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ON THE HOMOMORPHISM OF VON NEUMANN ALGEBRA

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1. Introduction. The purpose of this paper is to study the σ-weak continuity of homomorphism of von Neumann algebra. For the σ-weak continuity of homomorphism of von Neumann algebra, we have many results that have been shown recently. In this paper, we shall show the following:

Let M be a properly infinite von Neumann algebra with the separable predual M_*, then any homomorphism π from M onto a von Neumann algebra N is σ-weakly continuous.

Moreover, we shall show some results by using the above fact. Before going into the discussions, the author wishes to express his hearty to Prof. M. Fukamiya and Prof. M. Takesaki in the presentation of this paper.

2. Main result. We start with the following lemma in [8] (Theorem 7 in [8]). In this paper, we shall give a simple proof of Theorem 7 in [8].

LEMMA 1. *Let M be a σ-finite, properly infinite von Neumann algebra, then any *-homomorphism π from M into a σ-finite von Neumann algebra N is σ-weakly continuous.*

PROOF. From Theorem 5 in [8], we have the following decomposition of π; $\pi = \pi_1 \oplus \pi_2$ where π_1 is the σ-weakly continuous part of π and π_2 is the singular part of π. If π is not σ-weakly continuous, then the singular part π_2 of π is not zero. Therefore, we assume that π is singular and derive a contradiction. Since π is singular, there exists a normal faithful positive linear functional ψ on N such that ${}^t\pi(\psi) = \varphi$ is a singular positive linear functional on M. Then, from the characterization of a singular positive linear functional in [9] and the σ-finiteness of M, we can find a family $\{e_n\}_{n=1}^{\infty}$ of countable orthogonal projections in M such that $1 = \sum_{n=1}^{\infty} e_n$ and $\varphi(e_n) = 0$ for all n. Since $0 = \varphi(e_n) = {}^t\pi(\psi)(e_n) = \psi(\pi(e_n))$ and ψ is faithful, $\pi(e_n) = 0$.

Therefore, there exists a family $\{e_n\}_{n=1}^{\infty}$ of countable orthogonal projections such that $\sum_{n=1}^{\infty} e_n = 1$ and $\pi(e_n) = 0$ for all n.

Now, since M is properly infinite, there exists in M a family $\{p_n\}_{n=1}^{\infty}$ of orthogonal projections such as $p_n \sim 1$. Take a partial isometry v_n such that $v_n^* v_n = p_n$ and $v_n v_n^* = 1$ for each n. Define the family $\{q_n\}_{n=1}^{\infty}$ of orthogonal projections by $q_n = v_n^*\left(\sum_{k=1}^{n} e_k\right) v_n$.

Let $\{n_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers, then we have

$$q_{n_{i+1}} = v_{n_{i+1}}^*\left(\sum_{k=1}^{n_{i+1}} e_k\right) v_{n_{i+1}} \geqq v_{n_{i+1}}^*\left(\sum_{k=n_i+1}^{n_{i+1}} e_k\right) v_{n_{i+1}} \sim \sum_{k=n_i+1}^{n_{i+1}} e_k\,.$$

Hence $\sum_{i=1}^{\infty} q_{n_i} \succsim \sum_{k=n_1+1}^{\infty} e_k$, and $\pi\left(\sum_{i=1}^{\infty} q_{n_i}\right) \succsim \pi\left(\sum_{k=n_1+1}^{\infty} e_k\right) = \pi\left(1-\sum_{k=1}^{n_1} e_k\right) = \pi(1) \neq 0$.
Therefore $\pi\left(\sum_{i=1}^{\infty} q_{n_i}\right) \neq 0$. On the other hand $\pi(q_n)=\pi(v_n)*\left(\sum_{k=1}^{n}\pi(e_k)\right)\pi(v_n)=0$.

Next, let $\{r_n\}_{n=1}^{\infty}$ be a countable set of all rational numbers. For each real number s, we can choose an infinite subsequence $\{r_{n_i}\}_{i=1}^{\infty}$ of $\{r_n\}_{n=1}^{\infty}$ such that, for each positive integer i, we have $0 < |r_{n_i}-s| < 1/i$ and $n_j < n_i$ for every $j < i$. Now, let us correspond to s the index set $\{n_i\}_{i=1}^{\infty}$ of the above sequence $\{r_{n_i}\}_{i=1}^{\infty}$. Then if $s \neq s'$ for real numbers s and s', $\{n_i\} \cap \{n_i'\}$ is at most a finite set where $\{n_i\}$ and $\{n_i'\}$ are corresponding index sets of s and s' respectively, because $\{r_{n_i}\}_{i=1}^{\infty}$ and $\{r_{n_i'}\}_{i=1}^{\infty}$ converge to s and s' respectively. Therefore, if we set $q_s = \sum_{i=1}^{\infty} q_{n_i}$ for a real number s where $\{n_i\}_{i=1}^{\infty}$ is an index set corresponding to s, we get $\pi(q_s) \neq 0$ and $\pi(q_s q_{s'}) = 0$ if $s \neq s'$. Hence the family $\{\pi(q_s);\ s \in R\}$ where R is the set of all real numbers is a family of orthogonal non-zero projections of N with the power of continuum $\mathfrak{c}$. But, N is σ-finite, this is a contradiction. Therefore, π does not admit the singular part, hence π is σ-weakly continuous. This completes the proof of Lemma 1.

Next, we shall need a result due to T. Okayasu [4].

LEMMA 2. (The Polar Decomposition Theorem). *Let A be a C^*-algebra with the identity* 1, *then any isomorphism π (not necessary *-preserving) from A onto a von Neumann algebra M has the decomposition*: $\pi = \pi_1 \circ \pi_2$, *where π_1 is an inner automorphism of M and π_2 is a *-isomorphism from A onto M.*

By Lemma 2, we have

LEMMA 3. *Let M and N be von Neumann algebras acting on Hilbert spaces H and K respectively. If any *-homomorphism from M onto N is σ-weakly continuous, then any homomorphism π from M onto N is σ-weakly continuous.*

PROOF. Since $\pi(M) = N$, by the Rickart's theorem, the kernel $\pi^{-1}(0)$ of π is a uniformly closed two-sided ideal, hence the quotient algebra $M/\pi^{-1}(0)$ is a C^*-algebra. Let δ be the canonical mapping from M onto $M/\pi^{-1}(0)$, then δ is a *-homomorphism. Let $\tilde{\pi}$ be the mapping from $M/\pi^{-1}(0)$ onto N which is induced canonically by π, then $\tilde{\pi}$ is an isomorphism from $M/\pi^{-1}(0)$ onto N. Therefore, by Lemma 2, we have the following decomposition of $\tilde{\pi}$: $\tilde{\pi} = \tilde{\pi}_1 \circ \tilde{\pi}_2$ where $\tilde{\pi}_1$ is an inner automorphism of N and $\tilde{\pi}_2$ is a *-isomorphism from $M/\pi^{-1}(0)$ onto N, whence we have: $\pi = \tilde{\pi} \circ \delta = \tilde{\pi}_1 \circ (\tilde{\pi}_2 \circ \delta)$. Since $\tilde{\pi}_1$ is the inner automorphism of N, it is σ-weakly continuous. Furthermore, since, by the assumption, $\tilde{\pi}_2 \circ \delta$ is a *-homomorphism from M onto N, it is σ-weakly continuous. Therefore, π is σ-weakly continuous. This completes the proof of Lemma 3.

COROLLARY. *Let M be a σ-finite, properly infinite von Neumann algebra, then any homomorphism π from M onto a σ-finite von Neumann algebra N is σ-weakly continuous.*

By using the above results, we can show the main result.

THEOREM I. *Let M be a properly infinite von Neumann algebra acting on a Hilbert space H and having the separable predual M_*, then any homomorphism π from M onto a von Neumann algebra acting on a Hilbert space K is σ-weakly continuous.*

PROOF. To prove Theorem I, we may suppose, by Lemma 1 and Lemma 3, that π is a *-homomorphism. At first, we shall show that the cardinal number of the set of all projections in M is not larger than $\mathfrak{c}$. For each non-zero projection e in M, there exists a normal state ψ on M such that $\mathrm{supp}(\psi) \leqq e$, and so, for any family $\{e_\lambda\}_{\lambda \in \Lambda}$ of mutually orthogonal projections in M, there exists a family $\{\psi_\lambda\}_{\lambda \in \Lambda}$ of normal states on M such that $\mathrm{supp}(\psi_\lambda) \leqq e_\lambda$ for each $\lambda \in \Lambda$. Since $\{e_\lambda\}_{\lambda \in \Lambda}$ is mutually orthogonal, we have: if $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$, $\|\psi_\lambda - \psi_\mu\| = \|\psi_\lambda\| + \|\psi_\mu\| = 2$. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a countable dense subset of M_*, then, for each $\lambda \in \Lambda$, there exists an element φ_{n_λ} in $\{\varphi_n\}_{n=1}^{\infty}$ with $\|\psi_\lambda - \varphi_{n_\lambda}\| < 1/2$. Furthermore, for two distinct λ and μ of Λ, $\varphi_{n_\lambda} \neq \varphi_{n_\mu}$. Therefore, we see that any family of mutually orthogonal non-zero projections in M is at most countable and M is a σ-finite von Neumann algebra. Since M is σ-finite, for each non-zero projection e in M, there exists a normal state ψ_e on M with $\mathrm{supp}(\psi_e) = e$, and it is clear that, for two distinct projections e and f in M, $\psi_e \neq \psi_f$. That is, the cardinal number of the set of all projections in M is not larger than the cardinal number of M_*. Now, since M_* is separable, the cardinal number of M_* is $\mathfrak{c}$. Therefore, the cardinal number of the set of all projections in M is not larger than $\mathfrak{c}$.

Next, we shall show that N is a σ-finite von Neumann algebra. If otherwise, the cardinal number of the set of all projections in N will be exactly larger than $\mathfrak{c}$. In fact, we can take a family $\{e_\alpha\}_{\alpha \in A}$ of orthogonal projections in N where A is the index set with the cardinal number β which is exactly larger than $\aleph_0$, because N is supposed not to be σ-finite. Then, the cardinal number of the set of all subsets of $\{e_\alpha\}_{\alpha \in A}$ is exactly larger than $\mathfrak{c}$, and so the cardinal number of the set of all projections in N will be exactly larger than $\mathfrak{c}$. Therefore, if N is not a σ-finite von Neumann algebra, then the cardinal number of the set of all projections in N is exactly larger than the continuum cardinal number $\mathfrak{c}$. But, for each projection p in N, there exists a projection e in M such that $\pi(e) = p$ (Theorem 3.2 in [10]). As we have proved above, the cardinal number of the set of all projections in M is not larger than the continuum cardinal number $\mathfrak{c}$. Therefore, we have a contradiction, and so N is a σ-finite von Neumann algebra.

By the above argument, M and N are σ-finite von Neumann algebras. Therefore, by Lemma 1, if π is any *-homomorphism from M onto N, then π is σ-weakly continuous. Furthermore, by Corollary of Lemma 3, even if π is not *-preserving, π is σ-weakly continuous. This completes the proof of Theorem I.

REMARK 1. The above result has some applications in the reduction theory ([7]). Besides we can show the following Theorem II.

Following the notation in J. Glimm [3], let M be a von Neumann algebra, Z be the center of M and X be the spectrum of Z. If $\zeta \in X$, let $I(\zeta)$ be the uniform closure of the set $\left\{\sum_{i=1}^{n} z_i a_i\,;\ n \text{ is a positive integer, } a_i \text{ is an element of } M \text{ and } z_i \text{ is a central element with } z_i\hat{}(\zeta) = 0\right\}$ where $z_i\hat{}$ is the Gelfand representation of z_i. Then $I(\zeta)$ is a uniformly closed two-sided ideal in M, and the quotient algebra $M(\zeta) = M/I(\zeta)$ is a C^*-algebra. However, it has not yet been certain whether $M(\zeta)$ is a von Neumann algebra or not. In the remaning part, we shall show that $M(\zeta)$ is not necessary a von

Neumann algebra.

Let π_ζ be the canonical mapping from M onto $M(\zeta)$. Then, for $\zeta \in X$, $\pi_\zeta^{-1}(0) \cap Z = \{z \in Z ; z^\wedge(\zeta) = 0\}$ where $\pi_\zeta^{-1}(0)$ is the kernel of π_ζ. Therefore, π_ζ is σ-weakly continuous if and only if the set $\{\zeta\}$ is a closed and open set in X, and π_ζ is σ-weakly continuous for all $\zeta \in X$ if and only if Z is an atomic abelian von Neumann algebra. By considering the above argument, we have the following result.

THEOREM II. *Let M be a properly infinite von Neumann algebra with the separable predual M_* and the center Z wich is non-atomic. Then there exists an element ζ of the spectrum X of Z for which $M(\zeta)$ is not a von Neumann algebra.*

PROOF. Since Z is non-atomic, there exists an element ζ of X such that π_ζ is not σ-weakly continuous. If $M(\zeta)$ is a von Neumann algebra for such an element ζ, then π_ζ is a *-homomorphism from M onto a von Neumann algebra $M(\zeta)$, and, by Theorem I, π_ζ must be σ-weakly continuous; this is a contradiction. Therefore, $M(\zeta)$ can't be a von Neumann algebra, which completes the proof.

We may construct a von Neumann algebra which satisfies the condition in Theorem II. Let H be a countably infinite dimensional Hilbert space, $B(H)$ the von Neumann algebra of all bounded operators on H and $L^\infty(0,1)$ a von Neumann algebra of all essentially bounded functions under the Lebesgue measure μ on the open interval $(0,1)$. As $B(H)$ is properly infinite and $L^\infty(0,1)$ is a finite von Neumann algebra, the W^*-tensor product $M = L^\infty(0,1) \otimes B(H)$ is a properly infinite von Neumann algebra (p. 3.40 in [6]) and the center Z of M is $L^\infty(0,1) \otimes \boldsymbol{C}_H$ where $\boldsymbol{C}_H$ is the von Neumann algebra of all scalar multiples of the identity operator on H. Therefore, Z is a non-atomic abelian von Neumann algebra. Since the Hilbert space $L^2(0,1) \otimes H$ is separable, the predual M_* of M is separable.

REMARK 2. Prof. M. Takesaki kindly communicated me that Theorem I in the present paper may be applied to show the statement that: if H is a countably infinite dimensional Hilbert space, then the quotient algebra $B(H)/C(H)$ is not a von Neumann algebra where $B(H)$ is the von Neumann algebra of all bounded operators on H and $C(H)$ is the uniformly closed two-sided ideal of all completely continuous operators on H. In fact, $B(H)$ is a properly infinite von Neumann algebra and the canonical mapping π from $B(H)$ onto $B(H)/C(H)$ is not σ-weakly continuous (cf.[11]).

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