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ON *GCR*-OPERATORS

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We may call an operator acting on a Hilbert space a *GCR-operator* if it generates a *GCR*-algebra. The purpose of this paper is to examine *GCR*-operators. Two results are shown. One of them asserts that the product of two *GCR*-operators which commute *doubly* is also a *GCR*-operator and the other that any von Neumann algebra of type I acting on a separable Hilbert space is generated by a *GCR*-operator. The latter is extremely connected with the result of C. Pearcy [10].

1. Definitions and Theorem 1. Throughout this paper, we mean by an operator a bounded linear operator on a Hilbert space and by a representation of a *-algebra a *-representation as an algebra of operators. Given families $F, G, \cdots$ of operators on a Hilbert space H, $A(F, G, \cdots)$ means the smallest C^*-algebra of operators on H containing $F, G, \cdots$ and the identity operator I on H; and $R(F, G, \cdots)$ the smallest von Neumann algebra on H containing $F, G, \cdots$ and, automatically, I. A C^*-algebra A on H is said to be generated by $F, G, \cdots$ if $A(F, G, \cdots)=A$; and a von Neumann algebra R on H is said to be generated as a von Neumann algebra, or simply to be generated unless we are thrown into confusion, by $F, G, \cdots$ if $R(F, G, \cdots)=R$.

We call a C^*-algebra A a *GCR*-algebra if any representation of A is of type I, in other words, if for any representation π of A the von Neumann algebra $R(\pi(A))$ is of type I ([3], [6], [7], and [12]). On the other hand, by an *NGCR*-algebra we mean a C^*-algebra in which there are no non-zero closed two-sided ideals which are *GCR*-algebras ([3], [6]). It is known that several C^*-algebras of interest are *GCR*-algebras and Glimm's uniformly hyperfinite algebras are *NGCR*-algebras ([5]).

Now we define a notion of *GCR*-operators together with that of *NGCR*-operators: An operator T on a Hilbert space is said to be a *GCR*-operator, an *NGCR*-operator, if the C^*-algebra $A(T)$ generated by T is a *GCR*-algebra, an *NGCR*-algebra, respectively. When we say, following some authors, that an operator T is of type I, of type II, of type III if the von Neumann algebra $R(T)$ generated by T is of type I, of type II, of type III, respectively, we can assert that all GCR-operators are of type I and, as Prof. J. Tomiyama kindly remarked to the author, that operators of type II and of type III are $NGCR$-operators. The reason of the latter is as following. If $R(T)$ has no portions of type I and if a non-zero closed two-sided ideal J in $A(T)$ is a GCR-algebra, then the weak closure $\tilde{J}$ of J in $R(T)$ becomes a weakly closed two-sided ideal in $R(T)$ which produces a portion of type I contradicting the assumption.

Normal operators, compact operators and isometries are GCR-operators (for isometries, [14] for instance), and there are $NGCR$-operators since operators of type II and of type III exist ([15], [20]). Moreover, it must be remarked that there is an $NGCR$-operator of type I. This fact is known immediately from D. Topping's result which says that there is an operator T such that $A(T)$ is uniformly hyperfinite (see [18]).

Hereafter we see

THEOREM 1. *If S and T are GCR-operators on a Hilbert space such that $ST=TS$ and $S^*T=TS^*$, then ST is a GCR-operator.*

The proof is easy from the following lemma, because in general any sub-C^*-algebra of a GCR-algebra is a GCR-algebra (4.3.5 in [3]).

LEMMA 1. *Let A and B be C^*-algebras on a Hilbert space which commute elementwise. Then, $A(A, B)$ is a GCR-algebra if and only if A and B are GCR-algebras.*

In the proof, some parts of arguments of tensor products of C^*-algebras are employed, so we recall here them. The α-norm in the algebraic tensor product $A \odot B$ of A and B is defined by

$$\|X\|_\alpha = \|\Sigma_k \pi_1(S_k) \otimes \pi_2(T_k)\| \quad \text{for} \quad X = \Sigma_k S_k \otimes T_k \quad \text{in} \quad A \odot B\,,$$

using arbitrarily chosen faithful representations π_1, π_2 of A, B, respectively, and the ν-norm in $A \odot B$ by

$$\|X\|_\nu = \sup\{\|\pi(X)\| : \pi \text{ taken over all representations } \pi \text{ of } A \odot B \text{ such that}$$

$$\|\pi(S \otimes T)\| \leqq \|S\|\,\|T\|\}$$

(cf. [9]). The following are known: The α-norm coincides with the ν-norm if A is a GCR-algebra ([16]); and the α-product $A \widehat{\otimes}_\alpha B$ of A and B, the completion of $A \odot B$ with respect to the α-norm in $A \odot B$, is a GCR-algebra if and only if A and B are GCR-algebras ([17]).

PROOF. Suppose that A and B are GCR-algebras in which we may assume the identity operator is contained. Since the α-norm in $A \odot B$ coincides with the ν-norm, the *-homomorphism

$$\Sigma_k S_k \otimes T_k \longrightarrow \Sigma_k S_k T_k$$

of $A \odot B$ to the smallest *-algebra containing A and B can be extended to a *-homomorphism φ of the α-product $A \hat{\otimes}_\alpha B$ of A and B to $A(A, B)$. When a representation π of $A(A, B)$ is given, the composition $\pi \circ \varphi$ of φ and π is a representation of a GCR-algebra $A \hat{\otimes}_\alpha B$, then it is of type I and also, so is π. Therefore $A(A, B)$ is a GCR-algebra. The converse is trivial and the proof is completed.

Here remark that an analogous argument shows that $A(A, B)$ is a CCR-algebra (see [3], [6]) if and only if A and B are CCR-algebras.

If an operator T commutes with T^*T, T is said to be nearly normal. Since such T is written in the form $T = SV$ with S a self-adjoint operator and with V an isometry commutes with S ([1]), we know that a nearly normal operator is a GCR-operator (cf. [21]), as an application of Theorem 1.

2. **Theorem 2.** In [10] C. Pearcy showed that any von Neumann algebra of type I on a separable Hilbert space is generated by an operator. On the other hand, it is seen that any von Neumann algebra of type I contains a weakly dense sub-C^*-algebra which is a GCR-algebra, though itself is sometimes not a GCR-algebra (cf. [13]). Then there arises a question whether we can find on a separable Hilbert space a GCR-operator by which a given von Neumann algebra of type I is generated. In the following we answear this affirmatively.

The next lemma is a key to our discussion. Its proof is essentially same as that of a lemma in [4].

LEMMA 2. *Let $\{A_i\}$ be a sequence of C^*-algebras with identities. If each A_i is generated by an operator, then the C^*-algebra obtained by adjoining the identity to the $C^*(\infty)$-sum of A_i's is generated by an operator.*

The $C^*(\infty)$-sum $\Sigma \oplus^{C^*(\infty)} A_\alpha$ of A_α's means the C^*-algebra of all formal sums $\Sigma \oplus T_\alpha$ with $T_\alpha \in A_\alpha$ and with all but finite number of $\|T_\alpha\|$'s less than ε for any $\varepsilon > 0$, in which algebraic operations are defined coordinatewise and in which norm is defined by $\|\Sigma \oplus T_\alpha\| = \sup_\alpha \|T_\alpha\|$.

PROOF. We may prove only the case when the sequence $\{A_i\}$ is infinite because an easy modification proves the other case.

We regard each A_i as a C^*-algebra acting on some Hilbert space H_i on which identity operator always denoted by I coincides with the identity of A_i. Then, $A=\Sigma\oplus^{C^*(\infty)}A_i$ is a C^*-algebra on the direct sum $H=\Sigma\oplus H_i$ of H_i's.

For each i, let T_i be an operator such that $A(T_i)=A_i$. We can choose sequences $\{\lambda_i\}$, $\{\mu_i\}$ of complex numbers and $\{K_i\}$ of closed discs in the complex plane which satisfy the following conditions:

(a) $\lambda_i \neq 0$ for each i.

(b) Let us put $S_i = \lambda_i T_i + \mu_i I$, then the spectrum $\sigma(S_i)$ of S_i is contained in the interior of K_i for each i, and

(c) $\{S_i\}$ converges uniformly to O.

(d) $K_i \cap K_j = \emptyset$ if $i \neq j$.

(e) Let γ_i be the center of K_i and δ_i the radius, then each γ_i is positive real, and

(f) $\{\gamma_i\}$ and $\{\delta_i\}$ converge monotone to 0.

Let i_0 be any positive integer and put $L = \Sigma\oplus_{i>i_0}H_i$ and $Q = \Sigma\oplus_{i>i_0}S_i$. Then we know that

$$\sigma(Q) = \bigcup_{i>i_0}\sigma(S_i) \cup \{0\} .$$

In fact, Theorem 1.6.17 in [11] teaches us that, for any neighborhood V of the origin 0, there is a $\delta > 0$ such that $\sigma(Q) \subset \sigma(P) + V$ for any operator P commutes with Q and satisfies $\|P-Q\| < \delta$, and we can choose $Q_n = S_{i_0+1}\oplus \cdots \oplus S_n \oplus O \oplus O \oplus \cdots$ as the above P when n is sufficiently large, then

$$\sigma(Q) \subset \sigma(Q_n) + V \subset \bigcup_{i>i_0}\sigma(S_i) \cup \{0\} + V ,$$

therefore, together with the obvious inclusion, the desired identity is obtained.

Next choose a closed disc K with its center at 0, disjoint with K_i if $i \leqq i_0$ and containing K_i if $i > i_0$. Define a function f on $M = \bigcup_{i>i_0} K_i \cup K$ as

$$f(z) = \begin{cases} 0, & \text{if } z \notin K_{i_0}; \\ 1, & \text{if } z \in K_{i_0}. \end{cases}$$

Then, from the theorem of Mergelyan (for instance [19]), there is a sequence $\{p_k\}$ of polynomials which converges to f uniformly on M. Since

$$\frac{1}{2\pi i}\int_{\partial K_i}(p_k(z)-f(z))(zI-S_i)^{-1}\,dz = p_k(S_i) \qquad \text{for } i < i_0 ,$$

$$\frac{1}{2\pi i}\int_{\partial K}(p_k(z)-f(z))(zI-Q)^{-1}\,dz = p_k(Q) = \Sigma \oplus_{i>i_0} p_k(S_i)\,,$$

and

$$\frac{1}{2\pi i}\int_{\partial K i_0}(p_k(z)-f(z))(zI-S_{i_0})^{-1}\,dz = p_k(S_{i_0}) - I\,;$$

we have

$$\|p_k(S_i)\| \leqq \delta_i \|p_k - f\| \sup_{z\in K_i}\|(zI-S_i)^{-1}\| \qquad \text{for} \quad i < i_0\,,$$

$$\|p_k(S_i)\| \leqq \|p_k(Q)\| \leqq \delta \|p_k - f\| \sup_{z\in K}\|(zI-Q)^{-1}\| \qquad \text{for} \quad i > i_0\,,$$

where δ denotes the radius of K, and

$$\|p_k(S_{i_0})-I\| \leqq \delta_{i_0}\|p_k-f\| \sup_{z\in K_{i_0}}\|(zI-S_{i_0})^{-1}\|\,.$$

Thus, we know that $\{p_k(S_i)\}$ converges as $k\to\infty$ uniformly to O when $i\neq i_0$ and to I when $i=i_0$, while these convergences are uniform with respect to i's. So that $\{\Sigma\oplus p_k(S_i)\}$ converges as $k\to\infty$ uniformly to $E_{i_0} = \cdots\oplus O\oplus \overset{\overset{i_0}{\cup}}{I}\oplus O \oplus O\oplus\cdots$. Put here $S=\Sigma\oplus S_i$. Since $\Sigma\oplus p_k(S_i) = p_k(S)$ is in $A(S)$ for each k, we have $E_{i_0}\in A(S)$ and also $\cdots\oplus O\oplus S_{i_0}\oplus O\oplus O\oplus\cdots = SE_{i_0}\in A(S)$. Therefore, $\cdots\oplus O\oplus A_{i_0}\oplus O\oplus O\oplus\cdots$ is contained in $A(S)$. Since i_0 is arbitrary, we have finally $A=A(S)$ and the proof is completed.

LEMMA 3. *Any homogeneous von Neumann algebra on a separable Hilbert space is generated as a von Neumann algebra by a GCR-operator.*

PROOF. We may regard a von Neumann algebra given in the lemma as $Z\otimes B(L)$, where Z is an abelian von Neumann algebra on a separable Hilbert space and L separable ([2]).

There is an invertible positive operator P in Z such that $R(P)=Z$ by von Neumann's generation theorem in [8], and it is easy to see that the operator

$$S = \begin{cases} 1\,, & \text{if} \quad \dim L = 1\,; \\ \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & 1 & \cdot & \\ & & \cdot & \cdot \\ 0 & & 1 & 0 \end{pmatrix}, & \text{if} \quad 2\leqq \dim L < \aleph_0\,; \end{cases}$$

$$\left\|\begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & 1 & \cdot & \\ & & \cdot & \cdot \\ 0 & & & \cdot & \cdot \end{pmatrix}\right\|, \qquad \text{if} \quad \dim L = \aleph_0;$$

satisfies that $R(S)=B(L)$.

Next let us put $T = P \otimes S$, then this is a GCR-operator because P and S are GCR-operators. We want to see that $R(T) = R$. When $\dim L = 1$, it is trivial. When $2 \leqq \dim L < \aleph_0$, by direct computations we have

$$\sqrt{T^*T}\, E + \sqrt{TT^*} = P \otimes I \quad \text{and} \quad T(P\otimes I)^{-1} = I \otimes S,$$

where $E = \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & 0 & \\ & & & \cdot \\ 0 & & & 0 \end{pmatrix}$. Since E is in $R(T)$, we know that $P\otimes I$ is in $R(T)$ and so is $I\otimes S$. Therefore, $R(T)=R(P\otimes I, I\otimes S)=R$. At last, when $\dim L=\aleph_0$, we have

$$\sqrt{T^*T} = P \otimes I \quad \text{and} \quad T(P\otimes I)^{-1} = I \otimes S,$$

so $R(T) = R$ as above. Now the proof is completed.

LEMMA 4. *Any $C^*(\infty)$-sum of GCR-algebras is a GCR-algebra.*

PROOF. Let $\{A_\alpha\}$ be an indexed family of GCR-algebras and A their $C^*(\infty)$-sum. We may assume for our purpose that each A_α has an identity I. For each α put $E_\alpha = \cdots \oplus O \oplus \overset{\alpha}{\overset{\cup}{I}} \oplus O \oplus O \oplus \cdots$, then we can identify A_α and AE_α in a trivial way. If π is a representation of A, then

$$\pi_\alpha(T) = \pi(TE_\alpha) \qquad \text{for} \quad T \in A_\alpha$$

is a representation of A_α and $\pi(X) = \Sigma \oplus \pi_\alpha(XE_\alpha)$ for all X in A. Since $\pi(E_\alpha)$ makes an orthogonal family of projections in the center of $\pi(A)$ and each $R(\pi_\alpha(A_\alpha))$ is of type I, $R(\pi(A)) = \Sigma \oplus R(\pi_\alpha(A_\alpha))$ is of type I. Then the proof is completed.

Now we show

THEOREM 2. *Any von Neumann algebra of type I on a separable Hilbert space is generated as a von Neumann algebra by a GCR-operator.*

PROOF. We can find a family $\{R_i\}$ of von Neumann algebras indexed by a suitable set of cardinals $\leqq \aleph_0$ with each R_i i-homogeneous such that the von Neumann algebra given in the theorem is identified with the direct sum $\Sigma \oplus R_i$ of R_i's ([2]). By Lemma 3, there is a GCR-operator T_i with $R(T_i) = R_i$. Let A be the C^*-algebra obtained by adjoining the identity to $\Sigma \oplus^{C^*(\infty)} A_i$, where $A_i = A(T_i)$. Then, by Lemma 4, A is a GCR-algebra; and weakly dense in R. Finally by Lemma 2, there is a operator T such that $A(T) = A$. It is of course a GCR-operator and the proof is completed.

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