Tôhoku Math. Journ.
21(1969), 21-38.

THE AUTOMORPHISM GROUPS OF ALMOST CONTACT RIEMANNIAN MANIFOLDS

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(Received April 24, 1968)

1. Introduction. The maximum dimension of the group of isometries of an m-dimensional connected Riemannian manifold is $m(m+1)/2$. The maximum is attained if and only if the Riemannian manifold is of constant curvature and one of the following spaces (cf. [3], p. 308):

(i) an m-dimensional sphere S^m, or a real projective space RP^m,

(ii) an m-dimensional Euclidean space E^m,

(iii) an m-dimensional simply connected hyperbolic space H^m.

If M is a $2n$-dimensional connected almost Hermitian manifold, then the maximum dimension of the automorphism group of M is $n(n+2)$. The maximum is attained if and only if M is a homogeneous Kaehlerian manifold with constant holomorphic sectional curvature k and one of the following spaces (cf. [17]):

(i) a complex projective space CP^n with a Fubini-Study metric $(k>0)$,

(ii) a unitary space CE^n $(k=0)$,

(iii) an open ball CD^n with a homogeneous Kaehlerian structure of negative constant holomorphic sectional curvature $(k<0)$.

In this paper we consider the similar problem in almost contact Riemannian manifolds. To state the main theorem we prepare the followings. We denote by (ϕ, ξ, η, g) structure tensors of an almost contact Riemannian manifold N. An odd dimensional sphere S^{2n+1} (in E^{2n+2}) has the standard Sasakian structure (cf. [11]). An odd dimensional Euclidean space E^{2n+1} has also the standard Sasakian structure ([8], [9]). By T or L we denote a circle or a line. By (L, CD^n) we denote a line bundle over a CD^n (which is a product bundle). The space (L, CD^n) has a Sasakian structure (§8). In these three spaces ξ is an infinitesimal automorphism of the structure and generates a 1-parameter group $\exp t\xi$ $(-\infty<t<\infty)$ of automorphisms. Definitions of an $\mathcal{E}$-deformation

*) The author was partially supported by the Sakkokai Foundation.

and a D-homothetic deformation are given by (4.6)–(4.7) and (7.1)–(7.2).

THEOREM. *Let N be a connected almost contact Riemannian manifold of $(2n+1)$-dimension. Then the maximum dimension of the automorphism group is $(n+1)^2$. The maximum is attained if and only if the sectional curvature for 2-planes which contain ξ is a constant C and N is one of the following spaces*:

(i) $C>0$: *a homogeneous Sasakian manifold (or its ε-deformation) with constant ϕ-holomorphic sectional curvature H and*

(i-1) $H>-3$: *a space which is D-homothetically deformable to a unit sphere S^{2n+1} or its factor space $S^{2n+1}/F(t)$ where $F(t)$ denotes a finite group generated by* $\exp t\xi$ *($2\pi/t$ being an integer)*,

(i-2) $H=-3$: *a (Euclidean) space E^{2n+1} or its factor space $E^{2n+1}/F(t)$ where $F(t)$ is a cyclic group generated by* $\exp t\xi$ *(t being a real number)*,

(i-3) $H<-3$: *a space (L, CD^n) or its factor space $(L, CD^n)/F(t)$ where $F(t)$ is a cyclic group generated by* $\exp t\xi$ *(t being a real number)*,

(ii) $C=0$: *six global Riemannian products*:

$$T\times CP^n,\quad T\times CE^n,\quad T\times CD^n,$$

$$L\times CP^n,\quad L\times CE^n,\quad L\times CD^n,$$

(iii) $C<0$: *a product space $L\times_{ct}CE^n$ whose metric is given by*

$$g_{(t,x)}=(dt)^2_{(t)}+e^{2ct}G_{(x)}\qquad \text{(cf. Lemma 4.6)}.$$

As a corollary we have

COROLLARY. *Let N be a compact, connected and simply connected almost contact Riemannian manifold. If the maximum dimension of the automorphism group is attained, then N is a sphere with a Sasakian structure or its deformation.*

2. Preliminaries. An almost complex manifold M is defined by a structure tensor J of type (1,1), satisfying $JJX=-X$ for any vector field X on M. M is almost Hermitian if, moreover, it has a Riemannian metric G such that $G(JX,JY)=G(X,Y)$ for any vector fields X and Y. Then we have a 2-form W called the fundamental 2-form, which is defined by $W(X,Y)=G(X,JY)$. When the exterior derivative dW of W vanishes, M is called an almost Kaehlerian manifold. If we have $DJ=0$ for the Riemannian connection D defined by G, then M is a Kaehlerian manifold.

On the other hand, an almost contact structure on N is defined by three tensor fields : a (1, 1)-tensor ϕ, a vector field ξ and a 1-form η. They satisfy (cf. [9], [10], [11])

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1\,, \tag{2.1}$$

$$\phi\phi X = -X + \eta(X)\xi \tag{2.2}$$

for any vector field X on N. An almost contact structure is said normal if the torsion tensor N_{bc}^{a} (see (3.7)) vanishes. If N has an associated Riemannian metric g such that

$$g(\xi, X) = \eta(X)\,, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) \tag{2.4}$$

for any vector fields X and Y on N, then N is called an almost contact Riemannian manifold. Further, if $d\eta(X, Y) = 2g(X, \phi Y)$ is satisfied, then N is called a contact Riemannian manifold. When ξ is a Killing vector field, a contact Riemannian manifold is called a K-contact Riemannian manifold, and then ξ is an infinitesimal automorphism. Further if the structure is normal, then a contact Riemannian manifold N is called a Sasakian mainfold ([8], [11], etc.). A Sasakian manifold is always a K-contact Riemannian mainfold.

By $A(M)$ or $A(N)$ we denote the automorphism group of M or N. By ∇ we denote Riemannian connection defined by g.

3. The maximum dimension of the automorphism group of N. Let N be a $(2n+1)$-dimensional almost contact Riemannian manifold. Then the necessary and sufficient conditions for X to be an infinitesimal automorphism are

$$(L_X g)_{bc} = g_{cs}\nabla_b X^s + g_{bs}\nabla_c X^s = 0\,, \tag{3.1}$$

$$(L_X \xi)^a = X^s\nabla_s \xi^a - \xi^s\nabla_s X^a = 0\,, \tag{3.2}$$

$$(L_X \eta)_b = X^s\nabla_s \eta_b + \eta_s\nabla_b X^s = 0\,, \tag{3.3}$$

$$(L_X \phi)_b^a = X^s\nabla_s\phi_b^a - \phi_b^s\nabla_s X^a + \phi_s^a\nabla_b X^s = 0\,, \tag{3.4}$$

where a, b, c, s run from 1 to $2n+1$. In the sequel, indices i, j, k, r run from 1 to $2n$. We take a ϕ-basis $(e_1, \cdots, e_n, e_{1^*} = \phi e_1, \cdots, e_{n^*} = \phi e_n, e_\Delta = \xi)$ at a point P and its dual basis. Then any infinitesimal automorphism X vanishing at P satisfies $\nabla_\Delta X^a = 0$ and $\nabla_b X^\Delta = 0$ by (3.2) and (3.3). Non-vanishing components are $\nabla_j X^i$, and the set of all these is contained in the Lie algebra of the unitary group $U(n)$ by (3.1) and (3.4). Therefore it is at most n^2-dimensional. While the set of X non-vanishing at P is at most $(2n+1)$-dimensional. Thus we have (cf. [17])

LEMMA 3.1. *Let N be a $(2n+1)$-dimensional almost contact Riemannian manifold. Then we have* $\dim A(N) \leqq (n+1)^2$.

Now we show the following

LEMMA 3.2. *Let N be a $(2n+1)$-dimensional almost contact Riemannian manifold which admits the automorphism group $A(N)$ of the maximum dimension $(n+1)^2$. Assume that a tensor field $(K^{ab\cdots}{}_{cd\cdots})$ of type (p,q) is invariant by any infinitesimal automorphism. Then with respect to a ϕ-basis at P we have*

(i) $K^{ij\cdots}{}_{kl\cdots} = 0$ *if $p+q$ is odd,*
(ii) *If K is of type $(1,1)$ (or $(0,2)$), then*

$$K^i_j = C_1\delta^i_j + C_2\phi^i_j \quad (or \quad K_{ij} = C_1 g_{ij} + C_2\phi_{ij}),$$

where C_1 and C_2 are real numbers.

PROOF. If we consider the linear isotropy group of $A(N)$ at a point P with respect to a ϕ-basis, then it is $U(n)\times 1$ since $A(N)$ is of the maximum dimension. So it contains a 1-parameter group $e^{it}I\times 1$ and, in particular, $(-I)\times 1$ which is a map:

$$Y \longrightarrow (-Y + \eta(Y)\xi) + \eta(Y)\xi, \tag{3.5}$$

$$w \longrightarrow (-w + w(\xi)\eta) + w(\xi)\eta, \tag{3.6}$$

where Y is a tangent vector at P and w is a tangent covector. Therefore we have (i). On the other hand, (ii) may be known.

In an almost contact manifold N we have four torson tensors (which do not depend on the metric, but we write them using Riemannian connection of the associated metric):

$$N^a_{bc} = \phi^s_c(\nabla_s\phi^a_b - \nabla_b\phi^a_s) - \phi^s_b(\nabla_s\phi^a_c - \nabla_c\phi^a_s) + \eta_c\nabla_b\xi^a - \nabla_c\xi^a\eta_b, \tag{3.7}$$

$$N_{bc} = \phi^s_c(\nabla_b\eta_s - \nabla_s\eta_b) - \phi^s_b(\nabla_c\eta_s - \nabla_s\eta_c), \tag{3.8}$$

$$N^a_b = \xi^s\nabla_s\phi^a_b + \phi^a_s\nabla_b\xi^s - \phi^s_b\nabla_s\xi^a, \tag{3.9}$$

$$N_b = \xi^s(\nabla_b \eta_s - \nabla_s \eta_b). \tag{3.10}$$

There are relations among them (cf. [10]). The followings are required in the sequel.

$$N^a_{bs}\xi^s + \phi^a_s N^s_b + \xi^a N_b = 0, \tag{3.11}$$

$$\eta_s N^s_{bc} - N_{bs}\phi^s_c + N_b \eta_c = 0, \tag{3.12}$$

$$\eta_s N^s_b = N_{bs}\xi^s = N_s \phi^s_b, \tag{3.13}$$

$$N^a_s \xi^s = N_s \xi^s = 0, \tag{3.14}$$

$$\phi^a_s N^s_b + N^a_s \phi^s_b + \xi^a N_b = 0, \tag{3.15}$$

$$N_{bs}\phi^s_c - N_{sc}\phi^s_b - N_b \eta_c + N_c \eta_b = 0. \tag{3.16}$$

LEMMA 3.3. *If φ (or X) is an (infinitesimal) automorphism of an almost contact (Riemannian) manifold, then N^a_{bc}, N_{bc}, N^a_b and N_b are invariant by φ (or X).*

This is clear by (3.7)~(3.10).

LEMMA 3.4. *If an almost contact Riemannian manifold N admits the automorphism group of the maximum dimension $(n+1)^2$, then N is normal and homogeneous.*

PROOF. By Lemma 3.2 (i) we get $N^i_{jk} = 0$ and $N_j = 0$. By (3.14) we have $N_\Delta = 0$ and hence $N_b = 0$. Then by (3.13), (3.14) and (3.15) we have $N^\Delta_b = N^a_\Delta = 0$ and $\phi^a_s N^s_b + N^a_s \phi^s_b = 0$. Since N^a_b is invariant by $A(N)$, N^i_j is written as $N^i_j = c_1\delta^i_j + c_2\phi^i_j$ by Lemma 3.2 (ii). Thus we get $0 = \phi^i_r N^r_j + N^i_r \phi^r_j = 2\phi^i_r N^r_j$. Since ϕ^i_r is non-singular we have $N^r_j = 0$, and hence $N^a_b = 0$.

Similarly by (3.16) we get $N_{bc} = 0$.

Now $N^a_{b\Delta} = 0$ follows from (3.11). $N^\Delta_{bc} = 0$ follows from (3.12). Therefore we have $N^a_{bc} = 0$.

4. Classification.

We assume that spaces are connected.

LEMMA 4.1. *Let N be an almost contact Riemannian manifold which admits the automorphism group of the maximum dimension. Then the sectional curvature for 2-planes which contain ξ is equal to a constant C_1. More precisely we have*

$$R^a_{bcd}\xi^d = C_1\xi^a g_{bc} - C_1\eta_b\delta^a_c . \tag{4.1}$$

PROOF. Since the tensor field $\eta_a R^a_{bcd}\xi^d$ is invariant by $A(N)$, by Lemma 3.2 (ii), we have

$$R^{\Delta}_{jk\Delta} = C_1 g_{jk} + C_2\phi_{jk} \tag{4.2}$$

with respect to a ϕ-basis at P. As is well known, $R^{\Delta}_{jk\Delta}$ is symmetric with respect to j and k. Thus $R^{\Delta}_{jk\Delta} = C_1 g_{jk}$ at P. Since $R^{\Delta}_{\Delta k\Delta} = R^{\Delta}_{j\Delta\Delta} = 0$, we have

$$\eta_a R^a_{bcd}\xi^d = C_1(g_{bc} - \eta_b\eta_c) , \tag{4.3}$$

where C_1 may be a function on N. However, easily we see that C_1 is constant on N. We consider the tensor field

$$R^a_{bcd}\xi^d - C_1\xi^a g_{bc} + C_1\eta_b\delta^a_c . \tag{4.4}$$

If all indices a, b, c differ from Δ, then by Lemma 3.2 (i) (4.4) is vanishing. If $a=\Delta$, then (4.4) vanishes by (4.3). After putting $b=\Delta$, or $c=\Delta$, we see that (4.4) vanishes.

LEMMA 4.2. *Let N be an almost contact Riemannian manifold which admits the automorphism group of the maximum dimension. Then*

$$\nabla_b\eta_c = C_3(g_{bc} - \eta_b\eta_c) + C_4\phi_{bc} \tag{4.5}$$

holds for some constant C_3 and C_4.

PROOF. The tensor $\nabla_b\eta_c$ is invariant by $A(N)$ and hence $\nabla_j\eta_k$ is, by Lemma 3.2 (ii), of the form

$$\nabla_j\eta_k = C_3 g_{jk} + C_4\phi_{jk}$$

at P for some real numbers C_3 and C_4. By $N_b = 0$, $\nabla_b(\eta_s\xi^s) = 0$ and $\nabla_b\xi^\Delta = \nabla_b\eta_\Delta$ we have $\nabla_b\eta_\Delta = \nabla_\Delta\eta_b = 0$. Then (4.5) follows. Easily we see that C_3 and C_4 are constant.

LEMMA 4.3. *In Lemma 4.2 if C_4 is non-zero, then C_3 is equal to zero and ξ is an infinitesimal automorphism.*

PROOF. By (4.5) we have $(d\eta)_{bc} = 2C_4\phi_{bc}$. On the other hand, we have $L_\xi\phi=0$ and $L_\xi\eta = 0$ by (3.9) and (3.10). Therefore we have $L_\xi d\eta = dL_\xi\eta = 0$

and $(L_\xi\phi)_{bc}=0$. Next taking Lie derivative of $\phi_{bc} = g_{bs}\phi^s_c$ and using $(L_\xi\phi)^s_c = 0$ we have

$$0 = (L_\xi g)_{bs}\phi^s_c = 2C_3(g_{bs} - \eta_b\eta_s)\phi^s_c\,.$$

That is, we get $2C_3 g_{bs}\phi^s_c = 0$, which implies $C_3=0$.

LEMMA 4.4. *Let N be an almost contact Riemannian manifold where $d\eta$ is not trivial ($C_4 \neq 0$). If N admits the automorphism group of the maximum dimension, then it is essentially a homogeneous Sasakian manifold.*

PROOF. By Lemma 4.3 we have $\nabla_b\eta_c = C_4\phi_{bc}$. We define an almost contact structure $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ by

$$(4.6)\qquad {}^*\phi^a_b = \varepsilon\phi^a_b,\quad {}^*\xi^a = \xi^a,\quad {}^*\eta_b = \eta_b\,,$$

$$(4.7)\qquad {}^*g_{bc} = \varepsilon C_4 g_{bc} + (1-\varepsilon C_4)\,\eta_b\eta_c\,,$$

where ε is the sign of C_4. Then we have $(d^*\eta)=2^*\phi_{bc}$, that is, the deformed structure is a Sasakian structure. Q.E.D.

Assume that $d\eta = 0$ at some point. Then it holds globally on N and we have

$$(4.8)\qquad \nabla_b\eta_c = C_3(g_{bc} - \eta_b\eta_c)\,.$$

There are two cases: $C_3 = 0$ (Lemma 4.5) and $C_3\neq 0$ (Lemma 4.6).

LEMMA 4.5. *Let N be an almost contact Riemannian manifold such that ξ is a parallel field. N admits the automorphism group of the maximum dimension if and only if N is a Riemannian product of one of the three spaces CP^n, CE^n, CD^n, and a real line or circle.*

PROOF. Let P be an arbitrary point of N. Then we define the distribution by $\eta = 0$ and we have the $2n$-dimensional maximal integral submanifold $M(P)$ through P. $M(P)$ is an almost Hermitain manifold by restriction of ϕ and g to $M(P)$. Since any automorphism φ of M leaves all structure tensors invariant, it is distribution-preserving and $\varphi M(P) = M(Q)$ where $\varphi P=Q$. Since ξ generates a 1-parameter group of automorphism $\exp t\xi$, we have $\exp t\xi\cdot Q \in M(P)$ for some t. Therefore $\exp t\xi\cdot\varphi$ is an automorphism of $M(P)$. Thereby the automorphism groups $A(M(P))$ and $A(N)$ differ only one dimension, which is caused by ξ. Hence $A(M(P))$ is $n(n+2)$-dimensional, and $M(P)$ is one of the three spaces: CP^n, CE^n, and CD^n ([17]). Since N is homogeneous each trajectory of ξ is homeomorphic to a real line L or a circle T. We show that each trajectory intersects $M(P)$ at only one point. Assume that $\exp t'\xi \cdot P = Q \in M(P)$ for some $t' \neq 0$. Let Y^* be an infinitesimal automorphism on $M(P)$ such that $Y^*_P \neq 0$. Since a small neighborhood U of P is a Riemannian product, $Y=(Y^*, 0)$ defines an infinitesimal automorphism on U. By $L_Y\xi = 0$ we have $\exp t\xi \cdot Y = Y$ (for any small t) on U. In order that $\dim A(N)=(n+1)^2$ holds Y must be globally defined on N so that the restriction of Y to $M(P)$ is Y^*. By this argument we must have $Y_Q^* \neq 0$. On the other hand, for any points P and Q in any one of the spaces CP^n, CE^n, CD^n, we have some infinitesimal automorphism Y^* such that $Y^*_P \neq 0$ and $Y^*_Q = 0$ (otherwise every geodesic starting at P goes to Q with the same length). Therefore N is globally a Riemannian product. The converse is clear. Q.E.D.

Now we come to the final case: $C_3 \neq 0$ and $C_4 = 0$. By the Ricci identity, (4. 8) and (4. 1), we have

$$-C_3^2(g_{bd}\eta_c - g_{bc}\eta_d) = -\eta_a R^a_{bcd} = -C_1(g_{bc}\eta_d - \eta_c g_{bd}) .$$

Thus $C_1 = -C_3^2 < 0$. This implies that the sectional curvature for 2-planes which contain ξ is negative. We define the distribution by $\eta=0$, which is also completely integrable by $d\eta = 0$. Let $M(P)$ be the maximal integral submanifold through P. By restriction of ϕ and g, $M(P)$ is an almost Hermitian manifold. Let X be an infinitesimal automorphism of N and denote by $\exp tX$ the 1-parameter group of automorphisms. Since $A(N)$ is transitive, we can assume that we have X which is not tangent at P (then X is not tangent at any point) to $M(P)$. For small t, if we put $\exp tX \cdot P = Q(t)$, then $\exp tX$ is an isomorphism of $M(P)$ to $M(Q(t))$, since the equation $\eta=0$ and the structure tensors are invariant by $\exp tX$. Now let $s(t)$ be a function of t such that $\exp s(t)\xi \cdot Q(t) = P$. Then $\exp s(t)\xi \circ \exp tX$ is a transformation of $M(P)$. Since $(L_\xi g)_{bc} = 2C_3(g_{bc} - \eta_b\eta_c)$, if $t \neq 0$, $\exp s(t)\xi$ is a non-isometric homothety with respect to the distribution $\eta = 0$. Thus $\exp s(t)\xi \circ \exp tX$ is a 1-parameter group of non-isometric homotheties of $M(P)$. Let X' be a vector field on $M(P)$ defined by this 1-parameter group: $L_{X'}G = C_5 G$, where C_5 is a non-zero constant and G is the restriction of g to $M(P)$. Let Y be another infinitesimal automorphism of N which is not tangent to $M(P)$. Then by the same argument we have Y' such that $L_{Y'}G = C_6 G$ on $M(P)$. Put $Y^* = Y' - (C_6/C_5)X'$. Then Y^* is an infinitesimal isometry on $M(P)$. On the other hand $\exp s(t)\xi \circ \exp tX$ leaves ϕ and η invariant for each t. Hence Y^* is an infinitesimal automorphism of $M(P)$. This means that any infinitesimal automorphism Y induces an infinitesimal automorphism Y^* on $M(P)$. So we can consider that only X is essential among infinitesimal automorphisms which are not tangent to $M(P)$. Therefore $A(M(P))$ is $n(n+2)$-dimensional, and $M(P)$ is a homogeneous Kaehlerian manifold. Since $M(P)$ admits an infinitesimal non-isometric homothety, $M(P)$ is flat and it is the unitary space. On the other hand, $\exp t\xi$ are homotheties with respect to the distribution $\eta=0$, whose proportional factor is monotonically increasing as t, and hence its trajectory is homeomorphic to a real line. Therefore we have

LEMMA 4.6. *Let N be an almost contact Riemannian manifold such that ξ is not parallel and $d\eta$ is trivial. Then the maximum dimension of the automorphism group is attained if and only if N is of the form $L\times_{ct}M$ where L is a real line and $M=CE^n$ is the unitary space with (J,G) and the metrics are related by*

$$g_{(t,x)} = (dt)^2_{(t)} + e^{2ct}G_{(x)} \tag{4.9}$$

for some constant c.

PROOF. We prove the converse. Let $N=L\times_{ct}M$. In this product we see that ξ is defined by (d/dt) and ϕ is defined by translation of J in M by $\exp t\xi$. Take a point P in M. Then we have a 1-parameter group of homotheties φ'_s such that $(\varphi'_s)^*G=e^{-2cs}G$ and they leave invariant J and the point P. Such φ'_s exist, because M is the unitary space. We identify M with $(0)\times M$ and consider J and φ'_s on both M and $(0)\times M$. By definition we have

$$\phi_{(t,x)} = \exp t\xi\cdot J_x\cdot \exp(-t)\xi\,,$$

where $\exp t\xi$ itself denotes the differential of $\exp t\xi$. Thus $\exp s\xi\cdot\phi=\phi\cdot\exp s\xi$ holds good. Since $\exp t\xi\cdot\xi=\xi$ we have also $(\exp s\xi)^*\eta=\eta$, where $\eta=(dt)$.

Let Z' be an infinitesimal automorphism on M. Then by $Z_{(t,x)}=\exp t\xi\cdot Z'_x$ we define a vector field Z on N. Since

$$\exp s\xi\cdot Z_{(t,x)} = \exp s\xi\cdot\exp t\xi\cdot Z'_x = Z_{(t+s,x)}\,,$$

we have $L_Z\xi=[Z,\xi]=-L_\xi Z=0$. Thus $\exp s\xi$ and $\exp tZ$ are commutative. Let Y be a vector field on N such that $\eta(Y)=0$. Then we get $\eta(\exp tZ\cdot Y)=0$. Therefore to prove $\exp tZ\cdot\phi=\phi\cdot\exp tZ$, it suffices to show for Y such that $\eta(Y)=0$.

$$\begin{aligned}\exp sZ\cdot\phi_{(t,x)}Y &= \exp sZ\cdot\exp t\xi\cdot J_x\cdot\exp(-t)\xi\cdot Y\\ &= \exp t\xi\cdot\exp sZ'\cdot J_x\cdot\exp(-t)\xi\cdot Y\end{aligned}$$

$$= \exp t\xi \cdot J_w \cdot \exp sZ' \cdot \exp(-t)\xi \cdot Y \quad (w = \exp sZ' \cdot x)$$

$$= \phi_{(t,w)} \cdot \exp sZ \cdot Y\,.$$

Since $((\exp sZ)^*g)(\xi,\xi) = 1$ and $((\exp sZ)^*g)(\xi, Y) = 0$ (if $\eta(Y)=0$) are clear, we calculate the following for Y, V such that $\eta(Y) = \eta(V) = 0$;

$$\begin{aligned}((\exp sZ)^*g)_{(t,x)}(Y,V) &= g_{(t,w)}(\exp sZ \cdot Y, \exp sZ \cdot V) \qquad (w = \exp sZ \cdot x)\\ &= e^{2ct}G_w(\exp(-t)\xi \cdot \exp sZ \cdot Y, \exp(-t)\xi \cdot \exp sZ \cdot V)\\ &= e^{2ct}G_w(\exp sZ' \cdot \exp(-t)\xi \cdot Y, \exp sZ' \cdot \exp(-t)\xi \cdot V)\\ &= e^{2ct}G_x(\exp(-t)\xi \cdot Y,\ \exp(-t)\xi \cdot V)\\ &= g_{(t,x)}(Y,V)\,.\end{aligned}$$

Therefore $\exp sZ$ is an isometry for each s, and hence Z is an infinitesimal automorphism on N. The set of all such vector fields is $n(n+2)$-demensional.

Next define transformations $\varphi_s : N \rightarrow N$ by $(t, x) \rightarrow (t+s, \varphi'_s x)$. Then (φ_s) is a 1-parameter group of transformations. Clearly φ_s and $\exp t\xi$ are commutative. So φ_s leaves ξ invariant. We also have $\eta(\varphi_s Y) = 0$ for any Y such that $\eta(Y) = 0$. To show $\varphi_s\phi = \phi\varphi_s$, it suffices to show the following for Y such that $\eta(Y) = 0$.

$$\begin{aligned}\varphi_s \cdot \phi_{(t,x)}Y &= \varphi_s \cdot \exp t\xi \cdot J_x \cdot \exp(-t)\xi \cdot Y\\ &= \exp t\xi \cdot \varphi_s \cdot J_x \cdot \exp(-t)\xi \cdot Y\\ &= \exp t\xi \cdot (\exp s\xi \cdot \varphi'_s) \cdot J_x \cdot \exp(-t)\xi \cdot Y\\ &= \exp s\xi \cdot \exp t\xi \cdot J_u \cdot \varphi'_s \cdot \exp(-t)\xi \cdot Y \qquad (u = \varphi'_s x)\\ &= \exp s\xi \cdot \exp t\xi \cdot J_u \cdot (\exp(-t)\xi \cdot \exp t\xi) \cdot \varphi'_s \cdot \exp(-t)\xi \cdot Y\\ &= \exp s\xi \cdot \phi_{(t,u)} \cdot \exp t\xi \cdot \varphi'_s \cdot \exp(-t)\xi \cdot Y\\ &= \phi_{(t+s,u)} \cdot \exp s\xi \cdot \exp t\xi \cdot \varphi'_s \cdot \exp(-t)\xi \cdot Y\\ &= \phi_{(t+s,u)} \cdot \varphi_s \cdot Y\,.\end{aligned}$$

Finally we prove that φ_s is an isometry. Since $(\varphi_s^*g)(\xi,\xi)=1$ and $(\varphi_s^*g)(\xi, Y) = 0$ (if $\eta(Y) = 0$) are clear, we show the following for Y, V such that $\eta(Y) = \eta(V) = 0$.

$$(\varphi_s^*g)_{(t,x)}(Y,V) = g_{(t+s,u)}(\varphi_s Y, \varphi_s V) \qquad (u = \varphi'_s x)$$

$$= e^{2c(t+s)} G_u(\exp(-t-s)\xi \cdot \varphi_s \cdot Y, \exp(-t-s)\xi \cdot \varphi_s \cdot V)$$
$$= e^{2c(t+s)} G_u(\exp(-s)\xi \cdot \varphi_s \cdot \exp(-t)\xi \cdot Y, \exp(-s)\xi \cdot \varphi_s \cdot \exp(-t)\xi \cdot V)$$
$$= e^{2c(t+s)} G_u(\varphi_s' \cdot \exp(-t)\xi \cdot Y, \varphi_s' \cdot \exp(-t)\xi \cdot V)$$
$$= e^{2ct} G_x(\exp(-t)\xi \cdot Y, \exp(-t)\xi \cdot V)$$
$$= g_{(t,x)}(Y, V) .$$

Therefore (φ_s) define an infinitesimal automorphism X which is not tangent to M, and we have $\dim A(N) = (n+1)^2$.

LEMMA 4.7. *Now we give the relation between the sectional curvature for 2-planes which contain ξ and the constants C_3, C_4.*

(i) $$C_3 = 0,\ C_4 \neq 0 \Longleftrightarrow C_1 = C_4^2 > 0 .$$

(ii) $$C_3 = 0,\ C_4 = 0 \Longleftrightarrow C_1 = 0 .$$

(iii) $$C_3 \neq 0,\ C_4 = 0 \Longleftrightarrow C_1 = -C_3^2 < 0 .$$

PROOF. For $(\Longrightarrow)$ part, (ii) is clear, and (iii) was proved already. We give a proof of (i) here. By $\nabla_b \eta_c = C_4 \phi_{bc}$, ξ is a Killing vector field, and so we have

$$\nabla_c \nabla_b \xi^a + R^a_{bcd} \xi^d = 0 .$$

We transvect the last equation with η_a and use (4.1). Then we get

$$-C_4^2(g_{bc} - \eta_b \eta_c) + C_1(g_{bc} - \eta_b \eta_c) = 0 .$$

Thus $C_1 = C_4^2 > 0$. Since (i)~(iii) expire all cases, the converse $(\Longleftarrow)$ is also true.

5. **Regular K-contact Riemannian manifolds.** Let $\pi : N \to M = N/\xi$ be the fibering of a regular K-contact Riemannian manifold N given by W. M. Boothby and H. C. Wang [1]. Then M is an almost Kaehlerian manifold with structure tensors J and G such that

(5.1) $$g = \pi^* G + \eta \otimes \eta ,$$

(5.2) $$(JX)^* = \phi X^* ,$$

where X^* is the horizontal lift with respect to η. And the fundamental

2-form W satisfies (cf. [2])

$$2\pi^* W = d\eta\,. \tag{5.3}$$

LEMMA 5.1. *In the fibering $\pi: N \to M$ of a regular simply connected K-contact Riemannian manifold N, if X is an infinitesimal automorphism on M, then we have some function f on N so that $X^* - f\xi$ is an infinitesimal automorphism of the K-contact Riemannian structure and f is unique up to an additive constant.*

PROOF. In the formula

$$\begin{aligned} dW(X,Y,Z) &= X\cdot W(Y,Z) + Y\cdot W(Z,X) + Z\cdot W(X,Y) \\ &\quad - W([X,Y],Z) - W([Z,X],Y) - W([Y,Z],X) \end{aligned} \tag{5.4}$$

we have $dW = 0$, where Y and Z are arbitrary vector fields on M. Let X^*, Y^*, Z^* be the horizontal lifts of X, Y, Z with respect to η, and consider the 1-form $i_{X^*}d\eta$. We notice that $[\xi, Y^*] = 0$ and

$$[Y^*, Z^*] = [Y, Z]^* + \eta([Y^*, Z^*])\,\xi \tag{5.5}$$

hold. We show that $i_{X^*}d\eta$ is a closed form.

$$\begin{aligned} d(i_{X^*}d\eta)(Y^*, Z^*) &= Y^*\cdot d\eta(X^*, Z^*) - Z^*\cdot d\eta(X^*, Y^*) - d\eta(X^*, [Y^*, Z^*]) \\ &= 2[Y\cdot W(X,Z)\cdot\pi - Z\cdot W(X,Y)\cdot\pi - W(X,[Y,Z])\cdot\pi]\,, \end{aligned}$$

which is seen to vanish by (5.4), since X is an infinitesimal automorphism of the almost Kaehlerian structure on M. Next easily we have

$$d(i_{X^*}d\eta)(Y^*, \xi) = 0\,. \tag{5.6}$$

Thus $i_{X^*}d\eta$ is closed, and locally it is a derived form. Since M is simply connected, we have some function f on N such that $i_{X^*}d\eta = df$. Now we prove that $X^* - f\xi$ is a Killing vector field with respect to g.

$$\begin{aligned} (L_{(X^*-f\xi)}g)(Y^*, Z^*) &= L_{(X^*-f\xi)}(G(Y,Z)\cdot\pi) - g([X^*-f\xi, Y^*], Z^*) \\ &\qquad - g(Y^*, [X^* - f\xi, Z^*]) \\ &= L_X G(Y,Z)\cdot\pi - G([X,Y],Z)\cdot\pi - G(Y,[X,Z])\cdot\pi\,, \end{aligned}$$

which vanishes, because X is a Killing vector field with respect to G on M.

Easily we have

$$(L_{(X^*-f\xi)}g)(Y^*, \xi) = -\eta([X^*, Y^*]) - Y^*f = 0\,,$$
$$(L_{(X^*-f\xi)}g)(\xi, \xi) = 0\,.$$

Thus $L_{(X^*-f\xi)}g=0$. On the other hand, we have

$$L_{(X^*-f\xi)}\eta = i_{(X^*-f\xi)}d\eta + di_{(X^*-f\xi)}\eta = i_{X^*}d\eta - df = 0\,.$$

Therefore $X^*-f\xi$ is an infinitesimal automorphism on N. Let f and f' be such two functions. Then the difference $f-f'$ is constant. Q.E.D.

Conversely, let φ be an automorphism of the K-contact Riemannian structure on N. Since φ leaves ξ invariant, we have some transformation Φ on M such that $\pi\varphi = \Phi\pi$. We show that Φ is an automorphism of the (J, G)-structure. Since $\varphi^*\eta = \eta$ and $\varphi^*g = g$, we have $\varphi^*(\pi^*G) = \pi^*G$. For any point P of N and for lifts Y^* and Z^* of Y and Z, we have

$$(\Phi^*G)_{\pi P}(Y, Z) = G_{\pi\varphi P}(\pi\varphi Y^*, \pi\varphi Z^*) = G_{\pi P}(Y, Z)\,.$$

That is $\Phi^*G=G$. Next by (5. 2) and other relations, we have

$$(J(\Phi Y))^*_{\varphi P} = \phi_{\varphi P}(\pi\varphi Y^*)^* = \varphi_P(JY)^*\,.$$

Operating π we have $J\Phi = \Phi J$. Thus

LEMMA 5.2. *If $\pi: N\to M$ is the fibering of a regular K-contact Riemannian manifold N, then φ of $A(N)$ induces Φ of $A(M)$. If u is an infinitesimal automorphism on N, then u is projectable and $\pi u = X$ is an infinitesimal automorphism on M. Thus* $\dim A(N) \leqq \dim A(M)+1$.

6. The relation of $A(N)$ and $A(N/\xi)$ of the fibering of K-contact Riemannian manifolds. Take an arbitrary point and a neighborhood U of the point such that U is a simply connected regular K-contact Riemannian manifold. On U we consider the Lie algebra $a(U)$ of all infinitesimal automorphisms of the structure. Let $\pi: U\to V$ be the fibering of U. Then for any $u\in a(U)$, we have an infinitesimal automorphism $X=\pi u$ on V. Then by Lemma 5.1 we have

$$\dim a(U) = \dim a(V) + 1\,, \tag{6.1}$$

where $a(V)$ is the Lie algebra of all infinitesimal automorphism on V and the difference 1 is caused by ξ. Of course we have $\dim A(N) \leqq \dim a(N) \leqq \dim a(U)$.

LEMMA 6.1. *Let N be a K-contact Riemannian manifold. And assume that N satisfies one of the following conditions:*

(i) *N is simply connected, regular and complete,*
(ii) *N is simply connected and homogeneous,*
(iii) *N is regular, compact and has vanishing first Betti number,*
(iv) *N is homogeneous, compact and has vanishing first Betti number,*
(v) $\dim A(N) = (n+1)^2$.

Then we have

$$\dim A(N) = \dim A(N/\xi) + 1 . \tag{6.2}$$

PROOF. First we note that any homogeneous contact manifold is regular ([1]). We need to prove only when N satisfies (iii). Since N is orientable and compact, a closed form $i_{X^*}d\eta$ on N must be a derived form on N, for the first Betti number vanishes. Thus we have $\dim a(N) = \dim a(N/\xi) + 1$. By completeness of N and N/ξ, we have (6.2). For (v) see Lemma 5.2 and notice $\dim A(N/\xi) \leqq n(n+2)$.

COROLLARY 6.2. *In Lemma* 6.1, *if N has property* (ii) *or* (iv), *then N/ξ is homogeneous. If N has property* (i) *or* (iii), *then N is homogeneous if and only if N/ξ is homogeneous.*

The unit $(2n+1)$-dimensional sphere S^{2n+1} is one of the standard Sasakian manifolds ([11]). S^{2n+1} is the circle bundle over the complex n-dimensional projective space CP^n. CP^n is one of the standard examples of irreducible Hermitian symmectric spaces.

PROPOSITION 6.3. $\dim A(S^{2n+1}) = (n+1)^2$.

7. *D*-homothety class of an almost contact Riemannian manifold. Let α be a positive number and define ϕ^*, ξ^*, η^* and g^* by

$$\phi^* = \phi, \quad \xi^* = (1/\alpha)\xi, \quad \eta^* = \alpha\eta , \tag{7.1}$$

$$g^* = \alpha g + (\alpha^2-\alpha)\eta \otimes \eta . \tag{7.2}$$

Then $(\phi^*, \xi^*, \eta^*, g^*, \alpha)$ is also an almost contact Riemannian structure on N. We call this deformation a D-homothety. By a D-homothety a K-contact

Riemannian structure is deformed to another K-contact Riemannian structure, and a Sasakian structure is deformed also to a Sasakian structure ([16]).

LEMMA 7.1. *Let N be an almost contact Riemannian manifold with (ϕ, ξ, η, g). Then the automorphism groups $A(N)$ and $A^*(N)$ with respect to (ϕ, ξ, η, g) and $(\phi^*, \xi^*, \eta^*, g^*, \alpha)$ coincide.*

PROOF. This follows from (7.1) and (7.2).

REMARK 7.2. By the Lemma we see that if N is homogeneous, then every D-homothetically deformed structure is also homogeneous. Thus S^{2n+1} gives an example of a homogeneous contact Riemannian (Sasakian) manifold whose curvatures take negative and positive values (cf. [4], [16]).

A Sasakian manifold N has constant ϕ-holomorphic sectional curvature $H(P)$ at P if every ϕ-holomorphic section at P, that is, 2-plane determined by Y_P such that $\eta(Y) = 0$ and ϕY_P, has a common sectional curvature $H(P)$. If H is constant on N, then N is said to have constant ϕ-holomorphic sectional curvature H. If $2n+1 \geqq 5$, then H is always constant on N. The necessary and sufficient condition for a Sasakian manifold N to have constant ϕ-holomorphic sectional curvature H is (cf. [6]).

$$(7.3) \qquad 4R_{abcd} = (H+3)(g_{da}g_{cb} - g_{db}g_{ca}) + (H-1)(\eta_b\eta_d g_{ac} + \eta_c\eta_a g_{bd} - \eta_d\eta_a g_{bc} - \eta_b\eta_c g_{da} + \phi_{db}\phi_{ac} - \phi_{da}\phi_{bc} + 2\phi_{dc}\phi_{ab}) \,.$$

It is known that, if H is constant > -3, we have a positive constant α so that N is of constant curvature 1 with respect to the deformed structure $(\phi^*, \xi^*, \eta^*, g^*)$ (cf [16]).

Next let $\pi : N \to N/\xi$ be the fibering of a regular Sasakian manifold with constant ϕ-holomorphic sectional curvature H. Then N/ξ is a Kaehlerian manifold with constant holomorphic sectional curvature $k=H+3$ (cf. [7]).

8. **Proof of the main theorem**. Assume that the maximum dimension of the automorphism group is attained in N. Then by Lemma 4.1 the sectional curvature for 2-planes which contain ξ is equal to a constant $C=C_1$. All possible cases are (i), (ii) and (iii) of Lemma 4.7.

(i) Suppose that $C > 0$ holds. Then by Lemma 4.4 N can be considered as a homogeneous Sasakian manifold after some deformation by (4.6)–(4.7). N has constant ϕ-holomorphic sectional curvature H, as is seen from the argument in proof of Lemma 3.2. Since N is regular it is a circle or line bundle over N/ξ. By Lemma 5.2 we have $\dim A(N/\xi) \geqq n(n+2)$ and hence N/ξ is one of the three spaces: CP^n, CE^n and CD^n according to $H > -3$, $H = -3$ and $H < -3$.

(i-1) When $H > -3$, N is D-homothetically deformable to a space N^* of constant curvature 1. Therefore N or N^* is a circle bundle over CP^n. N^* is S^{2n+1} or a factor space $S^{2n+1}/F(t_1)$ where $F(t_1)$ is a finite group generated by $\exp t_1\xi$. Conversely, $S^{2n+1}/F(t_1)$ admits the automorphism group of the maximum dimension. In fact, any infinitesimal automorphism on S^{2n+1} is either proportional to ξ or of the form $X^*-f\xi$ (for notations see Lemma 5.1) and it is invariant by $\exp t\xi$. So $X^*-f\xi$ can be considered as an infinitesimal automorphism on $S^{2n+1}/F(t_1)$.

(i-2) When $H=-3$ N is a T- or an L-bundle over CE^n. An L-bundle is a universal covering manifold of a T-bundle, and an L-bundle is considered as a (Euclidean) space E^{2n+1} with a suitable coordinates. The metric g and other tensors are given in terms of coordinates (cf. [8], [9]). Therefore N is E^{2n+1} or its factor space by $F(t)$, where $F(t)$ is a cyclic group generated by $\exp t\xi$ for a real number t. Conversely, by Lemma 6.1 E^{2n+1} admits the group of automorphisms of the maximum dimension (cf. [5]), and so does $E^{2n+1}/F(t)$ by the same argument as in (i-1).

(i-3) When $H < -3$ N is a T- or an L-bundle over CD^n. We consider the converse. Since the fundamental 2-form W (on CD^n) is closed, it is locally exact. However, since CD^n is an open ball W is globally an exact form, i.e., we have a 1-form w on CD^n such that $W=dw$. Let $\pi: (L, CD^n) \to CD^n$ be an L-product bundle over CD^n. Then $\eta = 2\pi^*w + dt$ is an invariant 1-form on (L, CD^n) which defines an infinitesimal connection. It defines a contact structure on (L, CD^n) which turns to a Sasakian structure by a suitable metric. Similarly to (i-1) or (i-2), (L, CD^n) or its factor space admits the automorphism group of the maximum dimension.

(ii) For the case $C = C_1 = 0$, see Lemmas 4.5 and 4.7.
(iii) For the case $C = C_1 < 0$, see Lemmas 4.6 and 4.7.

REMARK 8.1. The scalar curvature S^* in N and the scalar curvature S in $M=N/\xi$ are in the relation $S^* = S-2n$ ([15]). So we have

COROLLARY 8.2. *Let N be a simply connected contact Riemannian manifold which admits the automorphism group of the maximum dimension $(n+1)^2$ and has one of the following properties:*

(i) *N is compact,*
(ii) *the scalar curvature $S^* > -2n$,*
(iii) *the ϕ-holomorphic sectional curvature > -3.*

Then N is globally D-homothetic with the unit sphere.

REMARK 8.3. Roughly speaking, the maximum dimension of the automorphism group of a Sasakian manifold may be half of the dimension of the isometry group. The following fact may have some interest: Let N be a contact Riemannian manifold which is a symmetric space with respect to g. Then at any point P, the geodesic symmetry σ_P is not an automorphism, since $\sigma_P \xi_P = -\xi_P$.

9. ϕ-preserving transformations on contact Riemannian manifolds. We consider the group $\phi(N)$ of all ϕ-preserving transformations of a contact Riemannian manifold N. It is known that ([12]).

$$\dim \phi(N) \leqq \dim A(N) + 1\,. \tag{9.1}$$

If a contact Riemannian manifold is compact, we have

$$\phi(N) = A(N)\,. \tag{9.2}$$

These give the difference between M and N. Namely, in a compact contact Riemannian manifold N we have

$$I(N) \supset \phi(N)\,, \tag{9.3}$$

where $I(N)$ denotes the group of all isometries of N. While in a compact almost Kaehlerian manifold we have

$$\text{Lie algebra of } I(M) \subset \text{Lie algebra of } J(M)\,, \tag{9.4}$$

where $J(M)$ denotes the group of all J-preserving transformations of M.

THEOREM 9.1. *Let N be a $(2n+1)$-dimensional contact Riemannian manifold. Then we have* $\dim \phi(N) \leqq (n+1)^2 + 1$.

(i) *If the maximum is attained in a contact Riemannian manifold, then N is homeomorphic with the Euclidean space.*

(ii) *If N is compact, then* $\dim \phi(N) \leqq (n+1)^2$. *And if the maximum is attained in a compact and simply connected contact Riemannian manifold, then N is globally D-homothetic to the unit sphere.*

PROOF. The first follows from (9.1). (i) follows from [13] or [14], since N with $\dim \phi(N) = (n+1)^2 + 1$ $(\dim A(N) = (n+1)^2)$ is homogeneous and

Sasakian. (ii) follows from (9.2) and Corollary 8.2. Q.E.D.

We have considered $\phi(N)$ only for a contact Riemannian manifold N. If N is an almost contact manifold, then $\phi(N)$ is quite different from one we have treated in this section and it is too large. In order to get results analogous to $J(M)$ for an almost complex manifold M (cf. [17]), it is natural to consider the automorphism group of an almost contact structure, that is, the set of all transformations which leave ϕ, ξ and η invariant.

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