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ON THE EQUIVALENCE IN BANACH SPACES OF A CONVERGENCE THEOREM OF BERNSTEIN TYPE, A HAUSDORFF TYPE MOMENT PROBLEM AND A REGULARITY THEOREM

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1. Introduction. Suppose each of X and Y is a linear normed space and $B=B[X,Y]$ is the space of bounded linear transformations from X into Y. Following the notation in [3] and [4], we denote the weak sequential extension of B by B^+ and of Y by Y^+. We shall denote by C the space of continuous functions on [0, 1] with values in X, the space having the uniform norm topology, and by C_0 the subspace of C consisting of those functions f such that $f(0)=\theta_X$, the zero element of X.

For each $\alpha \geqq 0$, C_α will denote C_0 if $\alpha>0$ and it will signify C if $\alpha=0$. We shall suppose throughout that $\Phi=\{\varphi_k\}_{k=0}^{\infty}$ represents a sequence of elements of B.

DEFINITION 1.

$$B_n^\alpha[\Phi,f]=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}[\Delta^{n-k}\varphi_k]\cdot f\left(\frac{k}{n}\right)$$

will be called the (n,α)-Bernstein transform of f for each $f\in C$. Here we understand that

$$\begin{aligned}\binom{n+\alpha}{n-k}&=\frac{\Gamma(n+\alpha+1)}{(n-k)!\,\Gamma(k+\alpha+1)} && \text{for } 0\leqq k<n\,,\\ &=1 && \text{for } k=n\,,\\ &=0 && \text{for } k>n\,,\end{aligned}$$

and

$$\Delta^\nu\varphi_k=\sum_{m=0}^{\nu}(-1)^{\nu-m}\binom{\nu}{m}\varphi_{k+\nu-m}\qquad \text{for } \nu,k=0,1,2,\cdots.$$

DEFINITION 2. The statement that Φ satisfies condition A_α means that there exists a number $M > 0$ such that for any bounded sequence $\{x_k\}_{k=0}^{\infty}$ of elements of X and for each $n = 0, 1, 2, \cdots$,

$$\left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot x_k \right\|_Y < M \max_{0 \leqq k \leqq n} \|x_k\| .$$

In case both X and Y are the real (or complex) numbers, $\alpha = 0$ and $\varphi_k = t^k$, then $B_n^\alpha[\Phi, f]$ is the ordinary Bernstein polynomial of f of order n.

The vector-valued case for $\alpha = 0$ has been treated in [5] making use of the results in [3] and [4]. It happens that, in that case, it is possible to give a necessary and sufficient condition for the convergence for each f in C. The condition is that Φ should satisfy condition A_0. This is equivalent to the statement that Φ is a moment sequence, which is equivalent to the statement that a certain Hausdorff summability method from X into Y^+ is convergence preserving. Furthermore, in the case in which $Y \equiv C$, then $B_n^0[\Phi, f]$ converges (uniformly) to f for each f in C, if and only if $\varphi_k = t^k$ for $k = 0, 1, 2, \cdots$.

Endl [1] and Jakimovski and Ramanujan [2] have treated moment problems and summability methods of Hausdorff type for the case in which $X = Y =$ the scalar field and $\alpha > 0$. In this paper we consider the vector-valued case for $\alpha > 0$ and treat the above questions, obtaining results similar to those for $\alpha = 0$. There is one distinct difference, however, and that is that in the case for $\alpha > 0$, the related Hausdorff type summability methods are regular relative to an underlying linear transformation from X into Y.

2. Convergence Theorems.

LEMMA 1. *If $\alpha > 0$ and Φ satisfies condition A_α, then for each non-negative integer i,*

$$\sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot \left(\frac{k+i}{n+i} \right)^\alpha$$

converges in B-norm to φ_0 as $n \to \infty$.

REMARK. If $\alpha = 0$, then we have the identity

$$\sum_{k=0}^{n} \binom{n}{n-k} [\Delta^{n-k} \varphi_k] = \varphi_0$$

and condition A_0 is not necessary.

PROOF. Since

$$(2.1)\qquad \binom{n+\alpha}{n-k} = \binom{n}{k} \frac{\binom{n+\alpha}{n}}{\binom{k+\alpha}{k}}$$

we have that $\binom{n}{k} \leqq \binom{n+\alpha}{n-k}$ for every n and k such that $n \geqq k$ and hence by condition A_α we have that

$$(2.2)\qquad \left\| \sum_{k=0}^{n} \binom{n}{k} [\Delta^{n-k} \varphi_k] \cdot x_k \right\|_Y$$

$$= \left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot x'_k \right\|_Y < M \cdot \max \|x'_k\| \leqq M \cdot \max \|x_k\|$$

where $\binom{n+\alpha}{n-k} x'_k = \binom{n}{k} x_k$. Now consider the index of summation in (2.2) as changed to q, then, if we choose $x'_q = \theta_X$ for $q \neq k$ and $\|x'_k\| = 1$, we have that

$$(2.3)\qquad \binom{n+\alpha}{n-k} \|\Delta^{n-k} \varphi_k\|_B < M$$

for $n, k = 0, 1, 2, \cdots,$ where $n \geqq k$. An application of Stirling's formula shows that

$$\lim_{n\to\infty} \frac{\binom{n+\alpha}{n}}{\binom{k+\alpha}{k}} = \infty$$

for each $k \geqq 0$ and it then follows from (2.1) and (2.3) that

$$\lim_{n\to\infty} \binom{n}{k} \|\Delta^{n-k} \varphi_k\|_B = 0 \qquad \text{for } k \geqq 0 .$$

Suppose $\varepsilon > 0$ and choose N such that if $n \geqq k > N$, then

$$\left| \frac{\binom{n+\alpha}{n}}{\binom{k+\alpha}{k}} \cdot \left(\frac{k+i}{n+i} \right)^{\alpha} - 1 \right| < \frac{\varepsilon}{2M} .$$

That this is possible follows from the fact that $\binom{p+\alpha}{p} \sim \frac{p^\alpha}{\Gamma(\alpha+1)}$ which follows from Stirling's formula. Now choose K so that

$$\left| \frac{\binom{n+\alpha}{n}}{\binom{k+\alpha}{k}} \cdot \left(\frac{k+i}{n+i}\right)^{\alpha} - 1 \right| < K$$

for $n, k = 0, 1, 2, \cdots, n \geqq k$, and finally choose N' such that if $n > N'$, then

$$\left\| \binom{n}{k} \Delta^{n-k} \varphi_k \right\|_B < \frac{\varepsilon}{2(N+1)K}$$

for $k = 0, 1, \cdots, N$. Now since

$$\sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \varphi_k = \varphi_0 ,$$

we have for $n > N', N$ and $\|x\| \leqq 1$,

$$\left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \left(\frac{k+i}{n+i}\right)^{\alpha} \cdot x - \varphi_0 \cdot x \right\|_Y$$

$$= \left\| \sum_{k=0}^{n} \binom{n}{k} [\Delta^{n-k} \varphi_k] \left\{ \frac{\binom{n+\alpha}{n}}{\binom{k+\alpha}{k}} \left(\frac{k+i}{n+i}\right)^{\alpha} - 1 \right\} \cdot x \right\|_Y$$

$$\leqq \left\| \sum_{k=0}^{N} \binom{n}{k} [\Delta^{n-k} \varphi_k] \left\{ \frac{\binom{n+\alpha}{n}}{\binom{k+\alpha}{k}} \left(\frac{k+i}{n+i}\right)^{\alpha} - 1 \right\} \cdot x \right\|_Y$$

$$+ \left\| \sum_{k=N+1}^{n} \binom{n}{k} [\Delta^{n-k} \varphi_k] \left\{ \frac{\binom{n+\alpha}{n}}{\binom{k+\alpha}{k}} \left(\frac{k+i}{n+i}\right)^{\alpha} - 1 \right\} \cdot x \right\|_Y$$

$$\leqq \sum_{k=0}^{N} \left\| \binom{n}{k} \Delta^{n-k} \varphi_k \right\|_B \cdot \left| \frac{\binom{n+\alpha}{n}}{\binom{k+\alpha}{k}} \left(\frac{k+i}{n+i}\right)^{\alpha} - 1 \right|$$

$$+ \left\| \sum_{k=0}^{n} \binom{n}{k} [\Delta^{n-k} \varphi_k] \left\{ \frac{\binom{n+\alpha}{n}}{\binom{k+\alpha}{k}} \left(\frac{k+i}{n+i}\right)^{\alpha} - 1 \right\} \cdot x_k \right\|_Y$$

where $x_k = \theta_X$ for $k = 0, 1, \cdots, N$ and $x_k = x$ for $k = N+1, \cdots, n$,

$$< \sum_{k=0}^{N} \frac{\varepsilon}{2(N+1)K} \cdot K + \frac{\varepsilon}{2M} \cdot M = \varepsilon,$$

and hence

$$\left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \left(\frac{k+i}{n+i} \right)^{\alpha} - \varphi_0 \right\|_B$$

$$= \sup_{\|x\| \leqq 1} \left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \left(\frac{k+i}{n+i} \right)^{\alpha} \cdot x - \varphi_0 \cdot x \right\|_Y \leqq \varepsilon,$$

and the proof is complete.

REMARK. For the case $i=0$, one needs only that

$$\left\| \sum_{k=1}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot x_k \right\|_Y < M \max \|x_k\| .$$

THEOREM 1. (i) *If $\alpha > 0$ and Φ satisfies condition A_α, then $B_n^\alpha[\Phi, t^{\alpha+i}]$ converges in B-norm to φ_i, for $i = 0, 1, 2, \cdots$.*

(ii) *If $\alpha=0$, then $B_n^0[\Phi, t^i]$ converges in B-norm to φ_i, for $i=0, 1, 2, \cdots$, i.e., condition A_0 is not needed.*

PROOF. The proof is by induction on i. Lemma 1 establishes the result for $i=0$, hence we need only prove the induction step. In applying Lemma 1, we have made use of condition A_α, if $\alpha > 0$, but have not used condition A_0 for the case $\alpha=0$.

We shall give the proof for $\alpha \geqq 0$ and point out where condition A_α is required only when $\alpha > 0$.

Suppose the result holds for the integers $0, 1, \cdots, i-1$. Define b_j and c_j by the formulas

$$k(k-1) \cdots (k-i+1) = k^i - \sum_{j=1}^{i-1} b_j k^j \tag{2.4}$$

$$(k+\alpha)(k+\alpha-1) \cdots (k+\alpha-i+1) = k(k-1) \cdots (k-i+1) - \alpha \sum_{j=0}^{i-1} c_j k^j, \tag{2.5}$$

then

$$B_n^\alpha[\Phi, t^{\alpha+i}] = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \left(\frac{k}{n} \right)^{\alpha} \left(\frac{k}{n} \right)^{i}$$

$$= \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k}\varphi_k] \left(\frac{k}{n}\right)^{\alpha} \cdot \frac{k(k-1)\cdots(k-i+1)}{n^i}$$

$$+ \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k}\varphi_k] \left(\frac{k}{n}\right)^{\alpha} \frac{1}{n^i} \sum_{j=1}^{i-1} b_j k^j$$

$$= C_n + B_n .$$

We interchange the order of summation in B_n and apply the induction hypothesis obtaining

$$\lim_{n\to\infty} \|B_n\|_B \leqq \sum_{j=1}^{i-1} |b_j| \lim_{n\to\infty} \frac{1}{n^{i-j}} \left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k}\varphi_k] \left(\frac{k}{n}\right)^{\alpha+j} \right\|_B$$

$$\leqq \sum_{j=1}^{i-1} |b_j| \, \|\varphi_j\|_B \lim_{n\to\infty} \frac{1}{n^{i-j}} = 0 .$$

The induction hypothesis requires condition A_α only if $\alpha > 0$. Furthermore

$$C_n = \frac{(n+\alpha)\cdots(n+\alpha-i+1)}{n^i} \sum_{k=i}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k}\varphi_k] \left(\frac{k}{n}\right)^{\alpha} \frac{(k+\alpha)\cdots(k+\alpha-i+1)}{(n+\alpha)\cdots(n+\alpha-i+1)}$$

$$+ \sum_{k=i}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k}\varphi_k] \left(\frac{k}{n}\right)^{\alpha} \cdot \frac{\alpha}{n^i} \sum_{j=0}^{i-1} c_j k^j = A_n + D_n .$$

$$D_n = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k}\varphi_k] \left(\frac{k}{n}\right)^{\alpha} \frac{\alpha}{n^i} \sum_{j=0}^{i-1} c_j k^j$$

$$- \sum_{k=0}^{i-1} \binom{n+\alpha}{n-k} [\Delta^{n-k}\varphi_k] \left(\frac{k}{n}\right)^{\alpha} \frac{\alpha}{n^i} \sum_{j=0}^{i-1} c_j k^j = E_n + F_n .$$

Note that for $\alpha = 0$, we have that $D_n = 0$ and condition A_α is not needed. The fact that $\lim_{n\to\infty} \|E_n\|_B = 0$ can be demonstrated in much the same way that $\lim_{n\to\infty} \|B_n\|_B = 0$ was established and we omit the details.

In order to consider F_n, we choose $x \in X$ such that $\|x\| = 1$ and set

$$x_k = \left(\frac{k}{n}\right)^{\alpha} \frac{\alpha}{n^i} \sum_{j=0}^{i-1} c_j k^j \cdot x \qquad \text{for } k = 0, 1, \cdots, i-1$$

$$= \theta_X \qquad \text{for } k = i, i+1, \cdots, n ,$$

then

$$\|F_n\|_B = \sup_{\|x\|\leqq 1}\left\| \sum_{k=0}^{i-1}\binom{n+\alpha}{n-k}[\Delta^{n-k}\varphi_k]\left\{\left(\frac{k}{n}\right)^\alpha \frac{\alpha}{n^i}\sum_{j=0}^{i-1} c_j k^j\right\}\cdot x\right\|_Y$$

$$= \sup_{\|x\|\leqq 1}\left\| \sum_{k=0}^{n}\binom{n+\alpha}{n-k}[\Delta^{n-k}\varphi_k]\cdot x_k\right\|_Y < M \max_{0\leqq k\leqq i-1} \frac{k^\alpha \alpha}{n^{i+\alpha}}\left|\sum_{j=0}^{i-1} c_j k^j\right|$$

by condition A_α. But by (2.5) the c_j's are independent of n and hence the right side of the inequality converges to zero as $n\to\infty$, thus $\|D_n\|_B \to 0$.

We now note that

$$\binom{n-i+\alpha}{n-k} = \binom{n+\alpha}{n-k}\frac{(k+\alpha)\cdots(k+\alpha-i+1)}{(n+\alpha)\cdots(n+\alpha-i+1)} = \binom{n+\alpha}{n-k}(k,n,i)$$

where the last equality defines (k, n, i), and hence $k\leqq n$ implies that

$$\binom{n-i+\alpha}{n-k} \leqq \binom{n+\alpha}{n-k} \quad \text{since} \quad (k,n,i)\leqq 1\,.$$

We may now write

$$A_n = \frac{(n+\alpha)\cdots(n+\alpha-i+1)}{n^i}\sum_{k=i}^{n}\binom{n-i+\alpha}{n-k}[\Delta^{n-k}\varphi_k]\left(\frac{k}{n}\right)^\alpha$$

$$= \frac{(n+\alpha)\cdots(n+\alpha-i+1)}{n^i}\sum_{k=0}^{n-i}\binom{n-i+\alpha}{n-i-k}[\Delta^{n-i-k}\varphi_{k+i}]\left(\frac{k+i}{n}\right)^\alpha,$$

by replacing k by $k+i$ in the summation. Then setting $p=n-i$, we obtain

$$A_n = \frac{(n+\alpha)\cdots(n+\alpha-i+1)}{n^i}\sum_{k=0}^{p}\binom{p+\alpha}{p-k}[\Delta^{p-k}\varphi_{k+i}]\left(\frac{k+i}{p+i}\right)^\alpha.$$

The factor outside the summation converges to 1 as $n\to\infty$ and if we can show that

$$\left\|\sum_{k=0}^{p}\binom{p+\alpha}{p-k}[\Delta^{p-k}\varphi_{k+i}]\cdot x_k\right\|_Y < M\cdot \max\|x_k\|$$

holds for each bounded sequence $\{x_k\}$ in X, then Lemma 1 would show that $A_n \to \varphi_i$ in B-norm as $n\to\infty$ by simply replacing the original sequence $\varphi_0, \varphi_1, \cdots$, by $\varphi_i, \varphi_{i+1}, \cdots$. Note that for $\alpha=0$, the remark following Lemma 1 applies and the modified condition A_0 is not required, i.e., the proof of (ii) is complete.

Suppose $x_0, \cdots, x_p$ are given and set $x_{k-i} = \theta_X$ for $k=0, 1, \cdots, i-1$, then

$$\left\| \sum_{k=0}^{p} \binom{p+\alpha}{p-k} [\Delta^{p-k} \varphi_{k+i}] \cdot x_k \right\|_Y = \left\| \sum_{k=i}^{n} \binom{n-i+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot x_{k-i} \right\|_Y$$

$$= \left\| \sum_{k=i}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot (k, n, i)\, x_{k-i} \right\|_Y$$

$$= \left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot (k, n, i)\, x_{k-i} \right\|_Y$$

$$< M \max |(k, n, i)| \cdot \|x_k\| < M \max \|x_k\|$$

by condition A_α and the fact that $(k, n, i) \leqq 1$ for $k \leqq n$. The proof is now complete.

COROLLARY. *If Φ satisfies condition A_α (in case $\alpha > 0$) and $P(t) = \sum_{i=0}^{\nu} a_i t^i$ where $a_0, \cdots, a_\nu \in X$, then*

$$\lim_{n\to\infty} \left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k]\, P\left(\frac{k}{n}\right)\left(\frac{k}{n}\right)^\alpha - \sum_{i=0}^{\nu} \varphi_i a_i \right\|_Y = 0\,.$$

REMARK. Theorem 1 is in form very much like formula (22), page 150, of [2]. In a footnote on that same page the writers assert that indeed their formula (22)

$$\lim_{m\to\infty} \sum_{n=0}^{m} \binom{m+\alpha}{m-n} \Delta^{m-n} \mu_n \cdot \left(\frac{n+\alpha}{m+\alpha}\right)^{k+\alpha} = \mu_k$$

can be demonstrated with no restrictions on $\{\mu_k\}$. The following example shows this to be untrue.

Set $\mu_0 = 1$ and $\mu_k = 0$ for $k = 1, 2, 3, \cdots$, then $\Delta^{m-n}\mu_n = 1$ if $n = 0$ and $\Delta^{m-n}\mu_n = 0$ if $n \neq 0$. The formula then reduces to

$$\lim_{m\to\infty} \binom{m+\alpha}{m} \left(\frac{\alpha}{m+\alpha}\right)^{\alpha+k} = \mu_k\,.$$

For $k=0$, we have

$$\lim_{m\to\infty} \binom{m+\alpha}{m} \left(\frac{\alpha}{m+\alpha}\right)^\alpha = \frac{\alpha^\alpha}{\Gamma(\alpha+1)} \lim_{m\to\infty} \left(\frac{m}{m+\alpha}\right)^\alpha = \frac{\alpha^\alpha}{\Gamma(\alpha+1)}$$

which is not always $1=\mu_0$.

It is, of course, for $\alpha=0$ and $\alpha=1$. Theorem 1 shows that the assertion is always true for $\alpha=0$.

LEMMA 2. *If* $\alpha \geqq 0$, $f \in C_\alpha$ *and* $\varepsilon > 0$, *then there exists a polynomial* $P(t)$ *with coefficients in* X *such that*

$$\|t^\alpha P(t) - f(t)\|_X < \varepsilon \quad \textit{for every } t \in [0, 1].$$

PROOF. For $\alpha = 0$, the result follows from the fact that the Bernstein polynomials for f converge uniformly to f for each $f \in C$.

If $\alpha > 0$, then $f(0)=\theta_X$. Let $\delta > 0$ be chosen so that $\|f(t)\| < \varepsilon/3$ for $0 \leqq t < \delta$. Let $g(t) = f(t)$ for $\delta \leqq t \leqq 1$ and $g(t) = (t/\delta)^\alpha f(t)$ for $0 \leqq t < \delta$, then $g(t)/t^\alpha \in C$ and we may choose $P(t)$ such that $\|P(t) - g(t)/t^\alpha\| < \varepsilon/3$ for $0 \leqq t \leqq 1$. We now have that

$$\|t^\alpha P(t) - f(t)\| = t^\alpha \|P(t) - g(t)/t^\alpha\| < \frac{\varepsilon}{3} \quad \text{if} \quad \delta \leqq t \leqq 1$$

and

$$\begin{aligned}\|t^\alpha P(t) - f(t)\| &\leqq t^\alpha \|P(t) - g(t)/t^\alpha\| + \|g(t)\| + \|f(t)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

if $0 \leqq t < \delta$, and the proof is complete.

It has been shown (Theorem 1, [4]) that if T is a bounded linear transformation from C into Y, then there exists a function K on $[0, 1]$ such that $K(t) \in B^+$ for each t, K has the Gowurin ω-property with $W_0^1 K = \|T\|$ and for each f in C, $T(f) = \int_0^1 dK \cdot f$ where the integral converges in the Y^+ norm. Hereafter when we refer to such a K we shall understand that K does have the ω-property and generates such a T from C into Y. (In general the ω-property implies only that T maps C into the metric completion of Y^+). We also assume without loss that $K(0) = \theta_B$ which in turn implies that $K(1) \in B$.

We now state the main theorem of this section.

THEOREM 2. *Suppose* $\alpha \geqq 0$, $\Phi = \{\varphi_k\}_{k=0}^\infty$ *where* $\varphi_k \in B$ *and that* Y *is complete, then of the following four statements*:

(1) Φ *satisfies condition* A_α,

(2) *there exists a* K *such that*

$$\varphi_k = \int_0^1 dK(t) \cdot t^{k+\alpha} \quad for \quad k = 0, 1, 2, \cdots,$$

(3) *$B_n^\alpha[\Phi, f]$ converges in Y-norm for each $f \in C_\alpha$ (indeed, it converges to $\int_0^1 dK \cdot f$), and*

(4) *there exists $M' > 0$ such that*

$$\left\| \sum_{k=1}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot x_k \right\| < M' \max \|x_k\|$$

for every bounded sequence $\{x_k\}_{k=0}^{\infty}$ in X and $n = 0, 1, 2, \cdots$,

(1) *is equivalent to* (2) *and* (1) *implies* (3). *Furthermore, if X is complete, then for $\alpha > 0$,* (3) *implies* (4) *and for $\alpha = 0$,* (3) *implies* (1).

PROOF. (a) (1) $\Longrightarrow$ (3). Suppose $f \in C_\alpha$, $\varepsilon > 0$ and $P(t) = \sum_{i=0}^{v} t^{i+\alpha} a_i$ is such that $\|P(t) - f(t)\|_X < \varepsilon/3M$ for $0 \leqq t \leqq 1$. By the above corollary, there exists $N > 0$ such that if $n_1, n_2 > N$, then

$$\|B_{n_1}^\alpha[\Phi, P] - B_{n_2}^\alpha[\Phi, P]\|_Y < \varepsilon/3 .$$

we then have for $n_1, n_2 > N$ that

$$\begin{aligned}
&\|B_{n_1}^\alpha[\Phi, f] - B_{n_2}^\alpha[\Phi, f]\|_Y \\
&\quad \leqq \|B_{n_1}^\alpha[\Phi, f-P]\|_Y + \|B_{n_2}^\alpha[\Phi, f-P]\|_Y + \|B_{n_1}^\alpha[\Phi, P] - B_{n_2}^\alpha[\Phi, P]\|_Y \\
&\quad < M\left[\max\left\|f\left(\frac{k}{n_1}\right) - P\left(\frac{k}{n_1}\right)\right\| + \max\left\|f\left(\frac{k}{n_2}\right) - P\left(\frac{k}{n_2}\right)\right\|\right] + \frac{\varepsilon}{3} \\
&\quad < \varepsilon .
\end{aligned}$$

Therefore $B_n^\alpha[\Phi, f]$ is Cauchy and since Y is complete, it converges. From what we have just shown, we know that $B_n^\alpha[\Phi, f]$ converges for each $f \in C_\alpha$. Denote the limit by $L(f)$. L is clearly linear in f and (1) implies that $\|L\| \leqq M$.

If $\alpha = 0$, we apply Theorem 1 of [4] at once. If $\alpha > 0$, then we must take an extra step. Suppose $\alpha > 0$ and for each $f \in C$, define $L'(f) = L(f - f(0))$. L$'$ is clearly linear in f and agrees with L on C_α. Furthermore,

$$\|L'(f)\|_Y = \|L(f - f(0))\|_Y \leqq M\|f - f(0)\|_C \leqq 2M\|f\|_C$$

and hence L' is bounded by $2M$. In either case there exists K such that $L'(f)=\int_0^1 dK\cdot f$ for each $f\in C$ and for $f\in C_\alpha$ we now have that $\lim_{n\to\infty} B_n^\alpha[\Phi,f]$ $=\int_0^1 dK(t)\cdot f(t)$.

(b) (1) $\Longrightarrow$ (2). In the particular case that $f(t)=t^{\alpha+k}\cdot x$ for some fixed $x\in X$, then by Theorem 1 and the above remarks we have that

$$\varphi_k\cdot x=\int_0^1 dK(t)\cdot[t^{\alpha+k}\cdot x]=\left[\int_0^1 dK(t)\cdot t^{\alpha+k}\right]\cdot x$$

and since this is true for each $x\in X$, we have (2).

(c) (2) $\Longrightarrow$ (1). Suppose $\varphi_k=\int_0^1 dK(t)\cdot t^{k+\alpha}\in B$ where K has Gowurin constant M, then if we define $L(f)=\int_0^1 dK\cdot f$, we have that

$$\left\|\sum_{k=0}^n\binom{n+\alpha}{n-k}[\Delta^{n-k}\varphi_k]\cdot x_k\right\|_Y=\left\|L\left(\sum_{k=0}^n\binom{n+\alpha}{n-k}[\Delta^{n-k}t^{k+\alpha}]\cdot x_k\right)\right\|_Y$$

$$\leqq M\cdot\left\|\sum_{k=0}^n\binom{n+\alpha}{n-k}[\Delta^{n-k}t^{k+\alpha}]\cdot x_k\right\|_X.$$

We shall now demonstrate that

$$\left\|\sum_{k=0}^n\binom{n+\alpha}{n-k}[\Delta^{n-k}t^{k+\alpha}]\cdot x_k\right\|\leqq\max\|x_k\| \tag{2.6}$$

which is a special case of condition A_α, namely the case in which $\Phi=\Phi(t)$ $=\{t^{k+\alpha}\}_{k=0}^\infty$ and $Y=X$. This will be of fundamental importance in the remainder of this paper.

Setting $\nu=n-k$ we obtain

$$\left\|\sum_{k=0}^n\binom{n+\alpha}{n-k}\Delta^{n-k}t^{k+\alpha}\cdot x_k\right\|_X=\left\|\sum_{\nu=0}^n\binom{n+\alpha}{\nu}[\Delta^\nu t^{\alpha+n-\nu}]\cdot x_{n-\nu}\right\|_X$$

$$=\left\|\sum_{\nu=0}^n\binom{n+\alpha}{\nu}t^{n-\nu+\alpha}(1-t)^\nu\cdot x_{n-\nu}\right\|_X$$

$$= \left\| t^{\alpha} \sum_{\nu=0}^{n} \binom{n+\alpha}{\nu} t^{n-\nu}(1-t)^{\nu} \cdot x_{n-\nu} \right\|_X .$$

It is clear that for $0 \leqq t \leqq 1$, each term of $t^{\alpha} \sum_{\nu=0}^{n} \binom{n+\alpha}{\nu} t^{n-\nu}(1-t)^{\nu}$ is non-negative and Endl [1, p. 441] has shown that the sum is not greater than 1, hence by convexity, (2.6) follows and (1) holds.

(d) $\alpha > 0$, (3) $\Longrightarrow$ (4). Define $L_n(f) = B_n^{\alpha}[\Phi, f]$ for each $f \in C_{\alpha}$. For $\alpha > 0$, we then have, since $f(0) = \theta_X$, that

$$\begin{aligned} \|L_n(f)\|_Y &= \left\| \sum_{k=1}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot f\left(\frac{k}{n}\right) \right\|_Y \\ &\leqq \sup_{\|x_k\| \leqq 1} \left\| \sum_{k=1}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot x_k \right\|_Y \cdot \|f\|_{C_{\alpha}} \\ &= M_n \cdot \|f\|_{C_{\alpha}} \end{aligned}$$

where the last equality defines M_n which is clearly finite since each $\varphi_k \in B$. Now suppose $\varepsilon > 0$. There exist points $x_1, x_2, \cdots, x_n$ in X such that $\|x_i\| \leqq 1$ and

$$\left\| \sum_{k=1}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot x_k \right\| > M_n - \varepsilon .$$

Define $f(k/n) = x_k$ for $k = 1, 2, \cdots, n$, $f(0) = \theta_X$ and f linear otherwise so that $f \in C_{\alpha}$. We then have that $\|f\|_{C_{\alpha}} \leqq 1$ and $\|L_n f\| > M_n - \varepsilon$. It follows that $\|L_n\| = M_n$. The assumption that X is complete implies that C_{α} is a Banach space and (3) then allows us to invoke the uniform boundedness principle. Hence, there exists an $M > 0$ such that $M_n \leqq M$ for all n and the result follows.

(e) $\alpha = 0$, (3) $\Longrightarrow$ (1). Since it need not be true that $f(0) = \theta_X$, we must sum from $k=0$ and the required modifications to the argument in (d) are obvious.

COROLLARY. *If both X and Y are complete, then* (1), (2) *and* (3) *are equivalent.*

PROOF. We need only show that (3) implies (1) for $\alpha > 0$. Again we set $L_n(f) = B_n^{\alpha}[\Phi, f]$ and then (3) implies that $L_n(f) \to L(f)$ where the L_n

are uniformly bounded and hence L is bounded on the Banach space C_α. Suppose $\|L\| = M$, then as in (a) above, we may extend L to L' on C such that $\|L'\| \leqq 2M$ whence we may again apply the representation theorem of [4] and obtain $L'(f) = \int_0^1 dK \cdot f$ for each $f \in C$ where $W_0^1 K = \|L'\|$, and as before, Theorem 1 applies to show that $\varphi_k = \int_0^1 dK(t) \cdot t^{\alpha+k}$ for $k = 0, 1, 2, \cdots$. Therefore, we may write

$$\begin{aligned}\left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot x_k \right\|_Y &= \left\| L'\left(\sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} t^{\alpha+k}] \cdot x_k \right) \right\| \\ &\leqq 2M \left\| \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} t^{\alpha+k}] \cdot x_k \right\| \\ &\leqq 2M \max \|x_k\|\end{aligned}$$

by (2. 6) and thus (1) holds.

3. The Case in which Y is a Function Space. We shall now consider some special cases of the above results. These considerations will equip us with the tools to attack some questions in summability theory that are related to the convergence and moment theorems obtained in the previous section.

Suppose X is a Banach space, E is a Banach space and Y is the space of bounded functions from $[0, 1]$ into E, endowed with the uniform norm topology. Y is then a Banach space. $\Phi = \{\varphi_k\}_{k=0}^{\infty}$ is again a sequence in $B[X, Y]$ and clearly each φ_k may be considered as a bounded function from $[0, 1]$ into $B[X, E]$. For example, if each φ_k is a continuous function from $[0, 1]$ into $B[X, E]$, then Theorem 2 above generalizes the main result in [5], where the case $\alpha = 0$ was considered. We now obtain a generalization of the corollary of that paper.

THEOREM 3. *If E is X, then $B_n^\alpha[\Phi, f]$ converges uniformly to f for each $f \in C_\alpha$, if and only if $\varphi_k = t^{k+\alpha}$ for $k = 0, 1, 2, \cdots$. Furthermore, the uniform convergence to f for each $f \in C$ implies that $\alpha = 0$.*

PROOF. In order to obtain the sufficiency we note that equation (2. 6) shows that $\Phi = \{t^{\alpha+k}\}_{k=0}^{\infty}$ satisfies (1) of Theorem 2, and hence by (3) of Theorem 2 we have that $\lim_{n\to\infty} B_n^\alpha[\Phi, f] = L(f) = \int_0^1 dK(t, s) f(s)$ for each $f \in C_\alpha$. However, by the corollary to Theorem 1 and Lemma 1, we have that for each polynomial

$P(t)=\sum_{i=0}^{\nu} t^{i+\alpha}a_i$, $L(P)=P$ and these polynomials are dense in C_α, hence $L(f)=f=\lim_{n\to\infty} B_n^\alpha[\Phi, f]$ for each $f\in C_\alpha$.

In order to obtain the converse, we observe that from Theorem 2, convergence for each $f\in C_\alpha$ implies that $\varphi_k(t)=\int_0^1 dK(t,s)\cdot s^{\alpha+k}$ and that $B_n^\alpha[\Phi, f]$ converges to $\int_0^1 dK(s,t)\cdot f(s)$. However, since by hypothesis, $B_n^\alpha[\Phi, f]$ converges to f, we have that $f(t)=\int_0^1 dK(t,s)\cdot f(s)$ for each $f\in C_\alpha$ and the result follows.

The final statement follows from the fact that for $\alpha>0$, $B_n^\alpha[\Phi, f](s)=\theta_X$ at $s=0$ when $\Phi=\{t^{k+\alpha}\}_{k=0}^\infty$. This completes the proof of the theorem.

In the present setting we may also consider the question of pointwise convergence. If we consider s, $0\leqq s\leqq 1$ as a parameter and apply the corollary to Theorem 2 at each point s, we obtain:

THEOREM 4. *For each $\alpha\geqq 0$ and each $s\in[0,1]$, the following three statements are equivalent*:

(1′) *$\Phi(s)$ satisfies condition A_α where $M=M(s)$,*
(2′) *For each $s\in[0,1]$, there exists a $K(s,\cdot)$ such that*

$$\varphi_k(s)=\int_0^1 dK(s,u)\,u^{k+\alpha} \quad \textit{for} \quad k=0,1,2,\cdots,$$

and

(3′) *$B_n^\alpha[\Phi(s), f]$ converges in E norm for each $f\in C_\alpha$ (and it converges to $\int_0^1 dK(s,u)\cdot f(u)$).*

It happens, that in certain cases, pointwise convergence implies uniform convergence. Suppose E is X and $B_n^\alpha[\Phi(s), t^{k+\alpha}]$ converges in B-norm, for each $s\in[0,1]$, to $s^{k+\alpha}$. Then for each $P(t)=\sum_{i=0}^{\nu} t^{i+\alpha}a_i$ we have that $B_n^\alpha[\Phi(s), P]$ converges in E norm to $P(s)$ for each $s\in[0,1]$. Since these polynomials are uniformly dense in C_α and C_α is a Banach space, we have by the uniform boundedness principle that the operators defined by $L_{n,s}(f)=B_n^\alpha[\Phi(s), f]$ are uniformly bounded for each s and as in the proof of Theorem 2, can be extended to all of C. Then also as in that proof we conclude that there exists

$K(s,\cdot)$ such that $\varphi_k(s) = \int_0^1 dK(s,u)\cdot u^{\alpha+k} = \lim_{n\to\infty} B_n^\alpha[\Phi(s), u^{\alpha+k}]$. We further conclude as in Theorem 3, that indeed $\varphi_k(s) = s^{k+\alpha}$ for $k=0,1,2\cdots$ and hence for each $f\in C_\alpha$, $B_n^\alpha[\Phi, f]$ converges uniformly to f. We state this result as a theorem.

THEOREM 5. *If $B_n^\alpha[\Phi(s), t^{k+\alpha}]$ converges in B-norm to the limit $s^{k+\alpha}$, for each $s\in[0,1]$, then $\varphi_k(s)=s^{k+\alpha}$ for $k=0,1,2,\cdots$, and $B_n^\alpha[\Phi, f]$ converges uniformly to f for each $f\in C_\alpha$.*

As we have seen, the case in which $\Phi=\{t^{k+\alpha}\}_{k=0}^\infty$ plays a particular role in this type of approximation for then we obtain uniform convergence to f for each $f\in C_\alpha$. We have also seen that, in this case, the added requirement of uniform convergence to f for each $f\in C$ demands that $\alpha=0$. We shall now show that for $\alpha>0$ we obtain, on each interval $[\delta,1]$, $0<\delta<1$, uniform convergence to f for each $f\in C$. In order to establish this we need a lemma which proves to be essential in our later considerations of summability methods.

We define

$$f_{n-j}^\alpha(\varphi_j) = \sum_{k=0}^{n-j}\binom{n-j+\alpha}{n-j-k}[\Delta^{n-j-k}\varphi_{k+j}] \qquad \text{for} \quad n\geqq j$$

and

$$h_{n,j}^\alpha(\varphi_j) = \binom{n+\alpha}{n-j}[\Delta^{n-j}\varphi_j] \qquad \text{for} \quad n\geqq j\,.$$

LEMMA 3. *For each $t\in(0,1]$, and $\alpha>0$*

(i) $$\lim_{n\to\infty} f_n^\alpha(t^\alpha) = \lim_{n\to\infty}\sum_{k=0}^n\binom{n+\alpha}{n-k}\Delta^{n-k}t^{k+\alpha} = 1$$

(ii) $$\lim_{n\to\infty} h_{n,k}^\alpha(t^{k+\alpha}) = \lim_{n\to\infty}\binom{n+\alpha}{n-k}\Delta^{n-k}t^{k+\alpha} = 0\,,$$

the limits holding uniformly in t over $[\delta,1]$ for any $\delta\in(0,1)$. In addition,

(iii) $$\sup_{0\leqq t\leqq 1}\sup_{n\geqq 0}\sum_{k=0}^n\left|\binom{n+\alpha}{n-k}\Delta^{n-k}t^{k+\alpha}\right|\leqq 1\,.$$

REMARK. This is a Toeplitz type result and shows that for each $t\in(0,1]$, the matrix summability method $A(t)=(a_{n,k}(t))=(h_{n,k}^\alpha(t^{k+\alpha}))$ is regular.

PROOF. We first observe that (iii) is a special case of (2.6) where we take $x_k=1$.

In order to establish the uniform convergence in (i) we write

$$\frac{d}{dt}f_n^\alpha(t^\alpha)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}(k+\alpha)t^{k+\alpha-1}(1-t)^{n-k}-\sum_{k=0}^{n-1}\binom{n+\alpha}{n-k}(n-k)t^{k+\alpha}(1-t)^{n-k-1}$$

$$=\sum_{k=0}^{n}\frac{\Gamma(n+\alpha+1)}{(n-k)!\,\Gamma(k+\alpha)}t^{k+\alpha-1}(1-t)^{n-k}-\sum_{k=1}^{n}\frac{\Gamma(n+\alpha+1)}{(n-k)!\,\Gamma(k+\alpha)}t^{k+\alpha-1}(1-t)^{n-k}$$

$$=\alpha\frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}t^{\alpha-1}(1-t)^n$$

$$\leqq\alpha\binom{n+\alpha}{n}(1-\delta)^n \quad \text{for} \quad t\in[\delta,1]$$

$$=O(1)n^\alpha(1-\delta)^n\to 0 \quad \text{as} \quad n\to\infty\,.$$

Hence $\frac{d}{dt}f_n^\alpha(t^\alpha)$ converges uniformly to zero in $[\delta,1]$ and it follows that

$$\int_t^1\frac{d}{du}f_n^\alpha(u^\alpha)\,du=f_n^\alpha(1)-f_n^\alpha(t^\alpha)=1-f_n^\alpha(t^\alpha)$$

converges uniformly to zero in $[\delta,1]$ and (i) is established.

The uniform convergence in (ii) follows from the fact that

$$\binom{n+\alpha}{n-k}\Delta^{n-k}t^{k+\alpha}=\binom{n+\alpha}{n}\frac{n!}{(n-k)!}\frac{\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}t^{k+\alpha}(1-t)^{n-k}$$

$$=O(1)\,n^{k+\alpha}(1-\delta)^{n-k}$$

which converges to zero for $0<\delta<1$.

THEOREM 6. *If $\alpha>0$, $\Phi=\{t^{\alpha+k}\}_{k=0}^\infty$ and $f\in C$, then $B_n^\alpha[\Phi,f]$ converges uniformly to f on $[\delta,1]$ for $\delta\in(0,1)$.*

PROOF. By Theorem 5, $B_n^\alpha[\Phi,t^i\cdot x]$ converges uniformly on $[0,1]$ to $t^i\cdot x$ for each $i\geqq 1$ and each $x\in X$. By (i) of Lemma 3, $B_n^\alpha[\Phi,1\cdot x]$ converges uniformly on $[\delta,1]$ to $1\cdot x$ for each $x\in X$. The theorem is, therefore, true for polynomials.

Now for a fixed $f\in C$ and $\varepsilon>0$, choose a polynomial P such that $\|f(t)-P(t)\|<\varepsilon/2$ for $t\in[0,1]$, then

$$\|B_n^\alpha[\Phi, f] - f\| \leqq \|B_n^\alpha[\Phi, f - P]\| + \|B_n^\alpha[\Phi, P] - P\| + \|P - f\|$$
$$< \frac{\varepsilon}{2} + \|B_n^\alpha[\Phi, P] - P\| + \frac{\varepsilon}{2}$$

where the first $\varepsilon/2$ follows from (2. 6) and the second term converges uniformly to zero by the first part of the proof.

4. Summability Methods of Hausdorff Type. We suppose X and Y to be complete throughout this section.

Let $\delta^{(\alpha)} = \left((-1)^k \binom{n+\alpha}{n-k}\right)$, then $\delta^{(\alpha)}$ is an involutory matrix. If $\alpha=0$, then $\delta^{(\alpha)}$ is the classical differencing matrix associated with Hausdorff summability methods. As before, $\Phi=\{\varphi_k\}_{k=0}^\infty$ is a sequence in B and we define $H^{(\alpha)}=(h_{n\cdot k}^\alpha(\varphi_k))$. It is then true that $H^{(\alpha)} = \delta^{(\alpha)}\Phi\delta^{(\alpha)} = \left(\binom{n+\alpha}{n-k}[\Delta^{n-k}\varphi_k]\right)$.

In this section we shall discuss the questions of convergence preserving and regularity of such methods. The case $\alpha = 0$ has been treated in [3] and the notions of convergence preserving and regularity are as in [3]. The scalar case has been treated in considerable detail by Endl [1].

THEOREM 7. *If $H^{(\alpha)}$ is a convergence preserving (or regular) method, then Φ satisfies conditicn A_α and hence (by Theorem 2) is a solution to the moment problem.*

PROOF. It suffices to note that by Theorem 1 (or Theorem 2) of [3], that the hypothesis implies that condition A of [3] must hold, and this is just condition A_α.

We now turn our attention to a converse for Theorem 7. Suppose that Φ satisfies condition A_α (with $\alpha>0$), then by Theorem 2 above, there exists a K such that $\varphi_k = \int_0^1 dK(t)\cdot t^{\alpha+k}$ for $k = 0, 1, 2, \cdots$. We wish to consider the question of regularity of $H^{(\alpha)}$ relative to some linear transformation L from X into Y or possibly into some larger space. It is sufficient to check conditions A', B' and C' of [3] and to apply Theorem 2 of that paper. It is shown in [3] that conditions A' and A are equivalent and condtion A is just condition A_α in the present situation.

In order to check condition B', suppose $x \in X$, k is fixed, $y' \in Y'$ and consider

$$\eta = \lim_{n\to\infty} y' \binom{n+\alpha}{n-k}\Delta^{n-k}\varphi_k \cdot x = \lim_{n\to\infty} y' \left[\int_0^1 dK(t)\binom{n+\alpha}{n-k}\Delta^{n-k}t^{k+\alpha}\right]\cdot x$$
$$= \lim_{n\to\infty}\int_0^1 \binom{n+\alpha}{n-k}\Delta^{n-k}t^{k+\alpha}\, dy'\, K(t)\cdot x\,.$$

Denote the scalar-valued function $y' K(t) \cdot x$ by $g(t)$. By the Lemma in [4], g is of bounded variation on $[0, 1]$ and for $0 < \delta < 1$, we have by (ii) of Lemma 3 above that

$$\eta = \lim_{n\to\infty} \int_0^\delta \binom{n+\alpha}{n-k} \Delta^{n-k} t^{k+\alpha} \, dg(t) .$$

Since the integrand is continuous and has the value zero at $t = 0$, there is no change in the value of the integral if we change (if necessary) the value of $g(0)$ to be $g(0+)$ to make g continuous at $t = 0$. The integrand has its maximum value at $t=(k+\alpha)/(n+\alpha)$ which is less than δ for sufficiently large values of n. Applying Stirling's formula to $\binom{n+\alpha}{n-k}$ we can then write

$$\begin{aligned} |\eta| &\leq \lim_{n\to\infty} e \cdot \frac{(n+\alpha+1)^{1/2}}{(n-k+1)^{1/2}} \cdot \frac{(k+\alpha)^{k+\alpha}}{(k+\alpha+1)^{k+\alpha+\frac{1}{2}}} \cdot V_0^\delta g \\ &= O(1) \cdot V_0^\delta g \end{aligned}$$

which converges to zero with δ and hence condition B' holds.

We now check condition C'. Suppose $x \in X$ and $y' \in Y'$ and set

$$\begin{aligned} \nu &= \lim_{n\to\infty} y' \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k} \varphi_k] \cdot x \\ &= \lim_{n\to\infty} \int_0^1 \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \Delta^{n-k} t^{k+\alpha} \, dy' K(t) \cdot x \\ &= \lim_{n\to\infty} \int_0^\delta \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \Delta^{n-k} t^{k+\alpha} \, dy' K(t) \cdot x \\ &\qquad + \lim_{n\to\infty} \int_\delta^1 \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \Delta^{n-k} t^{k+\alpha} \, dy' K(t) \cdot x \\ &= \lim_{n\to\infty} \int_0^\delta \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \Delta^{n-k} t^{k+\alpha} \, dy' K(t) \cdot x + y' K(1) \cdot x - y' K(\delta) \cdot x \end{aligned}$$

where we have applied (i) of Lemma 3 in the second integral. The remaining integral has a non-negative integrand which, by (2.6), is bounded above by 1 and hence if we again adjust $y' K(0) \cdot x$ to be $y' K(0+) \cdot x$ we obtain

$$\nu = y' K(1) \cdot x - y' K(0+) \cdot x$$

by an argument like that for condition B'.

For fixed $x \in X$, this limit then exists for each y' and since each $\varphi_k \in B[X, Y]$ we see that

$$y_n = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} [\Delta^{n-k}\varphi_k] \cdot x \in Y$$

and hence we have that $L(x) = K(1) \cdot x - K(0+) \cdot x$ can be considered as a point in Y''. It is understood, of course, that in the equation $L(x) = K(1) \cdot x - K(0+) \cdot x$, $K(0+)$ is considered only as a suggestive symbol, being defined by the equation itself, $L(x)$ having been defined by the limit process which yields ν. (Recall that $K(0) = \theta_B$ and $K(1)$ were already elements of $B[X, Y]$).

Hence we now see by considering Y as a subspace of Y' under the natural mapping, we have that $H^{(\alpha)}$ is regular relative to L as a summability method from X into Y'' where the convergence in Y'' is that of its weak*-topology.

If it happens that K is continuous at $t = 0$ in the sense that $y'K(0+) \cdot x = y'K(0) \cdot x = 0$ for each pair y', x, then we would actually have regularity relative to $K(1)$ from X into Y with the convergence being in the weak topology on Y. We state these results formally as a theorem.

THEOREM 8. *If Φ satisfies condition A_α, $\alpha > 0$, then $H^{(\alpha)}$ is regular relative to $L(x) = K(1) \cdot x - K(0+) \cdot x$ from X into Y'' where the convergence in Y'' is in the weak*-topology. Furthermore, if $y'K(0+) \cdot x = 0$ for each pair y', x, then $H^{(\alpha)}$ is regular relative to $K(1)$ from X into Y where the convergence in Y is in the weak topology.*

We may now summarize our main results as follows:

If X and Y are Banach spaces and $\alpha > 0$, then the following four statements are equivalent:

(1) Φ satisfies condition A_α,

(2) $\varphi_k = \int_0^1 dK(t) \cdot t^{k+\alpha}$,

(3) $B_n^\alpha[\Phi, f]$ converges for each f in C_α

and

(4) $H^{(\alpha)}$ is regular relative to $L(x) = K(1) \cdot x - K(0+) \cdot x$ from X into Y'' with the weak*-topology on Y''.

The case $\alpha = 0$ is different in that (4) must be replaced by "$H^{(\alpha)}$ is convergence preserving" since condition B' may not hold. These results are contained in [3] and [5].

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