda\mu}$ is a metric tensor on M^n. A Sasakian space is orientable and odd dimensional. We define a 2-form φ by $\varphi_{\lambda\mu} = \nabla_\lambda \eta_\mu$. Then φ coincides with $(1/2)d\eta$. We denote the outer (or inner) product with respect to the 1-form η by $e(\eta)$ (or $i(\eta)$). Next we define the operators L and Λ by

$$L = e(\eta)d + de(\eta), \qquad \Lambda = i(\eta)\delta + \delta i(\eta),$$

where d (or δ) is the exterior differential (or co-differential) operator. Then L (or Λ) is the outer (or inner) product with respect to the 2-form $d\eta$. Let $u = (u_{\lambda_1\cdots\lambda_p})$ be a p-form. Then we define some operators as follows:

$$(\Phi u)_{\lambda_1\cdots\lambda_p} = \sum_{i=1}^{p} \varphi_{\lambda_i}{}^\rho u_{\lambda_1\cdots\overset{i}{\rho}\cdots\lambda_p}, \qquad (p \geqq 1) \tag{1.2}$$

$$(\nabla_\eta u)_{\lambda_1\cdots\lambda_p} = \eta^\rho \nabla_\rho u_{\lambda_1\cdots\lambda_p}, \qquad (p \geqq 0) \tag{1.3}$$

$$(\Gamma u)_{\lambda_0\cdots\lambda_p} = \sum_{\alpha=0}^{p} (-1)^\alpha \varphi_{\lambda_\alpha}{}^\rho \nabla_\rho u_{\lambda_0\cdots\hat{\alpha}\cdots\lambda_p}, \quad (p \geqq 0) \tag{1.4}$$

$$(Du)_{\lambda_2\cdots\lambda_p} = \varphi^{\rho\sigma} \nabla_\rho u_{\sigma\lambda_2\cdots\lambda_p}, \qquad (p \geqq 1) \tag{1.5}$$

where $u_{\lambda_1\cdots\overset{i}{\sigma}\cdots\lambda_p}$ means the subscript σ appears at the i-th position and $u_{\lambda_0\cdots\hat{\alpha}\cdots\lambda_p}$ means the α-th subscript λ_α is omitted. We denote by $\theta(X)$ the Lie derivative with respect to a vector field X.

When M^n is compact, let the volume element of M^n be ω. Then for any p-forms u and v, we define the global inner product by

$$(u, v) = \int_{M^n} \langle u, v \rangle \, \omega$$

where $\langle u, v \rangle$ shows the local inner product of u and v. It is known that the following relations hold good [6]

$$(\Phi u, v) = -(u, \Phi v), \tag{1.6}$$

$$(\nabla_\eta u, v) = -(u, \nabla_\eta v), \tag{1.7}$$

$$(\Gamma u, v) = (u, Dv) - (n-1)(u, i(\eta) v), \tag{1.8}$$

where u and v are p-forms in (1.6) and (1.7), and u is p-form and v is $(p+1)$-form in (1.8).

Now in a compact Sasakian space, we take a special C-Killing form u which is characterized by

$$\nabla_\lambda u_\mu + \nabla_\mu u_\lambda = 2u^\rho(\varphi_{\rho\lambda} \eta_\mu + \varphi_{\rho\mu} \eta_\lambda). \tag{1.9}$$

Then we know [6]

PROPOSITION 1.1. *In a compact Sasakian space, Φu is a closed form for any C-Killing form u.*

PROPOSITION 1.2. *In a compact Sasakian space, du is hybrid with respect to φ for any special C-Killing form u, that is, it is valid that*

$$\varphi_\lambda{}^\rho \varphi_\mu{}^\sigma (du)_{\rho\sigma} = (du)_{\lambda\mu}, \qquad \eta^\rho (du)_{\rho\lambda} = 0.$$

2. Contravariant *C*-analytic 1-form. In this section we suppose that M^n is an n-dimensional compact Sasakian space, and consider the contravariant C-analytic 1-form u, i.e., a 1-form satisfying

$$\nabla_\lambda u_\mu = \varphi_\lambda{}^\rho \varphi_\mu{}^\sigma \nabla_\rho u_\sigma + u^\rho(\varphi_{\rho\lambda} \eta_\mu + \varphi_{\rho\mu} \eta_\lambda). \tag{2.1}$$

In the following we call in short C-analytic for contravariant C-analytic. It is evident that the 1-form η is C-analytic. We denote by u' the η-component of a 1-form u.

LEMMA 2.1. *We have for any C-analytic 1-form u,*

$$\nabla_\lambda u' = 0\,, \tag{2.2}$$

$$\theta(\eta)u = 0\,. \tag{2.3}$$

PROOF. Contracting (2.1) by η^μ or η^λ, we can obtain easily (2.2) or (2.3) respectively.

LEMMA 2.2. *A special C-Killing form u is C-analytic.*

PROOF. If u is a special C-Killing form, du is hybrid by virtue of Proposition 1.2. Hence we have

$$\nabla_\lambda u_\mu - \nabla_\mu u_\lambda = \varphi_\lambda{}^\rho \varphi_\mu{}^\sigma(\nabla_\rho u_\sigma - \nabla_\sigma u_\rho)\,.$$

From (1.9), the right hand side is equal to $2\varphi_\lambda{}^\rho\varphi_\mu{}^\sigma\nabla_\rho u_\sigma$. Therefore again using (1.9), we have (2.1) immediately.

LEMMA 2.3. *A C-analytic* 1-*form u satisfies*

$$\Phi du = 0\,. \tag{2.4}$$

PROOF. Making use of (2.1) and (2.3), we can get

$$\varphi_\lambda{}^\rho\nabla_\rho u_\mu + \varphi_\mu{}^\rho\nabla_\lambda u_\rho = \eta_\lambda u_\mu + \eta_\mu u_\lambda - 2u'\eta_\lambda\eta_\mu\,. \tag{2.5}$$

Exchanging the subscripts λ and μ in (2.5) and subtracting side by side, we have

$$\varphi_\lambda{}^\rho(\nabla_\rho u_\mu - \nabla_\mu u_\rho) + \varphi_\mu{}^\rho(\nabla_\lambda u_\rho - \nabla_\rho u_\lambda) = 0$$

which means

$$(\Phi du)_{\lambda\mu} = 0\,.$$

LEMMA 2.4. *A C-analytic* 1-*form u is special C-Killing if and only if it satisfies*

$$d\Phi u = 0\,. \tag{2.6}$$

PROOF. We have from (2.1)

$$\nabla_\lambda u_\mu + \nabla_\mu u_\lambda = 2u^\rho(\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda) + \varphi_\lambda{}^\rho\varphi_\mu{}^\sigma(\nabla_\rho u_\sigma + \nabla_\sigma u_\rho)$$

and therefore u is special C-Killing if and only if it satisfies

$$\varphi_\lambda{}^\rho \varphi_\mu{}^\sigma(\nabla_\rho u_\sigma + \nabla_\sigma u_\rho) = 0 . \tag{2.7}$$

On the other hand, for any 1-form u, the following relation

$$\Phi du - d\Phi u = \Gamma u + e(\eta) u$$

is valid. Now we suppose that a C-analytic 1-form u satisfies (2.6). Then taking account of (2.4), we have $\Gamma u + e(\eta) u = 0$, and hence

$$\varphi_\lambda{}^\rho \nabla_\rho u_\mu - \varphi_\mu{}^\rho \nabla_\rho u_\lambda + \eta_\lambda u_\mu - \eta_\mu u_\lambda = 0$$

is obtained. Then we can get

$$\nabla_\lambda u_\mu = -\varphi_\lambda{}^\rho \varphi_\mu{}^\sigma \nabla_\sigma u_\rho + u^\rho(\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda)$$

from which we have (2.7) because of (2.1). Therefore u is special C-Killing. The converse statement is a direct result of Proposition 1.1.

LEMMA 2.5. *A C-analytic 1-form u satisfies*

$$(\Gamma u, e(\eta) u) = -(e(\eta) u, e(\eta) u) , \tag{2.8}$$

$$(\Gamma u, \Gamma u) = (e(\eta) u, e(\eta) u) + (\delta u, \delta u) . \tag{2.9}$$

PROOF. For a C-analytic 1-form u, we have $u' = $constant. Hence (2.8) comes from Lemma 4.1 in [6]. We show (2.9). Differentiating (2.1) by ∇_κ, we have

$$\begin{aligned}\nabla_\kappa \nabla_\lambda u_\mu = {} & \varphi_\lambda{}^\rho \varphi_\mu{}^\sigma \nabla_\kappa \nabla_\rho u_\sigma + \varphi_\lambda{}^\rho(\nabla_\rho u_\kappa - \nabla_\kappa u_\rho)\eta_\mu + u^\rho(\varphi_{\rho\lambda}\varphi_{\kappa\mu} + \varphi_{\rho\mu}\varphi_{\kappa\lambda}) \\ & - u_\mu g_{\kappa\lambda} - u_\lambda g_{\kappa\mu} - 2u_\kappa \eta_\lambda \eta_\mu + 2u'(\eta_\mu g_{\kappa\lambda} + \eta_\lambda g_{\kappa\mu}) .\end{aligned} \tag{2.10}$$

Using the fact that

$$\begin{aligned}\varphi^{\lambda\rho}\nabla_\lambda \nabla_\rho u_\sigma &= -\frac{1}{2}\varphi^{\lambda\rho} R_{\lambda\rho\sigma}{}^\tau u_\tau \\ &= -(R_{\sigma\rho}\varphi^{\tau\rho} + (n-2)\varphi_\sigma{}^\tau) u_\tau ,\end{aligned}$$

we transvect (2.9) by $g^{\kappa\lambda}$; then

$$\nabla^\rho \nabla_\rho u_\mu + R_{\mu\rho} u^\rho = -4u_\mu - 2(Du - (n+1)u')\eta_\mu \tag{2.11}$$

holds good. Moreover we can calculate the following relations from (2.10)

$$\varphi^{\kappa\mu} \nabla_\kappa \nabla_\lambda u_\mu = -\varphi_\lambda{}^\rho R_{\rho\sigma} u^\sigma - (n+2)\varphi_\lambda{}^\rho u_\rho + \varphi_\lambda{}^\rho \nabla_\rho(\delta u)\,, \tag{2.12}$$

$$\eta^\kappa \eta^\lambda \nabla_\kappa \nabla_\lambda u_\mu = -u_\mu + u' \eta_\mu\,. \tag{2.13}$$

Making use of (2.11), (2.12) and (2.13), we can get

$$\begin{aligned}(D\Gamma u)_\lambda &= \varphi^{\rho\sigma} \nabla_\rho(\varphi_\sigma{}^\tau \nabla_\tau u_\lambda - \varphi_\lambda{}^\tau \nabla_\tau u_\sigma) \\ &= (Du-u')\eta_\lambda - (n-2)u_\lambda + (d\delta u)_\lambda - \nabla_\eta(\delta u)\eta_\lambda\,.\end{aligned}$$

Then with the aid of (1.7) and (1.8), we have

$$\begin{aligned}(u, D\Gamma u) &= (u', Du-u') - (n-2)(u,u) + (\delta u, \delta u) - (u', \nabla_\eta \delta u) \\ &= (\Gamma u', u) + (n-2)(u', u') - (n-2)(u,u) + (\delta u, \delta u) \\ &= -(n-2)(e(\eta)u, e(\eta)u) + (\delta u, \delta u)\,.\end{aligned}$$

On the other hand, the left hand side of this relation becomes by virtue of (1.8)

$$\begin{aligned}&= (\Gamma u, \Gamma u) + (n-1)(e(\eta)u, \Gamma u) \\ &= (\Gamma u, \Gamma u) - (n-1)(e(\eta)u, e(\eta)u)\,,\end{aligned}$$

and consequently we can obtain

$$(\Gamma u, \Gamma u) = (e(\eta)u, e(\eta)u) + (\delta u, \delta u)\,.$$

THEOREM 2.6. *If a C-analytic* 1-*form u is coclosed, then it is special C-Killing.*

PROOF. We have from the assumption that $\delta u=0$ and (2.9)

$$(\Gamma u, \Gamma u) = (e(\eta)u, e(\eta)u)\,.$$

Hence it follows that

$$\begin{aligned}(\Gamma u+e(\eta)u, \Gamma u+e(\eta)u) &= (\Gamma u, \Gamma u) + 2(\Gamma u, e(\eta)u) + (e(\eta)u, e(\eta)u) \\ &= 0\,,\end{aligned}$$

which means that $\Gamma u+e(\eta)u=0$. Therefore by virtue of Lemma 2.3 and 2.4,

u is special C-Killing.

LEMMA 2.7. *If u is a C-analytic 1-form, then Φu is also C-analytic.*

PROOF. From (2.6), we can get

$$\begin{aligned}\nabla_\lambda(\varphi^\rho{}_\mu u_\rho) &= \eta_\mu u_\lambda - u' g_{\lambda\mu} + \varphi_\mu{}^\rho \nabla_\lambda u_\rho \\ &= -\varphi_\lambda{}^\rho \nabla_\rho u_\mu + \eta_\lambda u_\mu + 2\eta_\mu u_\lambda - u' g_{\lambda\mu} - 2u' \eta_\lambda \eta_\mu ,\end{aligned}$$

and hence we have

$$\begin{aligned}\varphi_\lambda{}^\rho \varphi_\mu{}^\sigma \nabla_\rho(\varphi_\sigma{}^\tau u_\tau) &= \varphi_\mu{}^\sigma \nabla_\lambda u_\sigma - u' g_{\lambda\mu} - \eta_\lambda u_\mu + 2u' \eta_\lambda \eta_\mu \\ &= \nabla_\lambda(\varphi_\mu{}^\sigma u_\sigma) - \eta_\mu u_\lambda - \eta_\lambda u_\mu + 2u' \eta_\lambda \eta_\mu .\end{aligned}$$

On the other hand, it is valid that

$$(\varphi^{\rho\tau} u_\tau)(\varphi_{\rho\lambda} \eta_\mu + \varphi_{\rho\mu} \eta_\lambda) = \eta_\lambda u_\mu + \eta_\mu u_\lambda - 2u' \eta_\lambda \eta_\mu .$$

Therefore for a 1-form $\bar{u} = \Phi u$, we have

$$\nabla_\lambda \bar{u}_\mu = \varphi_\lambda{}^\rho \varphi_\mu{}^\sigma \nabla_\rho \bar{u}_\sigma + \bar{u}^\rho(\varphi_{\rho\lambda} \eta_\mu + \varphi_{\rho\mu} \eta_\lambda) .$$

This shows that $\bar{u}$ is C-analytic.

LEMMA 2.8. *Let u be a 1-form satisfying $Du=0$. Then Φu is C-Killing if and only if u is C-analytic. In this case it holds good that $u'=0$ and $du=0$.*

PROOF. Suppose that u is C-analytic. Then we have

$$\delta\Phi u = (n-1)u' .$$

Since u' is constant, we can get integrating on M^n

$$0 = \int (\delta\Phi u)\,\omega = (n-1)u' \int \omega .$$

Therefore u' must be zero, and we have

$$\delta\Phi u = 0 .$$

As Φu is C-analytic, it is C-Killing by virtue of Theorem 2.6. Conversely, if Φu is C-Killing, then we have

$$0 = \delta\Phi u = Du - (n-1)u'$$
$$= (n-1)u',$$

hence $u'=0$ holds good. Therefore $u = -\Phi^2 u$ is also C-analytic and $du=0$ is valid.

N. B. If a 1-form u is closed, then it satisfies $Du = 0$. Thus we can apply Lemma 2.8 for a closed 1-form u.

Next we study the integral formula for a C-analytic form. We put for a 1-form u

$$U_{\lambda\mu} = \nabla_\lambda u_\mu - \varphi_\lambda{}^\rho \varphi_\mu{}^\sigma \nabla_\rho u_\sigma - u^\rho(\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda),$$

and

$$A_\lambda = \nabla^\rho \nabla_\rho u_\lambda + R_\lambda{}^\rho u_\rho + 4u_\lambda + 2(Du-(n+1)u')\eta_\lambda,$$
$$B_\lambda = \eta^\rho \nabla_\rho u^\sigma \varphi_{\sigma\lambda} + u_\lambda - u'\eta_\lambda,$$
$$C_\lambda = \eta^\rho \eta^\sigma \nabla_\rho \nabla_\sigma u_\lambda + u_\lambda - u'\eta_\lambda.$$

Then we can calculate

$$\int U_{\lambda\mu} U^{\lambda\mu} \omega = \int (-2A_\lambda + 2nB_\lambda + C_\lambda) u^\lambda \omega - 4(\Gamma u', u) - (du', du') + (\nabla_\eta u', \nabla_\eta u'). \tag{2.14}$$

Now we have for a 1-form u

$$\nabla^\rho \nabla_\rho u' = \eta^\sigma \nabla^\rho \nabla_\rho u_\sigma + 2Du - (n-1)u'.$$

Hence if u satisfies $A_\lambda=0$, then we see that $\nabla^\rho \nabla_\rho u'=0$, which shows that u' is constant on M^n. Therefore the following lemma is valid.

LEMMA 2.9. *If a 1-form u satisfies*

$$A_\lambda = 0, \tag{2.15}$$

then u' is a constant function.

From this lemma, it follows that for a 1-form u satisfying (2.15), we can get $\Gamma u'=0$, $du'=0$ and $\nabla_\eta u'=0$. Moreover if u satisfies

(2.16) $$\theta(\eta)u = 0\,,$$

then we have easily $B_\lambda u^\lambda = C_\lambda u^\lambda = 0$. Therefore if a 1-form u satisfies the relations (2.15) and (2.16) then it is valid that

$$\int (U_{\lambda\mu} U^{\lambda\mu})\,\omega = 0\,.$$

Hence we have $U_{\lambda\mu}=0$. This shows that u is a C-analytic 1-form. Conversely, a C-analytic 1-form satisfies the relations (2.11) and (2.3), which are the same as (2.15) and (2.16). Therefore the following theorem is proved.

THEOREM 2.10. *A 1-form u is C-analytic if and only if it satisfies*

$$\nabla^\rho \nabla_\rho u_\lambda + R_\lambda{}^\rho u_\rho + 4u_\lambda + 2(Du-(n+1)u')\,\eta_\lambda = 0\,,$$

$$\theta(\eta)\,u_\lambda = 0\,.$$

N. B. Calculating similarly, we have that a 1-form u is special C-Killing if and only if it satisfies (2.15), (2.16) and $\delta u=0$. This is the same result as Theorem 2.6.

For a later use, we show the following

LEMMA 2.11. *A C-analytic 1-form u satisfies*

(2.17) $$2\varphi^{\lambda\mu}\nabla_\kappa \nabla_\lambda u_\mu = (d\Lambda du)_\kappa\,.$$

PROOF. We have

$$\begin{aligned}2\varphi^{\lambda\mu}\nabla_\kappa \nabla_\lambda u_\mu &= \varphi^{\lambda\mu}\nabla_\kappa (du)_{\lambda\mu}\\ &= \nabla_\kappa(\Lambda du) - 2\eta^\lambda (du)_{\lambda\kappa} = (d\Lambda du)_\kappa\,.\end{aligned}$$

3. *C*-Einstein space. A Sasakian space is called C-Einstein if its Ricci tensor $R_{\lambda\mu}$ satisfies

(3.1) $$R_{\lambda\mu} = a\,g_{\lambda\mu} + b\,\eta_\lambda \eta_\mu\,.$$

A C-Fubini space is C-Einstein. In this section we assume that the dimension of the space > 3. It is known that in a C-Einstein space, a and b are constant scalar functions, and satisfy $a+b=n-1$. (M. Okumura [1]). In the following we consider a compact C-Einstein space and show that a C-analytic 1-form can be decomposed by special C-Killing forms.

The formula (2. 15) for a C-analytic 1-form u can be written in a C-Einstein space as

$$\Delta u_{\lambda} = 2(a+2)u_{\lambda} + 2(Du+(b-n-1)u')\eta_{\lambda}, \tag{3. 2}$$

where Δ denotes the Laplacian operator.

LEMMA 3.1. *In a compact C-Einstein space, if a 1-form u is C-analytic, then $(d\delta u)$ is C-analytic, too.*

PROOF. Since $\theta(\eta)$ commutes with the operators d and δ, we have for a C-analytic 1-form u

$$\theta(\eta)(d\delta u) = d\delta(\theta(\eta)u) = 0, \tag{3. 3}$$

and moreover it satisfies (3. 2). From (2. 17) we can get

$$\begin{aligned}\delta((Du+(b-n-1)u')\eta) &= -\nabla_{\eta}Du = -\frac{1}{2}(i(\eta)d\Lambda du)\\ &= -\frac{1}{2}(\Lambda d\theta(\eta)u - d\Lambda i(\eta)du) = 0.\end{aligned}$$

Therefore we have

$$\Delta\delta u = 2(a+2)\delta u, \tag{3. 4}$$

which shows that

$$\Delta d\delta u = 2(a+2)d\delta u.$$

We put $\bar{u}=d\delta u$. Then we have $i(\eta)\bar{u}=0$ and $D\bar{u}=0$ because $\bar{u}$ is closed. Therefore the relation (3. 2) becomes $\Delta\bar{u}=2(a+2)\bar{u}$. Together with (3. 4), $\bar{u}$ is a C-analytic 1-form by virtue of Theorem 2.10.

LEMMA 3.2. *In a compact C-Einstein space with $a+2\neq 0$,*

$$v_{\lambda} = u_{\lambda} - \frac{1}{2(a+2)}(d\delta u)_{\lambda}$$

is special C-Killing for a C-analytic 1-form u.

PROOF. By virtue of Lemma 3.1, v is C-analytic. We have from (3. 4)

$$\delta(d\delta u) = \Delta(\delta u) = 2(a+2)\delta u,$$

hence we can get

$$\begin{aligned}\delta v &= \delta u - (1/2(a+2))\,\delta(d\delta u)\\ &= \delta u - \delta u = 0\,.\end{aligned}$$

This shows that v is a special C-Killing form owing to Theorem 2.6.

THEOREM 3.3. *In a compact C-Einstein space with $a+2\neq 0$, a C-analytic 1-form u can be decomposed in the form*

$$u = v + \Phi w$$

where v and w are special C-Killing forms.

PROOF. Let u be a C-analytic 1-form. Putting $\bar{w} = d\delta u/2(a+2)$, there exists a special C-Killing form v such that

$$u = v + \bar{w}$$

holds good. Since $\bar{w}$ is closed, by virtue of Lemma 2.8, $\Phi\bar{w}$ $(= -w)$ is C-Killing and we have $i(\eta)\bar{w} = 0$. Therefore we can get $\Phi w = \bar{w}$. Thus we see that the C-analytic 1-form u can be written as

$$u = v + \Phi w\,,$$

where v and w are special C-Killing forms.

Unfortunately, the uniqueness of the decomposition in the theorem is negative. In the first place, we consider a 1-form u which is harmonic and special C-Killing simultaneously. Then it satisfies the following

$$\nabla_\lambda u_\mu = u^\rho(\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda)\,. \tag{3.5}$$

Conversely, a 1-form in a compact Sasakian space which satisfies (3.5) is harmonic and special C-Killing.

LEMMA 3.4. *In a compact C-Einstein space with $a+2\neq 0$, there exists no non-zero 1-form which satisfies* (3.5).

PROOF. We take a 1-form u satisfying (3.5). As u is harmonic, we know that

$$u' = i(\eta)u = 0 \tag{3.6}$$

owing to the theorem of Tachibana in [5]. Further, it satisfies

$$\nabla^\rho \nabla_\rho u_\lambda = R_\lambda{}^\rho u_\rho\,, \qquad Du = 0\,. \tag{3.7}$$

On the other hand, as u is special C-Killing, we have

$$\nabla^\rho \nabla_\rho u_\lambda + R_\lambda{}^\rho u_\rho = -4u_\lambda - 2(Du-(n+1)u')\eta_\lambda\,.$$

Taking account of (3.6) and (3.7), it follows that

$$R_\lambda{}^\rho u_\rho = -2u_\lambda\,.$$

In case of C-Einstein, it is equal to $(a+2)u_\lambda=0$, from which we can get $u_\lambda=0$ if $a+2\neq 0$.

Now let u be a C-analytic 1-form in a compact C-Einstein space with $a+2\neq 0$. Taking the two decompositions of Theorem 3.3, we can write

$$v_1 + \Phi w_1 = v_2 + \Phi w_2$$

where v_1, v_2, w_1 and w_2 are all special C-Killing 1-forms. Then we have

$$v_1 - v_2 + \Phi(w_1 - w_2) = 0\,.$$

Since w_1-w_2 is C-Killing, the form $\Phi(w_1-w_2)$ is closed by virtue of Proposition 1.1. Therefore v_1-v_2 is at the same time closed and coclosed, and hence v_1-v_2 is harmonic. Similarly v_1-v_2 and $\Phi(w_1-w_2)$ are harmonic and special C-Killing. Making use of Lemma 3.4, we have

$$v_1 - v_2 = 0\,,$$

$$\Phi(w_1 - w_2) = 0\,.$$

Hence we can conclude that there exists some constant c such that

$$w_1 = w_2 + c\eta\,.$$

Thus we have the following

THEOREM 3.5. *In the decomposition of a C-analytic 1-form u in Theorem 3.3, v is uniquely determined, and w is determined up to $c\eta$ (c is* *a constant*).

Now we shall treat with the case of C-Einstein space with $a+2=0$. Then

THEOREM 3.6. *In a compact C-Einstein space with $a+2=0$, any C-analytic 1-form is special C-Killing.*

PROOF. For a C-analytic 1-form u, we have (3.4). Then if $a+2=0$,

$$\Delta\delta u = 0$$

holds good. Thus δu is a constant scalar function. While it is codifferential, we see that δu must be zero on a compact space. By virtue of Theorem 2.6, u is C-Killing.

BIBLIOGRAPHY

[1] M. OKUMURA, Some remarks on space with a certain contact structure, Tôhoku Math, Journ., 14(1962), 135–142.
[2] S. TANNO, Note on infinitesimal transformations over contact manifolds, Tôhoku Math. Journ., 14(1962), 416–430.
[3] Y. MATSUSHIMA, Sur la structure du groupe d'homeomorphismes analytiques d'une crtaine variété kählérienne, Nagoya Math. Journ., 11(1957), 145–150.
[4] K. YANO, Sur un théorème de M. Matsushima, Nagoya Math. Journ., 12(1957), 167–170.
[5] S. TACHIBANA, On harmonic forms in compact Sasakian spaces, Tôhoku Math. Journ., 17(1965), 271–284.
[6] Y. OGAWA, On C-Killing forms in a compact Sasakian space, Tôhoku Math. Journ., 19(1967), 267–296.

DEPARTMENT OF MATHEMATICS
OCHANOMIZU UNIVERSITY
TOKYO, JAPAN