Tôhoku Math. Journ.
21(1969), 558-572.

NUCLEARITY ON HARMONIC SPACES

CORNELIU CONSTANTINESCU AND AUREL CORNEA

(Received January 6, 1969)

Let X be a locally compact space and $\mathcal{H}$ be a sheaf on X such that for any open subset U of X $\mathcal{H}(U)$ is a real vector space of real continous functions on U called harmonic functions on U. We suppose that $\mathcal{H}$ satisfies the axioms $\mathrm{H}_0, \mathrm{H}_1, \mathrm{H}_2$ [3]. In order to obtain a harmonic space we have to assume that a supplementary axiom concerning the convergence of increasing sequences of harmonic functions is fulfilled. There are known in literature three such axioms: $\mathrm{K}_1, \mathrm{K}_D$ [1] and 3 [4] where $3 \Longrightarrow \mathrm{K}_D \Longrightarrow \mathrm{K}_1$. In [6] (resp. [2]) it was proved that axiom 3 (resp. axiom K_D and the axiom of countable basis) implies the property that for any open subset U of X, $\mathcal{H}(U)$ is nuclear [7] with respect to the topology of uniform convergence on compact sets. We shall call this property *axiom of nuclearity*. This axiom implies the axiom K_1. If axiom 3 is fulfilled then for any regular domain V and for any $x \in V$ the carrier of the harmonic measure μ_x^V is equal to the boundary of V. We shall call this proonerty *axiom of ellipticity*. In [2] it was proved that if the axiom K_D and the axiom of ellipticity are fulfilled then the axiom 3 is fulfilled.

In this paper we give some equivalent conditions for the axiom of nuclearity and show that the nuclearity does not follow from the axiom K_1 and the axiom of ellipticity and that K_D does not follow from the axiom of nuclearity and the axiom of ellipticity. We don't know whether the axiom K_D implies the axiom of nuclearity when the space has no countable basis.

*   *

Let X be a locally compact space and $\mathcal{H}$ be a presheaf on X such that for any open subset U of X $\mathcal{H}(U)$ is a real vector space of real continuous functions on U. The elements of $\mathcal{H}(U)$ will be called *harmonic functions on* U. We suppose that the following axiom is fulfilled:

AXIOM. *For any $x_0 \in X$ and any compact neighbourhood K of x there exist an open neighbourhood V of x_0, $V \subset K$, and a family $(\mu_x)_{x \in V}$ of positive (Radon) measures on K such that*

a) *for any real continuous function f on K the function on V $x \to \mu_x(f)$*

belongs to $\mathcal{H}(V)$;

b) *for any open neighbourhood* U *of* K *and any* $u \in \mathcal{H}(U)$ *we have* $u(x) = \mu_x(u)$ *for any* $x \in V$.

The example we have in mind introducing this structure is a harmonic space [3].

THEOREM. *Let* $\mathfrak{U}$ *be a basis of open subsets of* X. *The following assertions are equivalent*:*)

a) (*resp.* a′) *for any open subset* U *of* X (*resp. for any* $U \in \mathfrak{U}$) *the vector space* $\mathcal{H}(U)$ *endowed with the topology of uniform convergence on compact subsets of* U *is nuclear*;

b) (*resp.* b′) *for any* $U \in \mathfrak{U}$ *and any compact subset* K *of* U *there exists a positive measure* μ *on* U *with compact carrier such that for any harmonic function* (*resp. positive harmonic function*) u *on* U *we have*

$$\sup_K |u| \leqq \mu(|u|) \qquad (\textit{resp. } \sup_K u \leqq \mu(u));$$

c) (*resp.* c′) *for any* $U \in \mathfrak{U}$ *and any compact subset* K *of* U *there exists a measure* μ *on* U *with compact carrier such that the least upper bound of any increasing sequence of harmonic functions on* U *is finite continuous* (*resp. bounded*) *on* K *if it is* μ-*integrable*;

d) (*resp.* d′) *for any open subset* U *of* X (*resp. for any* $U \in \mathfrak{U}$) *and any sequence* $(u_n)_{n \in N}$ *of positive harmonic functions on* U *such that* $\sum_{n \in N} u_n$ *is locally bounded we have*

$$\sum_{n \in N} \sup_K u_n < \infty$$

for any compact subset K *of* U;

e) *For any open subset* U *of* X, *for any compact subset* K *of* U *and for any positive harmonic function* u *on* U *we have*

$$\sup \sum_{\iota \in I} u_\iota(x_\iota) < \infty ,$$

where $(x_\iota)_{\iota \in I}$ *is an arbitrary finite family in* K *and* $(u_\iota)_{\iota \in I}$ *is an arbitrary finite family of positive harmonic functions on* U *such that* $u = \sum_{\iota \in I} u_\iota$;

*) The implication $a \Rightarrow f$ may be deduced also from D. HINRICHSEN, Randintegrale und nukleare Funktionenräume, Ann. Inst. Fourier 17, 1(1967), 225–271.

f) *for any open subset* V *of* X, *any compact subset* K *of* X, $V \subset K$, *any family* $(\mu_x)_{x \in V}$ *of positive measures on* K *satisfying the conditions* a), b) *of the axiom and any compact subset* L *of* V *there exists a positive measure* μ *on* K *such that* $\mu_x \leqq \mu$ *for any* $x \in L$;

f′) *for any* $x_0 \in X$ *and any compact neighbourhood* K *of* x_0 *there exist an open neighbourhood* V *of* x_0, $V \subset K$, *and a family* $(\mu_x)_{x \in V}$ *of positive measures on* K *satisfying the conditions* a), b) *of the axiom and such that for any compact subset* L *of* V *there exists a positive measure* μ *on* K *such that* $\mu_x \leqq \mu$ *for any* $x \in L$.

a $\Longrightarrow$ a′ is trivial.

a′ $\Longrightarrow$ b. The set $\{u \in \mathcal{H}(U) \mid \sup_K |u| \leqq 1\}$ is a circled convex closed neighbourhood of the origin of $\mathcal{H}(U)$. Since $\mathcal{H}(U)$ is nuclear there exists a compact subset L of U, $K \subset L$, such that the map $u|_L \mapsto u|_K : \mathcal{H}(U)|_L \to \mathcal{H}(U)|_K$*) is nuclear ([7] page 63). Hence there exist a sequence $(l_n)_{n \in N}$ of continuous linear functionals of norm at most equal to 1 on the normed space $\mathcal{H}(U)|_L$ and a sequence $(u_n)_{n \in N}$ in $\mathcal{H}(U)$ such that

$$\sum_{n \in N} \sup_K |u_n| < \infty\,, \qquad u|_K = \sum_{n \in N} l_n(u|_L) u_n|_K\,.$$

By the Hahn-Banach theorem there exists for any $n \in \boldsymbol{N}$ a measure μ_n on L such that $\|\mu_n\| \leqq 1$ and $\mu_n(u|_L) = l_n(u|_L)$ for any $u \in \mathcal{H}(U)$. We denote by μ the positive measure on L

$$\mu := \sum_{n \in N} (\sup_K |u_n|) |\mu_n|\,.$$

Obviously

$$\sup_K |u| \leqq \sum_{n \in N} |l_n(u|_L)| \sup_K |u_n| \leqq \mu(|u|)\,.$$

b $\Longrightarrow$ a. Let U be an open subset of X. For any $x \in U$ we denote by l_x the element of the dual $\mathcal{H}(U)'$ of $\mathcal{H}(U)$ defined by $u \mapsto u(x)$. It is obvious that the map

$$x \mapsto l_x : U \to \mathcal{H}(U)'$$

is continuous for the $\sigma(\mathcal{H}(U)', \mathcal{H}(U))$ topology. Let L be a compact subset of U and W be the circled convex closed neighbourhood of the origin of $\mathcal{H}(U)$

*) $u|_L$ (resp. $\mathcal{H}(U)|_L$) means the restriction of u (resp. the normed vector space of the restrictions of $\mathcal{H}(U)$) to L.

$$W: = \{u \in \mathcal{H}(U) | \sup_L |u| \leqq 1\} .$$

For any measure μ on L the map

$$f \mapsto \int f(l_x) d\mu(x) : C(W^o) \to \boldsymbol{R}$$

defines a measure ν on W^o, the polar of W.

Let K be a compact subset of U. There exist a finite family $(U_\iota)_{\iota \in I}$ in $\mathfrak{U}$ and a finite family $(K_\iota)_{\iota \in I}$ of compact sets such that

$$K_\iota \subset U_\iota \subset U, \qquad K = \bigcup_{\iota \in I} K_\iota .$$

From b) there exists for any $\iota \in I$ a positive measure μ_ι on U_ι with compact carrier such that

$$\sup_{K_\iota} |u| \leqq \mu_\iota(|u|)$$

for any $u \in \mathcal{H}(U_\iota)$. We set $\mu := \sum_{\iota \in I} \mu_\iota$. Let $u \in \mathcal{H}(U)$. Then, since $\mathcal{H}$ is a presheaf,

$$\sup_K |u| \leqq \sum_{\iota \in I} \sup_{K_\iota} |u| \leqq \sum_{\iota \in I} \mu_\iota(|u|) = \mu(|u|) .$$

The required implication follows now from the above considerations using SATZ 4.1.5 [7].

b $\Longrightarrow$ b$'$ is trivial.

b$'$ $\Longrightarrow$ c. Let K be a compact subset of U and μ be a measure stated in b$'$. Let further $(u_n)_{n \in N}$ be an increasing sequence in $\mathcal{H}(U)$ such that its least upper bound is μ-integrable. Then for any natural numbers $m, n, m < n$, we have

$$\sup_K (u_n - u_m) \leqq \mu(u_n - u_m) .$$

Hence the sequence $(u_n|_K)_{n \in N}$ is uniformly convergent.

c $\Longrightarrow$ c$'$ is trivial.

c$'$ $\Longrightarrow$ b$'$. Let K be a compact subset of U and μ be a measure stated in c$'$. We want to show that there exists a positive real number α such that the measure $\alpha\mu$ satisfies the conditions required in b$'$. Assume the contrary. Then for any $m \in \boldsymbol{N}$ there exists a positive harmonic function v_m on U such that

$$\sup_K v_m \geqq m , \qquad \mu(v_m) \leqq 1/m^2 .$$

Then $\left(\sum_{m \leqq n} v_m\right)_{n \in N}$ is an increasing sequence of harmonic functions on U whose least upper bound is μ-integrable but not bounded on K.

b′ $\Longrightarrow$ d′. Let K be a compact subset of U, μ be a measure stated in b′ and $(u_n)_{n \in N}$ be a sequence of positive harmonic functions on U such that $\sum_{n \in N} u_n$ is locally bounded. Then

$$\sum_{n \in N} \sup_K u_n \leqq \sum_{n \in N} \int u_n d\mu = \int \left(\sum_{n \in N} u_n\right) d\mu < \infty .$$

d′ $\Longrightarrow$ d follows immediately from the fact that $\mathfrak{U}$ is a basis of X.

d $\Longrightarrow$ e. Suppose that e) is not true. Then there exist an open subset U of X, a compact subset K of U and a positive harmonic function u on U such that for any $n \in \boldsymbol{N}$ there exists a finite family $(u_{n,\iota})_{\iota \in I_n}$ of positive harmonic functions on U such that

$$\sum_{\iota \in I_n} \sup_K u_{n,\iota} > 2^n , \qquad \sum_{\iota \in I_n} u_{n,\iota} = u .$$

This contradicts d) since

$$\sum_{n \in N} \sum_{\iota \in I_n} \frac{1}{2^n} u_{n,\iota}$$

is locally bounded and

$$\sum_{n \in N} \sum_{\iota \in I_n} \sup_K \frac{1}{2^n} u_{n,\iota} = \infty .$$

e $\Longrightarrow$ f. Let f be a positive real continuous function on K. By e)

$$\sup \sum_{\iota \in I} \mu_{x_\iota}(f_\iota) < \infty ,$$

where $(x_\iota)_{\iota \in I}$ is an arbitrary finite family in L and $(f_\iota)_{\iota \in I}$ is an arbitrary finite family of positive real continuous functions on K such that

$$f = \sum_{\iota \in I} f_\iota ,$$

the map

$$f \mapsto \sup \sum_{\iota \in I} \mu_{x_\iota}(f_\iota)$$

yields the required measure μ.

f $\Longrightarrow$ f′ follows immediately from the axiom.

f′ $\Longrightarrow$ b. Let U be an open subset of X and K be a compact subset of U. By f′) there exists a finite family $(K_\iota, V_\iota, (\mu_{\iota,x})_{x \in V_\iota})_{\iota \in I}$ such that K_ι are compact subsets of U, V_ι are open subsets of $K_\iota, (\mu_{\iota,x})_{x \in V_\iota}$ are families of positive measures on K_ι satisfying the conditions a), b) of the axiom and $K \subset \bigcup_{\iota \in I} V_\iota$. There exists a family $(L_\iota)_{\iota \in I}$ of compact subsets of U such that $L_\iota \subset V_\iota$ for any $\iota \in I$ and

$$K = \bigcup_{\iota \in I} L_\iota .$$

By f′) there exists for any $\iota \in I$ a positive measure μ_ι on K_ι such that $\mu_{\iota,x} \leqq \mu_\iota$ for any $x \in L_\iota$. We set

$$\mu := \sum_{\iota \in I} \mu_\iota .$$

Then for any harmonic function u on U we have

$$\sup_K |u| \leqq \sum_{\iota \in I} \sup_{L_\iota} |u| = \sum_{\iota \in I} \sup_{x \in L_\iota} |\mu_{\iota,x}(u)|$$

$$\leqq \sum_{\iota \in I} \sup_{x \in L_\iota} \mu_{\iota,x}(|u|) \leqq \sum_{\iota \in I} \mu_\iota(|u|) = \mu(|u|) .$$

COROLLARY 1. *Let X be a harmonic space satisfying the axiom K_D and such that for any open subset U of X and any compact set K of U there exists a measure μ on U with compact carrier L for which any absorbent set containing L contains K. Then the axiom of nuclearity is fulfilled.*

Let U be an open subset of X and K be a compact subset of U. Let K' be a compact neighbourhood of K contained in U and μ be a measure on U with compact carrier L such that any absorbent set containing L contains K'. Then there exists a positive real number α such that for any positive harmonic function u on U we have

$$\sup_K u \leqq \alpha\mu(u).$$

Indeed, if this assertion is not true then there exists a sequence $(u_n)_{n\in N}$ of positive harmonic functions on U such that

$$\sup_K u_n \geqq n, \qquad \mu(u_n) \leqq 1/n^2$$

for any $n \in N$. The sequence $\left(\sum_{m\leqq n} u_m\right)_{n\in N}$ is an increasing sequence of harmonic functions on U whose least upper bound u is μ-integrable. Hence

$$L \subset \overline{\{x \in U \mid u(x) < \infty\}} \cap U.$$

This last set being absorbent ([3] Lemma 1.6 or [2] Satz 1. 4. 2) it contains K'. Hence u is harmonic on the interior of K' and therefore bounded on K which is a contradiction. The corollary follows now from b$'\Longrightarrow$a.

This corollary contains the result of P. Loeb and B. Walsh [6] (resp. that of H. Bauer [2]) that axiom 3 (resp. axiom $\mathrm{K_D}$ and the countable basis of X) implies (resp. imply) the axiom of nuclearity.

COROLLARY 2. *Let $(X, \mathcal{H})$ be a harmonic space, F be an absorbent set of X and $\mathcal{H}_F$ be the sheaf induced on F by $\mathcal{H}$ in the sence of* [5]. *If the axiom of nuclearity is fulfilled on X then it is fulfilled also on F.*

This corollary follows immediately from the theorem (a $\Longleftrightarrow$ f) using [5] Corollary 2. 1.

We shall construct now examples of harmonic spaces which will clear up the relations between the axiom of nuclearity and the axiom $\mathrm{K_1}$ and $\mathrm{K_D}$.

EXAMPLE. Let $(r_n)_{n\in N}$ be a decreasing sequence of strictly positive real numbers converging to 0 and such that for any $n \in \boldsymbol{N}$ $\dfrac{r_n}{r_{n+1}} \in \boldsymbol{N}$. Let further $(\theta_n)_{n\in N}$ be a sequence of pairwise different real numbers, $0 \leqq \theta_n < 2\pi$. We set

$$X := \{(x,y,z) \in \boldsymbol{R}^3 \mid y = z = 0\}) \bigcup$$

$$\left(\bigcup_{n\in N} \{(x,y,z) \in \boldsymbol{R}^3 \mid y = r_n \cos\theta_n,\ z = r_n \sin\theta_n\}\right) \bigcup$$

$$\left(\bigcup_{n\in N}\bigcup_{m\in Z}\{(x,y,z)\in \boldsymbol{R}^3 \mid x=mr_n,\ y=r\cos\theta_n,\right.$$

$$\left. z=r\sin\theta_n,\ 0<r<r_n\}\right);$$

here $\boldsymbol{N},\boldsymbol{Z},\boldsymbol{R}$ denote as usal the set of natural, integer, real numbers respectively. X endowed with the induced topology is a locally compact (connected and locally connected) space. We shall take as harmonic functions the real continuous functions which are locally linear on the axis $\{y=z=0\}$ and on any free segment and such that for any point (x,y,z) of the form $x=mr_n$, $y=r_n\cos\theta_n$, $z=r_n\sin\theta_n$, the sum of the derivatives taken in the three directions starting from this point is equal to zero. More precisely we denote for any open subset U of X by $\mathscr{H}(U)$ the set of real continuous functions on U such that:

a) the function $x\mapsto u(x,0,0)$ is locally linear on $\{x\in\boldsymbol{R}\mid(x,0,0)\in U\}$;

b) for any $n\in\boldsymbol{N}$ the function $x\mapsto u(x,r_n\cos\theta_n,\ r_n\sin\theta_n)$ is locally linear on $\{x\in\boldsymbol{R}\mid x\neq mr_n$ for any $m\in\boldsymbol{Z},\ (x,r_n\cos\theta_n,\ r_n\sin\theta_n)\in U\}$;

c) for any $n\in\boldsymbol{N},\ m\in\boldsymbol{Z}$ the function $r\mapsto u(mr_n,r\cos\theta_n,\ r\sin\theta_n)$ is locally linear on the set $\{0<r<r_n\mid(mr_n,\ r\cos\theta_n\ \ r\sin\theta_n)\in U\}$;

d) for any $m\in\boldsymbol{Z},\ n\in\boldsymbol{N}$ such that $(mr_n,\ r_n\cos\theta_n,\ r_n\sin\theta_n)\in U$

$$\lim_{\substack{r\to0\\ r>0}}(u(mr_n+r,r_n\cos\theta_n,r_n\sin\theta_n)-u(mr_n,r_n\cos\theta_n,r_n\sin\theta_n))/r$$

$$+\lim_{\substack{r\to0\\ r>0}}(u(mr_n-r,r_n\cos\theta_n,r_n\sin\theta_n)-u(mr_n,r_n\cos\theta_n,r_n\sin\theta_n))/r$$

$$+\lim_{\substack{r\to0\\ r>0}}(u(mr_n,(r_n-r)\cos\theta_n,(r_n-r)\sin\theta_n)-u(mr_n,r_n\cos\theta_n,r_n\sin\theta_n))/r$$

$$=0.$$

It is obvious that $\mathscr{H}$ is a sheaf on X such that for any open subset U $\mathscr{H}(U)$ is a real vector space of real continuous functions on U. Moreover for any $n\in\boldsymbol{N}$ the open set

$$X_n:=\{(x,y,z)\in X\mid\ y=r\cos\theta_n,\ z=r\sin\theta_n,\ 0<r\}$$

endowed with the restriction of $\mathscr{H}$ is a harmonic space satisfying the axiom 3.

Let V be a set of the form

$$V=\{(x,y,z)\in X\mid a<x<b,\ y^2+z^2<c^2\},$$

where $a, b, c \in \boldsymbol{R}$, and let f be a real bounded function on ∂V, the boundary of V in X. We denote for any $n \in \boldsymbol{N}$ such that $r_n < c$ by u_n (resp. v_n) the function on V equal to 0 on $V \cap \complement X_n$, harmonic on $V \cap X_n$ and such that

$$\lim_{x \to a} u_n(x, r_n \cos \theta_n, r_n \sin \theta_n) = 1, \quad \lim_{x \to b} u_n(x, r_n \cos \theta_n, r_n \sin \theta_n) = 0\,,$$

$$(\text{resp. } \lim_{x \to a} v_n(x, r_n \cos \theta_n, r_n \sin \theta_n) = 0, \quad \lim_{x \to b} v_n(x, r_n \cos \theta_n, r_n \sin \theta_n) = 1)$$

$$\lim_{q \to p} u_n(q) = 0 \qquad (\text{resp. } \lim_{q \to p} v_n(\mathrm{q}) = 0)$$

for any $p \in V \cap \partial(V \cap X_n)$. It is easy to see that u_n, v_n are harmonic functions on V and that

$$(1/3)\, u_n(mr_n, r_n \cos \theta_n, r_n \sin \theta_n) \leqq u_n((m+1)r_n, r_n \cos \theta_n, r_n \sin \theta_n)$$

$$\leqq (1/2)\, u_n(mr_n, r_n \cos \theta_n, r_n \sin \theta_n)\,,$$

$$(1/3)\, v_n((m+1)r_n, r_n \cos \theta_n, r_n \sin \theta_n) \leqq v_n(mr_n, r_n \cos \theta_n, r_n \sin \theta_n)$$

$$\leqq (1/2)\, v_n((m+1)r_n, r_n \cos \theta_n, r_n \sin \theta_n)$$

for any $m \in \boldsymbol{Z}$, $a < mr_n < (m+1)\, r_n < b$. Let a', b' be real numbers such that $a < a' \leqq b' < b$. From the above inequalities we deduce

$$1/3^{\frac{a'-a}{r_n}+2} \leqq \sup_{x \in [a', b']} u_n(x, y, z) \leqq 1/2^{\frac{a'-a}{r_n}-2}\,,$$

$$1/3^{\frac{b-b'}{r_n}+2} \leqq \sup_{x \in [a', b']} v_n(x, y, z) \leqq 1/2^{\frac{b-b'}{r_n}-2}\,.$$

Let $(\alpha_n)_{n \in N}, (\beta_n)_{n \in N}$ be two bounded sequences of real numbers. Using the last inequalities one may show that the function $\sum_{\substack{n \in N \\ r_n < c}} (\alpha_n u_n + \beta_n v_n)$ is harmonic on V and may be extended continuously to $\overline{V}$ if

$$\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0\,.$$

From this fact and the fact that the functions of the form $(x, y, z) \mapsto \alpha x + \beta$ $(\alpha, \beta \in R)$ are harmonic on X it follows that V is regular and that the Bauer convergence axiom K_1 is fulfilled. Moreover V is an MP-set.

The sequence $\left(\sum_{\substack{m \leqq n \\ r_m < c}} 3^{\frac{b-a}{r_m}} u_m\right)_{n \in N}$ is an increasing sequence of harmonic functions on V whose limit is finite everywhere but it is not locally bounded. Hence the axiom K_D (and even axiom K_2 [1]) is not fulfilled.

We prove now that if $(r_n)_{n \in N}$ is a strictly decreasing sequence then the axiom of nuclearity is fulfilled. Let first V be as above and K be a compact subset of V. Then there exist real numbers a', b' such that

$$a < a' < b' < b, \quad a' - a = b - b', \quad K \subset \{(x, y, z) \in V \mid a' < x < b'\}\ .$$

Let p be an isolated boundary point of V belonging to X_n with $r_n < c$ and let $(\mu_x^V)_{x \in V}$ be the family of harmonic measures of V. Then, by the above inequality,

$$\mu_x^V(\{p\}) \leqq 1/2^{\frac{a'-a}{r_n}-2}\ .$$

for any $x \in K$. If the sequence $(r_n)_{n \in N}$ is strictly decreasing then

$$\sum_{n \in N} 1/2^{\frac{a'-a}{r_n}-2} < \infty\ .$$

We deduce immediately that there exists a measure μ on the boundary of V such that $\mu_x^V \leqq \mu$ for any $x \in K$. A similar result is true for any regular set V in X_n for any $n \in \boldsymbol{N}$ since on X_n the axiom 3 is fulfilled. The axiom of nuclearity follows now from the theorem (f$'$ $\Longrightarrow$ a).

Suppose now that the sequence $(r_n)_{n \in N}$ is such that

$$\sum_{n \in N} 1/3^{1/r_n} = \infty\ .$$

In this case the axiom of nuclearity is not fulfilled. Indeed let V be the set

$$V := \{(x, y, z) \in X \mid 0 < x < 2\}\ .$$

We set

$$K := \{(x, y, z) \in V \mid x = 1\}\ .$$

K is a compact subset of V and we have

$$\sup_K u_n \geqq 1/3^{1/r_n+2},$$

where u_n is the function constructed above. Since

$$\sum_{n \in N} u_n \leqq 1, \qquad \sum_{n \in N} \sup_K u_n = \infty$$

the above assertion follows from the theorem (a $\Longrightarrow$ d).

In this example the axiom of ellipticity is not fulfilled. In the sequel we shall modify the sheaf $\mathcal{H}$ such that all the above properties are conserved and such that the axiom of ellipticity shall be satisfied.

Let $(\varepsilon_n)_{n \in N}$ be a sequence of strictly positive real numbers such that

$$\sum_{n \in N} \varepsilon_n / r_n^2 < \infty .$$

For any $n \in \boldsymbol{N} \cup \{\infty\}$ and any open subset U of X we denote by $\mathcal{H}_n(U)$ the set of real continuous functions u on U such that:

a) the restriction of u to $\{(x, y, z) \in U \mid (y, z) \neq (0, 0)\}$ belongs to $\mathcal{H}(\{(x, y, z) \in U \mid (y, x) \neq (0, 0)\})$;

b) for any two times continuously differentiable real function φ on $\boldsymbol{R}$ whose carrier is contained in $\{x \in \boldsymbol{R} \mid (x, 0, 0) \in U\}$ we have

$$\int_{\{x \in \boldsymbol{R} \mid (x,0,0) \in U\}} u(x, 0, 0) \varphi''(x) dx$$

$$+ \sum_{\substack{i \in N \\ i \leqq n}} \varepsilon_i \sum_{\substack{m \in Z \\ (mr_i, 0, 0) \in U}} \varphi(mr_i) \lim_{\substack{r \to 0 \\ r > 0}} (u(mr_i, r\cos\theta_i, r\sin\theta_i) - u(mr_i, 0, 0))/r$$

$$= 0 .$$

It can be shown as above that for any $n \in \boldsymbol{N}$ $(X, \mathcal{H}_n)$ is a harmonic space for which the above sets V are regular. For any $p \in V$ we denote by $\mu_{n,p}^V$ the harmonic measure associated to V, p and $\mathcal{H}_n$. For any bounded real function f on the boundary of V we denote by $H_{n,f}^V$ the function on V

$$p \longmapsto \int f d\mu_{n,p}^V .$$

For any real numbers $a, r, r > 0$, we set

$$V(a,r) := \{(x,y,z) \in X \mid |x-a| < r,\ y^2 + z^2 < r^2\}\ .$$

Let $(\delta_n)_{n\in N}$ be a sequence of strictly positive real numbers. We may construct inductively the above sequence $(\varepsilon_n)_{n\in N}$ such that for any $n \in \boldsymbol{N}$ we have

$$|H_{n,f}^{V(mr_j,r_j)} - H_{n+1,f}^{V(mr_j,r_j)}| < \delta_n\ ,$$

$$\sup_{|x-mr_j|<r_j} \mu_{n,(x,0,0)}^{V(mr_j,r_j)}(X_k) < \delta_k\ ,$$

for any $j \in \boldsymbol{N}$, $j \leqq n+1$, any $m \in N$, $0 < m \leqq r_0/r_j$, any real function f on the boundary of $V(mr_j, r_j)$, $|f| \leqq 1$, and any $k \in \boldsymbol{N}$. Then by simple considerations we deduce that we have

$$|H_{n,f}^{V(mr_j,r_j)} - H_{n+1,f}^{V(mr_j,r_j)}| < \delta_n,$$

$$(F) \qquad \sup_{|x-mr_j|<r_j} \mu_{n,(x,0,0)}^{V(mr_j,r_j)}(X_k) < \delta_k\ ,$$

for any $m \in \boldsymbol{Z}$ any $j \in \boldsymbol{N}$, any real function f on the boundary of $V(mr_j, r_j)$, $|f| \leqq 1$, and any $k \in \boldsymbol{N}$. If

$$\sum_{n\in N} \delta_n < \infty$$

then for any $m \in \boldsymbol{Z}$ any $j \in \boldsymbol{N}$ and any real bounded function f on the boundary of $V(mr_j, r_j)$ the sequence $(H_{n,f}^{V(mr_j,r_j)})_{n\in N}$ is uniformly convergent. We set

$$H_{\infty,f}^{V(mr_j,r_j)} = \lim_{n\to\infty} H_{n,f}^{V(mr_j,r_j)}\ .$$

It is easy to see that $H_{\infty,f}^{V(mr_j,r_j)} \in \mathcal{H}_\infty(V(mr_j, r_j))$. Moreover if f is continuous the function on $\overline{V(mr_j, r_j)}$ equal to $H_{\infty,f}^{V(mr_j,r_j)}$ on $V(mr_j, r_j)$ and equal to f on the boundary of $V(mr_j, r_j)$ is continuous.

Let U be an open relatively compact subset of X and u be a function of $\mathcal{H}_\infty(U)$ whose lower limit at the boundary of U is positive. We want to show that u is positive. Suppose the contrary and let p be a point of U where u takes its minimum. It is obvious that $p \notin \bigcup_{n\in N} X_n$. Hence p is of the form $(a, 0, 0)$. We may suppose that there exists an increasing sequence $(x_n)_{n\in N}$ in $\boldsymbol{R}$ converging to a such that $u(x_n, 0, 0) > u(a, 0, 0)$ for any $n \in \boldsymbol{N}$. Let ε be a strictly positive real number such that $\{(x,0,0) \in X \mid |x-a| \leqq \varepsilon\} \subset U$ and $u(x,0,0) \leqq 0$ for $|x-a| \leqq \varepsilon$. We set

$$\delta := \sup_{a-\varepsilon \leqq x < a} u(x,0,0) - u(a,0,0).$$

By the hypothesis $\delta > 0$. There exist real numbers, b, c, d such that

$$a - \varepsilon < d < c < b < a$$

and such that

$$\sup_{b \leqq x \leqq a} u(x,0,0) \leqq u(a,0,0) + \delta/3,$$

$$\inf_{d \leqq x \leqq c} u(x,0,0) \geqq u(a,0,0) + 2\delta/3.$$

Let φ be a two times continuously differentiable real function on $\boldsymbol{R}$ whose carrier lies in $]d, a+\varepsilon[$ such that: a) it is positive and $\varphi(a) = 1$, b) $\varphi' \geqq 0$ on $[a-\varepsilon, a]$, $\varphi' \leqq 0$ on $[a, a+\varepsilon]$ and $\varphi'(b) \geqq 1/\varepsilon$; c) $\varphi'' \geqq 0$ on $[d, b]$ and $\varphi'' \leqq 0$ on $[c, a]$. We have

$$\int_{a-\varepsilon}^{a} u(x,0,0)\varphi''(x)dx \geqq -(\delta/3)\int_b^a \varphi''(x)\,dx > \delta/3\varepsilon.$$

On the other hand

$$\sum_{i \in N} \varepsilon_i \sum_{\substack{m \in Z \\ (mr_i,0,0) \in U \\ mr_i \leqq a}} \varphi(mr_i) \lim_{\substack{r \to 0 \\ r > 0}} (u(mr_i, r\cos\theta_i, r\sin\theta_i) - u(mr_i,0,0))/r$$

$$\geqq -\delta\varepsilon \sum_{i \in N} \varepsilon_i/r_i^2.$$

Hence for a sufficiently small ε

$$\int_{a-\varepsilon}^{a} u(x,0,0)\,\varphi''(x)dx$$

$$+ \sum_{i \in N} \varepsilon_i \sum_{\substack{m \in Z \\ (mr_i,0,0 \in U \\ mr_i \leqq a}} \varphi(mr_i) \lim_{\substack{r \to 0 \\ r > 0}} (u(mr_i, r\cos\theta_i, r\sin\theta_i) - u(mr_i, 0,0))/r$$

$$> 0.$$

If u is constant on an interval of the form $[a, a+\eta]$ $(\eta > 0)$ then we may take φ such that

$$\int_a^{a+\varepsilon} u(x,0,0)\,\varphi''(x)dx$$

$$+\sum_{i\in N}\varepsilon_i \sum_{\substack{m\in Z\\ (mr_i,0,0)\in U\\ mr_i\geqq a}} \varphi(mr_i)\lim_{\substack{r\to 0\\ r>0}}(u(mr_i, r\cos\theta_i, r\sin\theta_i)-u(mr_i,0,0))/r$$

$$=0\,.$$

If this condition is not fulfilled we may show as before that the above expression is strictly positive for a sufficiently small ε. We get therefore the contradictory relation

$$0=\int_{\{x\in R\mid (x,0,0)\in U\}} u(x,0,0)\,\varphi''(x)dx$$

$$+\sum_{i\in N}\varepsilon_i \sum_{\substack{m\in Z\\ (mr_i,0,0)\in U}} \varphi(mr_i)\lim_{\substack{r\to 0\\ r>0}}(u(mr_i, r\cos\theta_i, r\sin\theta_i)-u(mr_i,0,0))/r$$

$$>0\,.$$

From all these considerations we deduce that $V(mr_j, r_j)$ is a regular domain with respect to $\mathcal{H}_\infty$ for any $m\in \boldsymbol{Z}$, $j\in \boldsymbol{N}$. Since the restriction of $\mathcal{H}_\infty$ to $\bigcup_{n\in N} X_n$ forms a harmonic space we deduce that there exists a basis of regular sets for $\mathcal{H}_\infty$. Let $(u_n)_{n\in N}$ be an increasing sequence in $\mathcal{H}_\infty(U)$ whose limit u is bounded. For any $m\in \boldsymbol{Z}$, $j\in \boldsymbol{N}$ such that $\overline{V(mr_j, r_j)}\subset U$ we have

$$u = H_{\infty,u}^{V(mr_j,r_j)}$$

on $V(mr_j, r_j)$. Hence u is continuous and $\mathcal{H}_\infty$ satisfies the axiom K_1. Since $(x,y,z)\to \alpha x+\beta: X\to R$ belongs to $\mathcal{H}_\infty(X)$ for any real numbers α, β we deduce that the set of harmonic functions on X separates X. We have proved therefore that $(X, \mathcal{H}_\infty)$ is a harmonic space. It is easy to see, since any ε_n $(n\in N)$ is strictly positive, that $(X, \mathcal{H}_\infty)$ satisfies the axiom of ellipticity.

Comparing the harmonic measures of $V(mr_j, r_j)$ with respect to $\mathcal{H}$ and $\mathcal{H}_\infty$ it is easy to see using the formula (F) and the theorem (a $\Longleftrightarrow$ f′) that $(X, \mathcal{H})$

and $(X, \mathcal{H}_\infty)$ satisfy simultaneously the axiom of nuclearity. If

$$\sum_{n \in N} 3^{\frac{r_0}{r_n}} \delta_n < \infty ,$$

then it can be shown as for $(X, \mathcal{H})$ that $(X, \mathcal{H}_\infty)$ does not satisfy the axiom K_D (an even axiom K_2 [1]).

BIBLIOGRAPHY

[1] H. BAUER, Axiomatische Behandlung des Dirichletschen Problems für elliptische und parabolishe Differentialgleichungen, Math. Ann., 146(1962), 1-59.

[2] H. BAUER, Harmonische Räume und ihre Potentialtheorie, Springer Verlag, Berlin-Heidelberg-New York, 1966.

[3] N. BOBOC, C. CONSTANTINESCU AND A. CORNEA, Axiomatic theory of harmonic functions. Non-negative superharmonic functions, Ann. Inst. Fourier, 15, 1(1965), 283-312.

[4] M. BRELOT, Lecture on potential theory, Part IV, Tata Institute of Fundamental Research, Bombay, 1960.

[5] C. CONSTANTINESCU, Harmonic spaces, absorbent sets and balayage, Rev. Roumaine Ma h. Pures Appl., 11(1966), 887-910.

[6] P. LOEB AND B. WALSH, Nuclearity in axiomatic potential theory, Bull. Amer. Math. Soc., 72(1966), 685-689.

[7] A. PIETSCH, Nukleare lokalkonvexe Räume, Akademie Verlag, Berlin, 1965.

INSTITUTUL DE MATEMATICA
CALEA GRIVIŢEI 21
BUCHAREST 12, RUMANIA