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ON COMPACT COMPLEX SUBMANIFOLDS OF THE COMPLEX PROJECTIVE SPACE

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1. Statement of results. Let $P_{n+p}(\boldsymbol{C})$ be the complex projective space of complex dimension $n+p$ with the Fubini-Study metric of constant holomorphic sectional curvature 1 and let M be an n-dimensional compact complex submanifold of $P_{n+p}(\boldsymbol{C})$ with the induced Kaehler structure.

Using a result of Simons, S. Tanno [2] has proved the following results:

PROPOSITION A. *Let R be the scalar curvature of M. If*

$$R > n(n+1) - \frac{n+\dfrac{1}{2}}{4-\dfrac{1}{p}},$$

then M is totally geodesic, that is, $M = P_n(\boldsymbol{C})$.

PROPOSITION B. *If every holomorphic sectional curvature of M is greater than* $1-\dfrac{n+\dfrac{1}{2}}{2n^2\left(4-\dfrac{1}{p}\right)}$, *then $M = P_n(\boldsymbol{C})$.*

In this note, we shall improve these results as follows:

THEOREM 1. *If $R > n(n+1) - \dfrac{n+2}{4-\dfrac{1}{p}}$ everywhere on M, then $M = P_n(\boldsymbol{C})$.*

THEOREM 2. *If every holomorphic sectional curvature of M is greater*

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than $1-\dfrac{n+2}{2n^2\left(4-\dfrac{1}{p}\right)}$, *then* $M=P_n(\boldsymbol{C})$.

2. Outline of Proofs. Let S be the square of the length of the second fundamental form of the immersion of M into $P_{n+p}(\boldsymbol{C})$. Then, in [1], we have proved the following

PROPOSITION 1. *If* $S\leqq\dfrac{n+2}{4-\dfrac{1}{p}}$ *everywhere on* M, *then either* $S=0$ (*i.e.*, M *is totally geodesic*) *or* $S=\dfrac{n+2}{4-\dfrac{1}{p}}$.

On the other hand, the equation of Gauss implies $R=n(n+1)-S$. This, together with Proposition 1, implies that if $R>n(n+1)-\dfrac{n+2}{4-\dfrac{1}{p}}$ everywhere on M, then $S=0$. This proves Theorem 1.

Let $K(X,Y)$ denote the sectional curvature determined by X and Y. If we put $\lambda=1-\dfrac{n+2}{2n^2\left(4-\dfrac{1}{p}\right)}$, then the assumption of Theorem 2 implies $\lambda<K(X,JX)\leqq 1$ for every X (the right hand equality is not necessarily attained), where J denotes the complex structure of M. Let $e_1,\cdots,e_n,Je_1,\cdots,Je_n$ be an orthonormal basis for $T_x(M)$. Then we have

$$R=2\sum_{i=1}^{n}\sum_{j\neq i}\{K(e_i,e_j)+K(e_i,Je_j)\}+2\sum_{i=1}^{n}K(e_i,Je_i).$$

On the other hand we have

$$K(e_i,e_j)+K(e_i,Je_j)=\frac{1}{4}\{H(e_i+e_j)+H(e_i-e_j)+H(e_i+Je_j)$$
$$+H(e_i-Je_j)-H(e_i)-H(e_j)\},$$

where $H(*)=K(*,J*)$.

Hence we have

$$K(e_i, e_j) + K(e_i, Je_j) > \frac{2\lambda - 1}{2}.$$

This implies

$$R > n(2n\lambda - n + 1) = n(n+1) - \frac{n+2}{4 - \frac{1}{p}}.$$

This, together with Theorem 1, implies Theorem 2.

BIBLIOGRAPHY

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