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ON COMPLETE NON-COMPACT RIEMANNIAN MANIFOLDS WITH CERTAIN PROPERTIES

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0. Introduction. Let M be a connected, complete and non-compact Riemannian manifold of dimension $n \geqq 2$ whose sectional curvature satisfies $K_\sigma \geqq 0$ for all plane sections σ. By virtue of completeness and non-compactness, there exists at least one ray starting from every point of M. Toponogov [12] proved that if there is a straight line in such M, M is isometric to $N \times R$ where N is a totally geodesic hypersurface. And if there exist k straight lines through a point of such M, M is isometric to $N^{n-k} \times R^k$, where N^{n-k} is an $(n-k)$ -dimensional totally geodesic submanifold. Recently, D. Gromoll and W. Meyer [4] have investigated some structures of complete and non-compact Riemannian manifold satisfying $K_\sigma > 0$ for all plane sections σ. Some results obtained in [4] is stated as follows:

(1) Every geodesic $\Gamma = \{\gamma(t)\}(-\infty < t < \infty)$ in M has conjugate pairs, especially M has no straight line.
(2) M does not contain any compact totally geodesic submanifold.
(3) M is contractible.

More recently J. Cheeger and D. Gromoll [3] investigated some structures of complete and non-compact Riemannian manifold satisfying $K_\sigma \geqq 0$ for all plane sections σ. One of the main results obtained in [3] is stated in the following:

(4) There is a compact totally convex set $S_M \subset M$ which is a compact totally geodesic submanifold of M without boundary (Theorem 3, [3]) which is called a soul of M.

The souls of M will give strong restrictions for the structures of M. Hence it might be interesting to investigate the isometric structure of a complete and non-compact Riemannian manifold with non-negative sectional curvature which contains a compact totally geodesic submanifold N and the relation between N and the souls of M. In this paper we only consider N being a hypersurface where the inclusion map $\iota: N \rightarrow M$ is an imbedding. Our main results obtained in the paper will be stated as follows.

THEOREM A. *Let M be a connected, complete and non-compact Riemannian manifold of class C^∞ with non-negative sectional curvature which has a compact totally geodesic hypersurface N. Suppose that there does not* *exist any normal vector field of* N *which is defined globally over* N. *Let* N_t *be defined by* $N_t=\{x \in M \mid d(x, N)=t\}$ *where* d *means the distance function of* M. *Then* N_t *is also a compact totally geodesic hypersurface for each* $t>0$, *and every* N_t *is the double covering of* N *and moreover* $M-N$ *is isometric to* $N_t \times (0, \infty)$. *Moreover,* N *is a soul of* M. *Especially* M *is isometric to an open Möbiusband if* dim $M=2$.

Considering the case where there is the unit normal vector field V of N which is defined globally over N, it will be proved that M is isometric to $N\times R$ if there exists a point $x \in N$ at which two geodesics defined by $t \to \exp_x tV(x)$ and $t \to \exp_x t(-V(x))$ are rays from N to ∞ respectively (Proposition 5). In this case the cut locus $C(N)$ of N is vacuous. Therefore we shall next consider the case $C(N) \neq \emptyset$. Let $F(N)$ be the first focal locus of N. Denoting the tangent cut locus and the tangent focal locus of N by C_N and F_N respectively, we shall prove

THEOREM B. *Let* M *be a connected, complete and non-compact Riemannian manifold of class* C^∞ *with non-negative sectional curvature. Let* N *be a compact totally geodesic hypersurface of* M. *Suppose that* N *has a unit normal vector field* V *which is defined globally over* N. *Assume that there is a normal vector* X *to* N *such that* $X \in C_N$, $X \notin F_N$ *and* $\|X\|=d(N, C(N))$. *Then there is a compact totally geodesic hypersurface* $\tilde{N}$ *which is a soul of* M *and coincides with* $C(N)$ *as a set and we have* $F(N)=\emptyset$. *Moreover, let* $\tilde{N}_t$ *be defined as* $\tilde{N}_t=\{x \in M \mid d(x, \tilde{N})=t\}$. $\tilde{N}_t$ *is isometric to* N *which is the double covering of* $\tilde{N}$ *for every* $t>0$ *and* $M-\tilde{N}$ *is isometric to* $N \times (0,\infty)$. *Especially* M *is isometric to an open Möbiusband if* dim $M=2$.

Theorems stated above have the extreme property $F(N)=\emptyset$ Hence we shall lastly consider N satisfying $F(N) \neq \emptyset$.

As for a compact Riemannian manifold, many people have investigated the structures of conjugate loci or cut loci of compact Riemannian manifolds under suitable conditions of M. And they investigated some structures of compact manifolds satisfying certain conditions for conjugate loci or cut loci. As an intuitive condition which is of course an interesting one, we see the one that the distance between a point (or a submanifold) of M and each point of its cut locus or its first conjugate locus (or its first focal locus) is constant. The manifold structures of compact manifold satisfying the above condition have been investigated by Bott [2], Nakagawa [5], [6], [7] and [8], Omori [9], Warner [14] and other people.

Turning to our situation that M is non-compact with non-negative sectional curvaure and $F(N) \neq \emptyset$, we shall consider M satisfying that for any tangent vector $X \in C_N$, $\|X\| = l$ holds, where l is a positive constant. Then we shall prove the following

THEOREM C. *Let M be a connected, complete and non-compact Riemannian manifold of class C^∞ with non-negative sectional curvature. Let N be a compact totally geodesic hypersurface of M. Assume that we have $\|X\| = l$ for all $X \in C_N$ and $F(N) \neq \emptyset$. Then $F(N)$ coincides with $C(N)$ as a set in M. Furthermore suppose that the multiplicity of the first focal point with respect to every geodesic normal to N is constant k. Then $C(N)$ becomes a compact totally geodesic submanifold of dimension $n-k-1$, which is a soul of M and every point of whose normal space is defined and diffeomorphic to R^{k+1}.*

COROLLARY TO THEOREM C. *If $k = n-1$, we have $F(N) = C(N) = \{q\}$ and M is diffeomorphic to R^n where $C(N) = \{q\}$ becomes a pole. And $C(N)$ is a 0-dimensional soul. Moreover N is diffeomorphic to S^{n-1}.*

1. Definitions and Notations. Throughout this paper let M be a conneted, complete and non-compact Riemannian manifold of dimension $n (n \geqq 2)$ and of class C^∞ which has an isometrically imbedded, compact totally geodesic hypersurface N. Geodesics are parametrized by arc-length. For any disjoint compact subsets A and B in M, let $G(A, B)$ be the set of all shortest geodesic segments starting from $x \in A$ and ending at $y \in B$ such that $d(x, y) = d(A, B)$, where d means the distance function with respect to the Riemannian metric tensor of M. For a compact subset A, there is a sequence of points $\{x_k\}$ in M such that $d(A, x_k) > k$ by non-compactness of M. There is a shortest geodesic segment $\Gamma_k \in G(A, x_k)$ for each k. Then we can choose a subsequence $\{\Gamma_j\}$ of $\{\Gamma_k\}$ in such a way that both $\{\gamma_j(0)\}$ and $\{\gamma_j'(0)\}$ converge to a point $y \in A$ and a unit vector $u \in M_y$ respectively. The geodesic $\Gamma_0 = \{\gamma_0(t)\}$ $(0 \leqq t < \infty)$ satisfying $\gamma_0(0) = y$ and $\gamma_0'(0) = u$ defines a ray from A to ∞. We denote by $G(A, \infty)$ the set of all rays from A to ∞. A point $p \in M$ is called a pole of M if $\exp_p : M_p \to M$ has maximal rank [4], where M_p is the tangent space at p. For two tangent vectors $u, v \in M_p$, we denote by $\sphericalangle(u, v)$ the angle between u and v. For a totally geodesic hypersurface N, we denote the tangent space of N at a point $x \in N$ by N_x. For any point $x \in N$, let Z be a unit normal vector to N_x and define a geodesic $\Gamma_x = \{\gamma_x(t)\}$ $(0 \leqq t \leqq a)$, $\gamma_x(0) = x$, $\gamma_x'(0) = Z$, where $\gamma_x'(t)$ means the tangent vector to Γ_x. A cut point q $(q = \gamma_x(a))$ of N along Γ_x is by definition the minimal point to N of Γ_x such that $\Gamma_x | [0, t] \in G(N, \gamma_x(t))$ for any $0 < t \leqq a$ and $\Gamma_x | [0, a+\varepsilon] \notin G(N, \gamma_x(a+\varepsilon))$ for any positive number ε. The cut locus $C(N)$ of N is by definition the set of all cut points of N along every Γ_x, $x \in N$ whose starting direction is normal to N. The tangent cut point of N with respect to Z is by definition $a \cdot Z$ and the tangent cut locus C_N of N is defined by the set of all tangent cut points of N with respect to every unit normal vector to N. A first focal point $q=\gamma_x(b)$ of N along Γ_x is defined in such a way that there exists a non trivial Jacobi field Y along Γ_x such that $<Y, \gamma_x'>=0$ with the initial condition $Y(0) \in N_x, Y'(0)=0$ and which satisfies $Y(b)=0$, and there does not exist any other non trivial Jacabi field Y_1 along Γ_x such that $<Y_1, \gamma_x'>=0$ with the initial condition $Y_1(0) \in N_x$, $Y_1'(0)=0$ whose zero point b_1 satisfies $b_1 < b$. The first focal locus $F(N)$ of N is defined by the set of all first focal points of N along every geodesic which starts from N and normal to N at the starting point. The tangent focal locus F_N of N is defined by the set of all normal vectors $b\gamma_x'(0)$. We have by definition, $C(N)=\exp \circ C_N$ and $F(N)=\exp \circ F_N$.

The tools for proofs of our results are the basic Lemma investigated by Gromoll and Meyer which plays an important role in [4] and the present paper, the basic theorem on triangles of Toponogov [11] and some property on cut locus of a submanifold investigated by Omori [9] which is stated in §3. In §2, §3 and §4, we shall prove Theorems A, B and C respectively. Some applications of the results will be stated in §5. Under our hypothesis of N, there is at least one plane section σ satisfying $K_\sigma=0$ by the statement (2) in §0.

2. The structure of M with certain condition for N. First of all, we shall prove the following lemma.

LEMMA 1. *Let Λ_p be a ray from N to ∞ such that $\lambda_p(0)=p \in N$. Then for any point $q \in N$, there is a ray Λ_q from N to ∞ which is obtained by Λ_p.*

PROOF. Let $\delta>0$ be the fundamental length of M on the compact set N and $B_\delta(p)$ be the open ball in M with center p and radius δ. Take any fixed point $r \in B_\delta(p) \cap N$, $\Gamma \in G(p, r)$ and $\Sigma_t \in G(r, \lambda_p(t))$ for each $t>0$. By definition of Λ_p, we have

$$(1) \qquad \mathcal{L}(\Lambda_p | [0, t]) = t < \mathcal{L}(\Sigma_t) \qquad \text{for all } t>0.$$

Let $(\tilde{\Gamma}, \tilde{\Lambda}_p | [0, t], \tilde{\Sigma}_t)$ be the triangle in R^2 corresponding to the geodesic triangle $(\Gamma, \Lambda_p | [0, t], \Sigma_t)$ with same side lengths, and the vertices be $\tilde{\lambda}_p(t)$, $\tilde{r}$ and $\tilde{p}$ respectively. The inequality obtained above shows us that $\sphericalangle(\tilde{r}, \tilde{p}, \tilde{\lambda}_p(t)) \geqq \sphericalangle(\tilde{\lambda}_p(t), \tilde{r}, \tilde{p})$ for any $t>0$. When $t \to \infty$, we can choose a subsequence $\{\sigma_{t_n}'(0)\}$ of $\{\sigma_t'(0)\}$, $t_1 < t_2 < \cdots < t_n < \cdots$, $\lim t_n = \infty$ which converges to some tangent vector $\sigma_r'(0) \in M_r$. Putting $\Lambda_r = \{\lambda_r(t)\}$ $(0 \leqq t < \infty)$, $\lambda_r(t) = \exp_r t \cdot \sigma_r'(0)$, we see that $\Lambda_r \in G(r,\infty)$ and moreover we must have $\sphericalangle(\lambda_r'(0), -\gamma'(a)) \geqq \pi/2$ by virtue of the basic theorem on triangles, where $\gamma(a)=r$. In fact for the sequence of triangles $(\widetilde{\Gamma}, \widetilde{\Lambda}_p|[0,t_n], \widetilde{\Sigma}_{t_n})$, we have $\lim_{n\to\infty}(\sphericalangle(\widetilde{r}, \widetilde{p}, \widetilde{\lambda}_p(t_n)) + \sphericalangle(\widetilde{\lambda}_p(t_n), \widetilde{r}, \widetilde{p})) = \pi$ and $\pi/2 \geqq \lim_{n\to\infty}\sphericalangle((\widetilde{r}, \widetilde{p}, \widetilde{\lambda}_p(t_n)) \geqq \lim_{n\to\infty}\sphericalangle(\widetilde{\lambda}_p(t_n), \widetilde{r}, \widetilde{p})$, from which we must have

$$(2) \qquad \lim_{n\to\infty}\sphericalangle(\widetilde{r}, \widetilde{p}, \widetilde{\lambda}_p(t_n)) = \lim_{n\to\infty}\sphericalangle(\widetilde{\lambda}_p(t_n), \widetilde{r}, \widetilde{p}) = \pi/2 .$$

The basic theorem on triangles implies $\sphericalangle(\lambda_r'(0), -\gamma'(a)) \geqq \lim_{n\to\infty}\sphericalangle(\widetilde{\lambda_p}(t_n), \widetilde{r}, \widetilde{p}) = \pi/2$. Suppose that $\sphericalangle(\lambda_r'(0), -\gamma'(a)) > \pi/2$. We shall derive a contradiction. In fact there are $x \in \Lambda_r$ and $y \in N$ satisfying $d(x,y)=d(x,N)<d(x,r)$ if $\sphericalangle(\lambda_r'(0), -\gamma'(a)) > \pi/2$. There exists sufficiently small $\varepsilon>0$ satisfying the following:

$$(3) \qquad d(r,x) - d(y,x) > 2\varepsilon.$$

The equality (2) is equivalent to $\lim_{n\to\infty}(d(r,\lambda_p(t_n)) - t_n) = 0$ from the argument in [10]. Then there is a large number k such that

$$(4) \qquad 0 < d(r,\lambda_p(t_n)) - t_n < \varepsilon \qquad \text{for all } n>k.$$

On the other hand, there is a point $x_{t_n} \in \Sigma_{t_n}$ for every n satisfying $d(r,x_{t_n}) = d(r,x)$ and $\lim_{n\to\infty} x_{t_n} = x$. Hence there is a number k_1 such that

$$(5) \qquad d(x, x_{t_n}) < \varepsilon \qquad \text{for all } n>k_1 .$$

Then we must have for every $n > \mathrm{Max}\{k, k_1\}$,

$$\begin{aligned}
& d(y, \lambda_p(t_n)) \\
& \leqq d(y,x) + d(x,x_{t_n}) + d(x_{t_n}, \lambda_p(t_n)) \\
& < (d(r,x) - 2\varepsilon) + \varepsilon + d(x_{t_n}, \lambda_p(t_n)), \qquad (3) \text{ and } (5) \\
& = d(r, x_{t_n}) + d(x_{t_n}, \lambda_p(t_n)) - \varepsilon \\
& = d(r, \lambda_p(t_n)) - \varepsilon < t_n = d(p, \lambda_p(t_n)), \quad (4)
\end{aligned}$$

which contradicts that Λ_p is a ray from N to ∞.

Next, we shall prove that Λ_r is a ray from N to ∞. Suppose that Λ_r is not a ray from N to ∞. There are points $x \in \Lambda_r$ and $y \in N$ such that $d(x,r) > d(x,y) = d(x,N)$. The argument developed above leads us to $\Lambda_p \notin G(N,\infty)$.

Hence there is at least one ray from N to ∞ through each point $q \in N$ by compactness of N. Q. E. D.

We shall denote a ray from N to ∞ through a point $x \in N$ by Λ_x. Let us note the following; We see from the convexity condition due to Alexandrov and Toponogov [11], that there are at most two rays starting from every point $x \in N$ to ∞ and we can observe that every ray in $G(x, \infty)$, $x \in N$ becomes a ray from N to ∞.

Now let $x \in N$ be a fixed point and Z be the unit normal vector field to N defined in an open neighborhood $\widetilde{U}_x \subset N$ of x which is differentiably defined by $Z(x) = \lambda_x'(0)$. For every point $x \in N$, assume that there is an open neighborhood $U_x \subset \widetilde{U}_x$ in which every geodesic defined by $t \to \exp_y tZ(y)$, $y \in U_x$ is a ray from N to ∞. By virtue of compactness of N and Lemma 1, we have under the assumption three cases for N and M as follows:

(a) There does not exist any normal vector field to N which is defined globally over N.

(b) N has a unit normal vector field V which is defined globally and the geodesics $t \to \exp_y tV(y)$ and $t \to \exp_y t(-V(y))$ are rays from N to ∞ for some point $x \in N$.

(c) N has a unit normal vector field V which is defined globally and $V(x)$ coincides with $\lambda_x'(0)$ for each $x \in N$ while the geodesic $t \to \exp_x t(-V(x))$ is no more a ray from N to ∞ for every $x \in N$.

Now, for each point $x \in N$ we shall prove the existence of an open neighborhood U_x of N in which the unit normal vector field Z is defined differentiably by $Z(x) = \lambda_x'(0)$ and for each point $y \in U_x$ the geodesic defined by $t \to \exp_y tZ(y)$ is a ray from N to ∞.

THEOREM 2. *For every point $x \in N$ there is an open neighborhood U_x of N in which the unit normal vector field Z is defined by $Z(x) = \lambda'_x(0)$ and for each point $y \in U_x$, the geodesic $t \to \exp_y tZ(y)$ is a ray from N to ∞.*

PROOF. We shall argue by contradiction. Suppose that there is a point $x \in N$ where there is no neighborhood with the property of Theorem 2. Let Z be a normal vector field to N defined in the convex neighborhood U at x such that $Z(x) = \lambda_x'(0)$ and δ be taken in the proof of Lemma 1. By the assumption of x, there is a point $y \in U$ at which the geodesic $t \to \exp_y tZ(y)$ is not a ray. Then there is the unique ray from y to ∞ whose starting direction is $-Z(y)$. Put $\Sigma_t^u \in G(\lambda_y(u), \lambda_x(t))$ where $t > 0$ and $u \in (-\delta, 0)$ are arbitrary taken. For every fixed $u \in (-\delta, 0)$, we have a subsequence $\{\Sigma_{t_i}^u\}$ of Σ_t^u converging to a ray $\Sigma_\infty^u \in G(\lambda_y(u), \infty)$ as $t_i \to \infty$. Then there is a small number $\alpha \in (-\delta, 0)$ which satisfies $\sphericalangle(\lambda_y'(u), \sigma_\infty^{u\prime}(0)) < \pi/2$ for any $u \in (\alpha, 0)$ and any $\Sigma_\infty^u \in G(\lambda_y(u), \infty)$. In fact, if otherwise stated there is a sequence $\{u_j\}$ converging to 0 and a sequence of rays $\{\Sigma_\infty^{u_j}\}$, $\Sigma_\infty^{u_j} \in G(\lambda_y(u_j), \infty)$ such that $\sphericalangle(\lambda_y'(u_j), \sigma_\infty^{u_j\prime}(0)) \geqq \pi/2$. Then we can choose a subsequence of $\{\Sigma_\infty^{u_j}\}$ converging to a ray Σ from y to ∞ which is different from Λ_y. But this is a contradiction.

We may consider α is taken so small that $\lambda_y(\alpha)$ is contained in a convex normal neighborhood centered at x. Next, fix $u_0 \in (\alpha, 0)$ in such a way that the angle $\sphericalangle(\lambda_y(u_0), x, \lambda_x(t))$ is less than $\pi/2$ for $t > 0$. Consider a geodesic triangle with vertices $(\lambda_y(u_0), x, \lambda_x(t))$ for each $t>0$. Then $\sphericalangle(\lambda_y(u_0), x, \lambda_x(t))<\pi/2$ implies the existence of a positive number ε which satisfies $\lim_{t\to\infty} [t-d(\lambda_y(u_0), \lambda_x(t))] = \varepsilon$. Consider another geodesic triangle with vertices $(\lambda_y(u_0), y, \lambda_x(t))$. The sequence of geodesics from y to $\lambda_x(t)$, $t>0$ has the limit ray Λ_y, as $t\to\infty$, and we can choose a subsequence $\{\Sigma_{t_i}^{u_0}\}$ of $\{\Sigma_t^{u_0}\}$ which converges to some ray $\Sigma_\infty^{u_0} \in G(\lambda_y(u_0), \infty)$. Then we have from above discussion $\sphericalangle(\widetilde{\lambda}_y'(u_0), \widetilde{\sigma}_\infty^{u_0\prime}(0))<\pi/2$ for sufficiently large i, where $\widetilde{\Lambda}_y|[u_0, 0]$ and $\widetilde{\Sigma}_{t_i}^{u_0}$ are sides of triangle in R^2 with vertices $(\widetilde{\lambda}_y(u_0), \widetilde{y}, \widetilde{\lambda}_x(t))$ corresponding to $\Lambda_y|[u_0, 0]$ and $\Sigma_{u_0}^{t_i}$ respectively. But this contradicts the basic theorem on triangles. Q. E. D.

Now take a point $x \in N$ and let U_x be stated in Theorem 2. It is not certain whether Λ_y coincides with the geodesic $t\to\exp_y t\cdot Z(y)$ or not, where Z is the unit normal vector field which is differentiably defined in U_x such that $Z(x)=\lambda_x'(0)$. In the following we shall denote the geodesic $t\to\exp_y t\cdot Z(y)$ which is a ray from N to ∞ by Λ_y^* for any point $y \in U_x$.

If N has a unit normal vector field V which is defined globally over N, the case (b) or (c) holds and we may consider $V|U_x$ coincides with the normal vector field Z which is defined in a small neighborhood $U_x \subset N$. We see from Theorem 2 that any geodesic starting from any point of N and normal to N at the starting point is a ray from N to ∞ if N has not a normal vector field defned globally over N.

In any case, we shall prove the following lemma which is essentially due to Lemma 1 of [4].

LEMMA 3. *Let* $\Lambda_y^* \in G(N, \infty)$ *be defined by* $\lambda_y^{*\prime}(0)=Z(y)$ *for any point* $y \in U_x$. *Then for any* $t \geqq 0$ *and any tangent vector* $X \in M_{\lambda^*_y(t)}$ *which is normal to* $\lambda_y^{*\prime}(t)$, *we have* $K(X, \lambda_y^{*\prime}(t))=0$.

PROOF. Suppose that there are $t_0>0$ and $X \in M_{\lambda^*_y(t_0)}$ such that $K(X, \lambda_y^{*\prime}(t_0))>0$ holds. Let $X(t)$ be the parallel vector field along Λ_x^* defined by $X(t_0) = X$.

Putting $K(t)=K(X(t), \lambda_y^{*\prime}(t))$, there is a differentiable function $H(t)\geqq 0$ satisfying $H(t)\leqq K(t)$ for all $t\geqq t_0$ and $H(t_0)<K(t_0)$. Consider the following differential equations:

$$\varphi'' + H\varphi = 0$$

$$\psi'' + K\psi = 0$$

with the initial conditions $\varphi(t_0)=\psi(t_0)=1$ and $\varphi'(t_0)=\psi'(t_0)=0$. There exists $\tau_1>0$ such that $\varphi(t_0+\tau_1)=0$. Let Y be a vector field along $\Lambda_y^*|[0, t_0+\tau_1]$ such that

$$Y(t) = \begin{cases} X(t) & 0\leqq t\leqq t_0 \\ \varphi\cdot X(t) & t_0\leqq t\leqq t_0+\tau_1 \,. \end{cases}$$

Then we have

$$\begin{aligned} I(Y, Y) &= \int_0^{t_0+\tau_1} (<Y', Y'> - K(Y, \lambda_y^{*\prime}(t))<Y, Y>)|_t\, dt \\ &= -\int_0^{t_0} K(X, \lambda_y^{*\prime}(t))\, dt + \int_{t_0}^{t_0+\tau_1} (\varphi'^2 - K\varphi^2)\, dt \\ &< \varphi\varphi'|_{t_0}^{t_0+\tau_1} - \int_{t_0}^{t_0+\tau_1} \varphi(\varphi'' + K\varphi)\, dt \\ &< -\int_{t_0}^{t_0+\tau_1} \varphi\cdot(\varphi'' + H\varphi)\, dt = 0 \,. \end{aligned}$$

This fact contradicts that $\Lambda_y^*|[0, t_0+\tau_1] \in G(N, \lambda_y^*(t_0+\tau_1))$. Q.E.D.

Theorem 2 implies together with Lemma 3 the following

COROLLARY TO THEOREM 2. *Let M be a connected, complete and non-compact Riemannian manifold with non-negative curvature. Let N be a compact totally geodesic hypersurface of M. Suppose that there does not exist any normal vector field which is defined globally over N. Then both $C(N)$ and $F(N)$ are vacuous.*

We see that if the orientation of N is coherent with that of M, the unit normal vector field Z defined in a small neighborhood U_x of N is extendable to one of the unit normal vector field V which is globally defined over N. Consider the map $x \to \lambda_x^{*\prime}(0)$ of N into TM. Then the map is at most two-valued.

PROPOSITION 4. *For every $t > 0$, let N_t^* be the set in M defined in such a way that N_t^* consists of all point $\lambda_x^*(t)$ for all Λ_x^*, $x \in N$. Then following statements hold for every $t > 0$.*

(1) *N_t^* is a compact totally geodesic hypersurface.*

(2) *N_t^* is locally isometric to N.*

PROOF. For any fixed point $x \in N$, let $B_\delta(x)$ be a sufficiently small convex ball in N with center x and radius δ which is contained in U_x. For any point $y \in B_\delta(x)$, there is the unique geodesic $\Gamma \in G(x, y)$ such that $\Gamma = \{\gamma(s)\}\,(0 \leqq s \leqq a)$, $\gamma(0) = x$, $\gamma(a) = y$. The vector field $s \to Z(\gamma(s))$ along Γ is a parallel vector field along Γ and normal to N. Let X_s be the unit parallel vector field along $\Lambda_{\gamma(s)}^*$ defined by $X_s(0) = \gamma'(s)$ for each $s \in [0, a]$. Then $K(X_s(t), \lambda_{\gamma(s)}^{*\prime}(t)) = 0$ holds for all $s \in [0, a]$ and all $t \geqq 0$, which implies that $\Lambda_{\gamma(s)}^*$ has no focal point. Because X_s is a Jacobi field along $\Lambda_{\gamma(s)}^*$ with the initial conditions $X_s(0) \in N_{\gamma(s)}$ and $X_s'(0) = 0$ where $X_s(0)$ can be considered as an arbitrary unit tangent vector to N at $\gamma(s)$. By virtue of Warner's metric comparison theorem [13], we have for any fixed $t > 0$, $d(x, \gamma(s)) = d(\lambda_x^*(t), \lambda_{\gamma(s)}^*(t))$ for any $s \in [0, a]$, and moreover the curve $s \to \lambda_{\gamma(s)}^*(t)$ is a geodesic in M which is also contained in N_t^*. Since x and y are any points in a convex ball, N_t^* is a totally geodesic hypersurface which is locally isometric to N. It is easily shown that N_t^* is complete. Furthermore the map $x \to \lambda_x^*(t)$ of N into M is also a continuous map of N onto N_t^* and at most two-valued, which implies that N_t^* is compact.

Q.E.D.

REMARK. N_t^* coincides with N_t which is defined in Theorem A (or $\widetilde{N}_t$ which is defined in Theorem B) if the case (a) or the case (b) occur. If the case (a) occurs N_t^* will be connected, on the other hand N_t will have two components if the case (b) occurs. In the case (c), N_t might have two components while N_t^* is connected. The last case seems to be more complicated than (a) or (b).

PROPOSITION 5. *Suppose that N has a unit normal vector field V which is defined globally over N, and the case* (b) *occurs. Then M is isometric to $N \times R$.*

PROOF. For any fixed $t > 0$, consider the map $f_t : N \to M$ defined by $f_t(x) = \exp_x t \cdot V(x)$. Then f_t is one-to-one because for any points $x, y \in N$, λ_x^*

and λ_y^* never intersect each other. By the hypothesis that $\Lambda_p|[0,\infty)\in G(N,\infty)$ and $\Lambda_p|(-\infty,0]\in G(N,\infty)$ hold for some point $p\in N$, we see that every geodesic starting from N and whose starting direction is normal to N is a ray from N to ∞ by Theorem 2. Proposition 4 leads f_t is a global isometry of N onto N_t^*. We can define $N_{-t}^*=\{\lambda_x^*(-t)\,|\,x\in N\}=\{\exp_x(-tV(x)(\,|\,x\in N\}$, which is also isometric to N. Of course we have $N_t=N_t^*\cup N_{-t}^*$. Hence we get $M=N\times R$. Q. E. D.

Now we shall prove Theorem A.

PROOF OF THEOREM A. It is evident from Theorem 2 that every geodesic starting from a point of N and normal to N is a ray from N to ∞. We see that the map of N into TM, $x\to\lambda_x^{*\prime}(0)$ is differentiable and two-valued. Hence N_t^* is a connected and compact totally geodesic hypersurface for each $t>0$. Let $\pi: N_t^*\to N$ be defined by $\pi(\lambda_x^*(\pm t)=x$. We see that π is a local isometry of N_t^* onto N and for any $x\in N$, there is an open neighborhood $W_x\subset N$ such that $\pi^{-1}(W_x)$ consists of two disjoint neighborhoods of $\exp_x t\cdot V(x)$ and $\exp_x(-tV(x))$ each of which is isometric to W_x. Therefore π is the covering map and N_t^* is the double covering of N for each $t>0$. It is easy to see that $N_{t_1}^*$ is globally isometric to $N_{t_2}^*$ for any $t_1,t_2>0$. Hence $M-N$ is isometric to $N_t\times(0,\infty)$. We easily see that N is a soul of M. Q. E. D.

3. The structure of M with $C(N)\neq\emptyset$ and $F(N)=\emptyset$. Throughout this section let M satisfy the assumption of Theorem B. If there is a point $x\in N$ at which both $\Lambda_x^*|(-\infty,0)$ and $\Lambda_x^*|[0,\infty)$ are rays from N to ∞, then every geodesic starting from any point of N and normal to it becomes a ray from N to ∞ by Theorem 2. Therefore $\Lambda_x^*|(-\infty,0]$ has a cut point to N along it for each point $x\in N$. And moreover for any point $p\in M$, a ray from p to ∞ is contained in some Λ_x^* or coincides with its extension because the set $\{\lambda_x^*(t)\,|\,x\in N,\ t>0\}$ is of the form $N\times(0,\infty)$. Let V be defined in §2 in such a way that $\lambda_x^{*\prime}(0)=V(x)$ and V is defined globally over N. Then M is decomposed into two components $\{\lambda_x^*(t)\,|\,x\in N,\ t\geqq 0\}$ and $M-\{\lambda_x^*(t)\,|\,x\in N,$ $t\geqq 0\}$ because $\{\lambda_x^*(t)\,|\,x\in N,\ t\geqq 0\}$ forms $N\times[0,\infty)$ with boundary N. Note that $M-\{\lambda_x^*(t)\,|\,x\in N,\ t\geqq 0\}$ is bounded.

Now we shall state an intersting theorem investigated by Omori which plays an important role for a proof of Theorem B.

THEOREM. (3.4 Proposition in [9]) *Let M be a connected and compact Riemannian manifold of class C^∞ and N be a connected, compact and differentiable Riemannian submanifold of M. Suppose that there is a point*

$p \in C(N)$ at which $d(p, N) = d(C(N), N)$ holds and there exist two different geodesics $\Gamma_1, \Gamma_2 \in G(p, N)$ satisfying $\gamma_1'(0) \neq \pm \gamma_2'(0)$. Then, putting $l = d(C(N), N)$, we have $\exp_p lv \in N$ for any $a \geqq 0$, $b \geqq 0$ and the tangent vector $v \in M_p$ defined by $v = \dfrac{a\gamma_1'(0) + b\gamma_2'(0)}{\|a\gamma_1'(0) + b\gamma_2'(0)\|}$. Hence p must be a focal point of N.

COROLLARY TO THE THEOREM. *Let M be a connected, complete and differentiable Riemannian manifold and N be a connected, compact and differentiable Riemannian submanifold. Suppose that there is a point $p \in C(N)$ at which $d(p, N) = d(C(N), N)$ holds and there is $\Gamma \in G(N, p)$, defined by $\Gamma = \{\gamma(t)\}$ $(0 \leqq t \leqq l)$, $\gamma(l) = p$, $\gamma(0) \in N$ along which p is not a focal point of N. Then there is $\Gamma_1 \in G(N, P)$ satisfying $\measuredangle(\gamma'(l), \gamma_1'(l)) = \pi$.*

A proof of Corollary follows immediately from the fact that there exists a $\Gamma_1 \in G(N, p)$ which is different from Γ by $p \notin F(N)$. Of course these Theorem and Corollary hold for the cut locus and conjugate locus of a point (in case $\dim N = 0$).

We shall prepare a few lemmas for a proof of Theorem B. Let X be the normal vector to N at $x \in N$ such that $X \in C_N$, $X \notin F_N$ and $l = \|X\| = d(N, C(N))$. Let Γ_x be the geodesic defined by $\Gamma_x = \{\gamma_x(t)\}$ $(0 \leqq t \leqq l)$, $\gamma_x(t) = \exp_x tX/\|X\|$. Theorem of Omori and our assumption imply that $\gamma_x(2l)$ is a point of N and $\gamma_x'(2l)$ is normal to N at $\gamma_x(2l)$. Hence we see both $\Gamma_x|[l, \infty)$ and $\Gamma_x|(-\infty, l]$ are rays from $\gamma_x(l)$ to ∞. By the hypothesis of Theorem B, there is a unit normal vector field V defined globally over N satisfying $-V(x) = X/\|X\|$ and $V(y) = \lambda_y^*(0)$ for any $y \in N$ where $\Lambda_y^* \in G(N, \infty)$. Let us denote the geodesic through a point $y \in N$ with tangent vector $-V(y)$ at y by $\Gamma_y = \{\gamma_y(t)\}$ $(0 \leqq t \leqq l)$. We note that the inverse extension $\Gamma_y|(-\infty, 0]$ of Γ_y coincides with Λ_y^* which is a ray from N to ∞. Recall that every ray in M is either containd entirely in some Λ_y^* or an extension of it.

LEMMA 6. *Let Γ_x be defined by $\Gamma_x = \{\gamma_x(t)\}$ $(0 \leqq t \leqq l)$, $\gamma_x(t) = \exp_x tX/\|X\|$, For any point $y \in N$ and any geodesic $\Phi \in G(\gamma_x(l), \gamma_y(l))$, $\Phi = \{\varphi(s)\}$ $(0 \leqq s \leqq a)$, we have $\langle\varphi'(0), \gamma_x'(l)\rangle = 0$.*

PROOF. First of all we shall prove that the inverse geodesic $\Gamma_y|(-\infty, l]$ is a ray from $\gamma_y(l)$ to ∞. In fact, let $\Theta = \{\theta(t)\}$ $(0 \leqq t \leqq \infty)$ be a ray such that $\Theta \in G(\gamma_y(l), \infty)$. Θ must intersect N at some point $\theta(l') \in N$ with right angle. Because $l = d(N, C(N))$, we must have $l' \geqq l$ from which the statement above is shown, Without loss of generality, we can assume that $\sphericalangle(\gamma_x'(l), \varphi'(0)) \leqq \pi/2$. For a geodesic triangle with vertices $(\gamma_x(l+t), \gamma_x(l), \gamma_y(l))$, let us denote the corresponding triangle in R^2 with same side length by $(\tilde{\gamma}_x(l+t), \tilde{\gamma}_x(l), \tilde{\gamma}_y(l))$. The basic theorem on triangles implies that $\lim_{t\to\infty} \sphericalangle(\tilde{\gamma}_x(l+t), \tilde{\gamma}_x(l), \tilde{\gamma}_y(l)) = \lim_{t\to\infty} \sphericalangle(\tilde{\gamma}_x(l+t), \tilde{\gamma}_y(l), \tilde{\gamma}_x(l)) = \pi/2$ because of $\lim_{t\to\infty} [d(\tilde{\gamma}_x(l+t), \tilde{\gamma}_x(l)) - d(\tilde{\gamma}_x(l+t), \tilde{\gamma}_y(l))] = 0$. Therefore the proof is completed from the inequality $\sphericalangle(\gamma_x'(l), \varphi'(0)) \geqq \lim_{t\to\infty} \sphericalangle(\tilde{\gamma}_x(l+t), \tilde{\gamma}_x(l), \tilde{\gamma}_y(l)) = \pi/2$. Q. E. D.

LEMMA 7. *For any fixed point $\varphi(s)$ on $\Phi = \{\varphi(s)\}$ $(0 \leqq s \leqq a)$, we have $d(\varphi(s), N) = l$.*

PROOF. For any fixed point $\varphi(s)$, take any $\Psi \in G(\varphi(s), N)$, $\Psi = \{\psi(t)\}$ $(0 \leqq t \leqq l'')$. Then $\Psi | [0, \infty)$ becomes a ray from $\varphi(s)$ to ∞. As a first step, suppose that $l'' > l$. For a geodesic triangle with vertices $(\gamma_x(2l+t), \varphi(s), \psi(l''+t))$, it follows from $\lim_{t\to\infty} \sphericalangle(\varphi(s), \gamma_x(2l+t), \psi(l''+t)) = \lim_{t\to\infty} \sphericalangle(\varphi(s), \psi(l''+t), \gamma_x(2l+t)) = \pi/2$ for $t > 0$ that $\lim_{t\to\infty} [d(\varphi(s), \gamma_x(2l+t)) - (l''+t)] = 0$. Then we must have $\sphericalangle(\tilde{\gamma}_x(2l+t), \tilde{\gamma}_x(l), \tilde{\varphi}(s)) > \pi/2$ for sufficiently large $t > 0$, which contradicts the basic theorem on triangles. Next, suppose that $l'' < l$. We may assume that $\sphericalangle(\varphi'(s), \psi'(0)) \leqq \pi/2$ without loss of generality. An analogous argument for a geodesic triangle $(\psi(l''+t), \gamma_y(l), \varphi(s))$ leads us to a contradiction. Q. E. D.

We note that for any $s \in [0, a)$ and any $\Psi \in G(\varphi(s), N)$ stated in the proof of Lemma 7, we have $\sphericalangle(\varphi'(s), \psi'(0)) = \pi/2$. Moreover, we have $K(X(t), \psi'(t) = 0$ for all $t \geqq 0$ where X is the unit parallel vector fied along Ψ defined by $X(0) = \varphi'(s)$. This fact follows from Lemma 3.

LEMMA 8. *There exists a totally geodesic hypersurface of M which is defined locally as a small piece containing $\gamma_x(l)$.*

PROOF. Since $X \notin F_N$, the map $(\pi, \exp): TM \to M \times M$ has maximal rank in a neighborhood $\widetilde{W} \subset TM$ of X, where TM is the tangent bundle of M and π is the projection map of TM onto M. We may consider that $(\pi, \exp) | \widetilde{W}$ is a diffeomorphism of W onto $\pi(W) \times \exp(W)$. There is a small neighborhood $W \subset \widetilde{W}$ of X defined by $W = \{Z \in TM, \pi(Z) \in N, \|Z\| = l$ and Z is normal to N at $\pi(Z)\}$. Then, it is clear that W is an $(n-1)$-dimensional submanifold of TM. Hence, $\exp(W)$ becomes a hypersurface in M. By virtue of Lemmas 6 and 7, the hypersurface $\exp(W)$ is contained in the hypersurface S defined by $S = \{\exp_{\gamma_x(l)} sv \mid \|v\| = 1,\ v \in M_{\gamma_x(l)},\ <v, \gamma_x'(l)> = 0,\ -\delta < s < \delta\}$, where δ is the convex radius at $\gamma_x(l)$. Making use of Lemma 3 for the surface S and the geodesic $\Gamma_x|[l,\infty)$, we see that $K(\gamma_x'(l+t), Z)=0$ for any $t\geqq 0$ and any $Z\in M_{\gamma_x(l+t)}$.

On the other hand, let $X_1\in M_{x_1}$ be the normal vector to N at the point $x_1=\gamma_x(2l)\in N$ satisfying $\|X_1\|=l$ and $\gamma_x'(2l)=-X_1/\|X_1\|$. There is a small neighborhood $W_1\subset TM$ of X_1 which is defined in the same way as W for X. Then the hypersurface $\exp(W_1)$ is also contained in S.

Consider the connected component $S^*\subset S$ of $\exp(W)\cap\exp(W_1)$ containing $\gamma_x(l)$ and let $W^*\subset W$, $W_1{}^*\subset W_1$ be defined such that $\exp(W^*)=\exp(W_1{}^*)=S^*$. Then we find that for any point y in the neighborhood $\pi(W^*)\subset N$ of x, there is a point y_1 in the neighborhood $\pi(W_1{}^*)\subset N$ of x_1 satisfying $\gamma_y(l)=\gamma_{y_1}(l)$ and both $\gamma_y'(l)$ and $\gamma_{y_1}'(l)$ are normal to the hypersurface S^* at $\gamma_y(l)$. Though it might occur that $-lV(y_1)\in F_N$ for some $y_1\in\pi(W_1^*)$, the geodesic $\Gamma_y|(-\infty,\infty)$ is able to take place for $\Gamma_x|(-\infty,\infty)$ in both Lemmas 6 and 7. Hence for any two points $y,z\in\pi(W^*)$ and $\Psi\in G(\gamma_y(l),\gamma_z(l))$ we get $\Psi\subset S^*$ as a set. This fact shows that S^* is a piece of totally geodesic hypersurface. Q. E. D.

REMARK. For any tangent vector $X\in M_{\gamma_y(l)}$, $\langle X,\gamma_y'(l)\rangle=0$, and for the parallel vector field X along $\Gamma_y|(-\infty,\infty)$ satisfying $X(l)=X$ we have $K(X(t),\gamma_y'(t))=0$ for all $t\in(-\infty,\infty)$. Hence we get $-l\cdot V(y)\notin F_N$ and $-l\cdot V(y)\in C_N$ by the argument stated above. Therefore we also have $-l\cdot V(y_1)\notin F_N$.

Lemma 8 has stated that there exists an open neighborhood $\pi(W^*)$ in which for any point y, $\exp_y(-2lV(y))\in N$ holds. We also see by Theorem of Omori that for any point $\gamma_z(l)\in S^*$, we have just two rays from $\gamma_z(l)$ to ∞ which are defined by $\Gamma_z|[l,\infty)$ and $\Gamma_z|(-\infty,l]$. A theorem investigated by Sugimoto (stated in §4) implies that for any points $\gamma_y(l)$, $\gamma_z(l)$ in S^* there is a piece of two-dimensinal totally geodesic submanifold of M with constant curvature zero and boundaries $\Gamma_y|(-\infty,\infty)$ and $\Gamma_z|(-\infty,\infty)$.

We shall prove that the set $\pi(W^*)$ is closed in N. Remark stated above will play an important role for the proof.

LEMMA 9. *For any sequence of points $\{y_k\}$ satisfying $y_k\in\pi(W^*)$ and $\lim_{k\to\infty}y_k=y_0\in N$, let $\Gamma_k=\{\gamma_k(t)\}$ $(0\leqq t\leqq l)$ be the geodesic defined by $\gamma_k(t)=\exp_{y_k}(-tV(y_k))$, $(k=1,2,\cdots,)$. Then we have the following statements:*

(1) *$\gamma_0(2l)\in N$ and $\gamma_0'(2l)$ is normal to N at $\gamma_0(2l)$.*

(2) *For any $t\in(-\infty,\infty)$ and tangent vector $X\in M_{\gamma_0(t)}$ orthogonal to $\gamma_0'(t)$, we have $K(X,\gamma_0'(t))=0$ and hence $-l\cdot V(y_0)\notin F_N$ and $-l\cdot V(y_0)\in C_N$ hold.*

(3) *For each point $z\in N$, we have $\exp_z(-2lV(z))\in N$ and $-lV(z)\notin F_N$.*

PROOF. Since $\lim_{k\to\infty} \gamma'_k(0) = \gamma'_0(0)$ and for every $k = 1, 2, \cdots$, we have $\gamma_k(2l) \in N$ and $\gamma'_k(2l)$ is normal to N at $\gamma_k(2l)$, the first statement is evident. Suppose that there is $\bar{t} \in (-\infty, \infty)$ and $X \in M_{\gamma_0(\bar{t})}$ such that $<X, \gamma'_0(\bar{t})> = 0$ and $K(X, \gamma'_0(\bar{t})) > 0$. Let X_0 be the parallel vector field along $\Gamma_0 | (-\infty, \infty)$ defined by $X_0(\bar{t}) = X$. There is a large number k_0 such that for any $k > k_0$, there exists a unique $\Psi_k \in G(\gamma_0(0), \gamma_k(0))$. Translating $X(0)$ parallely along Ψ_k, we get $X_k \in M_{\gamma_k(0)}$. Let $X_k(t)$ be the parallel vector field along Γ_k defined by $X_k(0) = X_k$. Then it is clear that for each $t \in (-\infty, \infty)$, $\lim_{k\to\infty} X_k(t) = X_0(t)$ and $\lim_{t\to\infty} \gamma'_k(t) = \gamma'_0(t)$. Therefore we must have $K(X_k(\bar{t}), \gamma'_k(\bar{t})) > 0$ for sufficiently large k, from which we lead a contradiction. This fact implies $-l \cdot V(y_0) \notin F_N$ and $-l \cdot V(y_0) \in C_N$. Then Γ_{y_0} takes place for Γ_x in Lemma 8 which implies that there is a neighborhood $W_0^* \subset TM$ of $-lV(y_0)$, where $\exp(W_0^*)$ becomes a piece of totally geodesic hypersurface as is stated in Lemma 8 and it is contained in $C(N)$. By compactness of N, $C(N)$ is covered by finitely many open neighborhoods $\pi(W^*)$ defined as stated above. Hence the last statement is evident. Q. E. D.

PROOF OF THEOREM B. It suffices to show that the set $\tilde{N}$ defined by $\tilde{N} = \{\gamma_y(l) | y \in N\}$ becomes a compact totally geodesic hypersurface of M. We have found that for any point $y \in N$, Γ_y has the properties $\gamma_y(2l) \in N$ and $-lV(y) \in C_N - F_N$. Then for any point $\gamma_y(l) \in \tilde{N}$, there exists a piece of totally geodesic hypersurface $S^*_{\gamma_y(l)}$ which is contained entirely in $\tilde{N}$ and $C(N)$. This fact implies that the set $\tilde{N}$ is a hypersurface of M. Because every point $\gamma_y(l) \in \tilde{N}$ has an open neighborhood $S^*_{\gamma_y(l)} \subset \tilde{N}$ which is isometric to some open neighborhood of $y \in N$ in N, every geodesic in $\tilde{N}$ is able to extend infinitely in $\tilde{N}$. Therefore $\tilde{N}$ is complete and clearly compact. We also see that every geodesic starting from a point of $\tilde{N}$ and normal to it at the starting point is a ray from $\tilde{N}$ to ∞. Then $\tilde{N}$ satisfies the hypothesis of Theorem A. So the proof is completed by Theorem A. Q. E. D.

4. The structure of *M* with certain focal locus. Throughout this section let M satisfy the assumption of Theorem C. The assumption $F(N) \neq \emptyset$ implies that N has a unit normal vector field V which is defined globally over N. In fact, a contrapositive of Corollary to Theorem 2 implies the statement. Then there exists a family of rays $\{\Lambda_x^* | x \in N\}$ from N to ∞ stated in §3 in such a way that $\lambda_x^{*\prime}(0) = V(x)$ for any $x \in N$. The set defined by $\{\lambda_x^*(t) | x \in N, t \geqq 0\} \subset M$ forms $N \times [0, \infty)$ which is unbounded. On the other hand the set $M - \{\lambda_x^*(t) | x \in N, t \geqq 0\}$ is bounded and has boundary N, the form of which we shall study in the following.

LEMMA 10. *For any tangent vector X satisfying $X \in C_N$, we have $X \in F_N$. And $G(\exp X, N)$ contains at least two geodesics.*

PROOF. Suppose that there is $X \in C_N$ such that $X \notin F_N$. Then the hypothesis of Theorem B is satisfied from which we get $F(N) = \emptyset$. But this contradicts the assumption of Theorem C. Putting $x = \pi(X)$, $p = \exp_x X$ and $\Gamma_x = \{\gamma_x(t)\}$ $(0 \leqq t \leqq l)$ such that $\gamma_x'(0) = X/\|X\|$, where $l = d(N, C(N))$, we have a geodesic $\Gamma_y \in G(N, p)$ which satisfies $\sphericalangle(\gamma_x'(l), -\gamma_y'(l)) \leqq \pi/2$. In fact, there are sequence $\{t_i\}$ such that $\lim_{i\to\infty} t_i = 0$ and sequence of geodesics $\{\Psi_i\}$ such that $\Psi_i \in G(\gamma_x(l+t_i), N)$ satisfying $\sphericalangle(\gamma_x'(l+t_i), \psi_i'(0)) \leqq \pi/2$. For otherwise stated, there exists a point $\gamma_x(l+t_0)$ on Γ_x satisfying $d(\gamma_x(l+t_0), N) > l$ by virtue of the first variation formula. But this is a contradiction. By choosing a subsequence of $\{\Psi_i\}$ converging to geodesic $\Psi_0 \in G(p, N)$, we have $\sphericalangle(\psi_0'(0), \gamma_x'(l)) \leqq \pi/2$. Therefore the proof is completed. Q.E.D.

Lemma 10 implies that $C(N)$ coincides with $F(N)$ as a set without the assumption that the multiplicity of each focal point of N is constant k.

In the following let $p \in C(N)$ be a fixed point and $A_{p,N}$ be the set of all unit tangent vectors at p such that for any $v \in A_{p,N}$ we have $\exp_x lv \in N$. Lemma 10 shows that $A_{p,N}$ contains at least two elements. Suppose that for every point $q \in C(N)$, $A_{q,N}$ consists of just two elements. Then we must have $A_{q,N} = \{w, -w\}$ by use of Omori's Theorem, where $w \in M_q$, $\|w\| = 1$. Because we have a piece of totally geodesic surface of constant curvature zero with boundaries Γ_x and Γ_y (Remark after Lemma 8), we have again $F(N) = \emptyset$ from the argument developed in §3. This contradicts our assumption. Hence there is a point $p \in C(N)$ at which $A_{p,N}$ contains at least three distinct vectors.

Making use of the theorem investigated by Omori (see §3), we see that for any point $q \in C(N)$, $A_{q,N}$ has the following properties:

(1) For $u, v \in A_{q,N}$ and any non-negative numbers a and b, we have $(au+bv)/\|au+bv\| \in A_{q,N}$.

(2) For any tangent vector $Z \in M_q$, there exists $v \in A_{q,N}$ satisfying $\langle Z, v\rangle \geqq 0$.

(3) $A_{q,N}$ is closed in M_q.

From these properties for $A_{q,N}$ we find, after developing the same argument as [1].

(4) There exists $v_0 \in A_{q,N}$ which satisfies $-v_0 \in A_{q,N}$.

As a first step of a proof of Theorem C, we shall prove that $A_{p,N}$ is a k-dimensional unit sphere in M_p centered at origin. Taking account of the properties (1) and (4) of $A_{p,N}$, it will suffice to prove that for any $v \in A_{p,N}$

we have $-v \in A_{p,N}$.

For the purpose stated above we shall prepare the following theorem and lemmas.

THEOREM. (Sugimoto [10]) *Let M be a connected and complete Riemannian manifold of class C^∞ whose curvature is bounded below by a constant δ. Let $(\Gamma, \Lambda, \Sigma)$ and $(\widetilde{\Gamma}, \widetilde{\Lambda}, \widetilde{\Sigma})$ be geodesic triangles in M and the plane of constant curvature δ respectively which satisfy $\mathcal{L}(\Gamma) = \mathcal{L}(\widetilde{\Gamma})$, $\mathcal{L}(\Lambda) = \mathcal{L}(\widetilde{\Lambda})$ and $\mathcal{L}(\Sigma) = \mathcal{L}(\widetilde{\Sigma})$.*

Suppose that the angle between Γ and Λ is equal to the angle between $\widetilde{\Gamma}$ and $\widetilde{\Lambda}$. Then there exists a piece of surface in M with boundaries Γ, Λ and Σ_1 which is a two dimensional totally geodesic submanifold of M with coastant curvature δ, where Σ_1 is a shortest geodesic segment with same extremals as Σ.

LEMMA 11. *Assume that there exists $-v_1 \in A_{p,N}$ which satisfies $v_1 \notin A_{p,N}$. Then for any $s \in (0, c)$, we have $d(\exp_p sv_1, N) < l$, where c is a positive number such that $\exp_p cv_1$ is the cut point to p along the geodesic $s \to \exp_p sv_1$.*

PROOF. Suppose that there is $s_0 \in (0, c)$ satisfying $d(\exp_p s_0 v_1, N) \geqq l$. The hypothesis of Theorem C for $C(N)$ implies that $d(\exp_p s_0 v_1, N) = l$. Let $\Gamma_0 = \{\gamma_0(t)\}$ $(0 \leqq t \leqq \infty)$ be defined by $\gamma_0(t) = \exp_p tv_0$, $\Lambda = \{\lambda(s)\}$ $(0 \leqq s \leqq s_0)$ be defined $\lambda(s) = \exp_p sv_1$ and $\Sigma_t \in G(\lambda(s_0), \gamma_0(t))$. Then for each $t > 0$, we must have $\measuredangle(\tilde{\gamma}_0'(0), \tilde{\lambda}'(0)) = \pi/2$ by the convexity theorem of Toponogov [11], where $(\widetilde{\Gamma}_0 | [0, t], \widetilde{\Lambda}, \widetilde{\Sigma}_t)$ is the triangle in R^2 defined by $\mathcal{L}(\widetilde{\Gamma}_0 | [0, t]) = \mathcal{L}(\Gamma_0 | [0, t]) = t$, $\mathcal{L}(\widetilde{\Lambda}) = \mathcal{L}(\Lambda)$ and $\mathcal{L}(\widetilde{\Sigma}_t) = \mathcal{L}(\Sigma_t)$. Therefore we have a piece of surface in M with boundaries Γ_0, Λ and Σ which is a totally geodesic surface of constant curvature 0, where Σ is a ray from $\lambda(s_0)$ to ∞ obtained by a converging subsequence of $\{\Sigma_t\}$, $t > 0$. Consider the Jacobi field Y along Γ_0 defined by $Y(0) = 0$, $Y'(0) = v_1$. Let V_1 be the unit parallel vector field along Γ_0 defined by $V_1(0) = v_1$. Then we have $Y(t) = t \cdot V_1(t)$ because of $K(\gamma_0'(t), V_1(t)) = 0$ for all $t \geqq 0$. Hence $Y'(t) \neq 0$ holds for all $t \geqq 0$.

On the other hand, by virtue of property (1) of $A_{p,N}$, $\{\exp_p t(v_0 \cos \alpha - v_1 \sin \alpha) | t \geqq 0,\ 0 < \alpha < \pi\}$ becomes a surface in M with boundary $\Gamma_0 | (-\infty, \infty)$. For any $\alpha \in (0, \pi)$, let $\Gamma_\alpha = \{\gamma_\alpha(t)\}$ $(0 \leqq t \leqq \infty)$ be defined by $\gamma_\alpha(t) = \exp_p t(v_0 \cos \alpha - v_1 \sin \alpha)$ and Y_α be the Jacobi field along Γ_α defined by $Y_\alpha(0) = 0$, $Y_\alpha'(0) = -(v_0 \sin \alpha + v_1 \cos \alpha)$. Then we have $Y_\alpha'(t) = 0$ for any $t \geqq l$ and any $\alpha \in (0, \pi)$. Hence $\lim_{\alpha \to 0} Y_\alpha'(l) = 0$ follows, but we have $\lim_{\alpha \to 0} Y_\alpha = -Y$, from which we derive a contradiction. Q. E. D.

Lemma 11 implies that for any $s \in (0, c)$, there is a unique geodesic $\Phi_s \in G(\lambda(s), N)$ whose extension is defined by $\Phi_s = \{\varphi_s(t)\}$ $(0 \leqq t < \infty)$, $\varphi_s(\beta_s) = \lambda(s)$ and $\varphi_s(l) \in N$. There is a small positive number s_1 such that for any $s \in (0, s_1)$, there exists unique geodesic segment $\Psi_s \in G(p, \varphi_s(0))$. We get a piece of totally geodesic surface of constant curvature 0 with boundaries Γ_0, Ψ_s and Φ_s for all $s \in (0, s_1)$. Because this surface is flat and contains $\Lambda | [0, s]$, the sum of all angles of the geodesic triangle $(\Lambda | [0, s], \Psi_s, \Phi_s | [0, \beta_s])$ is just equal to π. But since the angle between Ψ_s and Φ_s at $\varphi_s(0)$ is just eqal to $\pi/2$, we get together with Lemma 11 $\sphericalangle(\varphi_s'(\beta_s), \lambda'(s)) < \pi/2$ for all $s \in (0, s_1)$.

LEMMA 12. *Assume that there exists* $-v_1 \in A_{p,N}$ *which satisfies* $v_1 \notin A_{p,N}$. *Then there exists* $\Phi_0 \in G(p, N)$ *with the properties* $\Phi_0 \neq \Lambda$ *and* $\sphericalangle(\varphi_0'(0), \lambda'(0)) < \pi/2$.

PROOF. Because $v \notin A_{p,N}$, the point $\varphi_s(0)$ is different from p for all $s \in (0, s_1)$. From the discussion stated in the proof of Lemma 11, there is a Jacobi field Y_s along Γ_0 defined by $Y_s(0)=0$ and $Y_s'(0)=\psi_s'(0)$ which is expressed by $Y_s(t)=t \cdot V_s(t)$ where V_s is the unit parallel vector field along Γ_0 defined by $V_s(0) = \psi_s'(0)$. We note again that $K(V_s(t), \gamma_0'(t)) = 0$ holds for all $t \geqq 0$ and $s \in (0, s_1)$. Suppose that there is a subsequence $\{\psi_{s_i}'(0)\}$ of the sequence $\{\psi_s'(0)\}$ which satisfies $\lim_{i\to\infty} \psi_{s_i}'(0) = \lambda'(0)$. Then we get a Jacobi field Y along Γ_0 defined by $Y(t)=t \cdot V_1(t)$ where V_1 is the unit parallel vector field along Γ_0 such that $V_1(0)=v_1$. The discussion in the proof of Lemma 11 derives a contradiction. Suppose that there is a subsequence $\{\psi_{s_j}'(0)\}$ of $\{\psi_s'(0)\}$ satisfying $\lim_{j\to\infty} \sphericalangle(\lambda'(0), \psi_{s_j}'((0)$ $=\pi/2$. Then the sequence $\{\Phi_{s_j}\}$ converges to Λ because $\lim_{j\to\infty} \sphericalangle(\varphi_{s_j}'(\beta_{s_j}), \lambda'(s_j))=0$ which contradicts our assumption that $v_1 \notin A_{p,N}$. Therefore we have $\lim_{i\to\infty} \sphericalangle(\lambda'(0), \psi_{s_i}'(0)) \in (0, \pi/2)$ for every converging subsequence $\{\psi_{s_i}'(0)\}$. Then we have $\lim_{i\to\infty} \sphericalangle(\varphi_{s_i}'(\beta_{s_i}), \lambda'(s_i)) = \pi/2 - \lim_{i\to\infty} \sphericalangle(\lambda'(0), \psi_{s_i}'(0)) \in (0, \pi/2)$. By choosing a subsequence of $\{\Phi_{s_i}\}$ converging to some $\Phi_0 \in G(p, N)$ the proof is completed. Q.E.D

PROOF OF THEOREM C. Let B_1 be defined by $B_1 = \{u \in A_{p,N} \mid \langle v_0, u \rangle = 0\}$. Then B_1 has the properties (1), (3) and moreover for any $u \in B_1$, there exists $u_1 \in B_1$ satisfying $\langle -u, u_1 \rangle > 0$ by Lemma 12. Hence there exists $v_1 \in B_1$ such that $-v_1 \in B_1$. Let B_2 be the set $B_2 = \{u \in B_1 \mid \langle v_1, u \rangle = 0\}$. It is clear that B_2 has the same properties as B_1, and hence we get that $A_{p,N}$ is a k-dimensional unit sphere in M_p by induction. We also see that there is a small neighborhood of p in which any point q of $C(N)$ has the property that $A_{q,N}$ is a k-dimensional unit sphere in M_q and moreover the intersection of $C(N)$ and the neighborhood becomes a totally geodesic submanifold of dimension $n-k-1$. Hence we find

that $C(N)$ is a compact totally geodesic submanifold of dimension $n-k-1$. Let T_q be the tangent space of $C(N)$ at $q \in C(N)$ which is a subspace of M_q. Let $T_q{}^{\perp}$ be the orthogonal complement of T_q in M_q. Then $\exp_q | T_q{}^{\perp}$ is a global diffeomorphism of $T_q{}^{\perp}$ onto the image $\exp_q(T_q{}^{\perp}) \subset M$. Putting $F_q = \exp_q(T_q{}^{\perp})$, we see that F_q is a $(k+1)$-dimensional Riemannian submanifold of M which is diffeomorphic to R^{k+1}. We may call F_q is the normal space to $C(N)$ at $q \in C(N)$.
Q. E. D.

REMARK. Because every geodesic Γ_q starting from $q \in C(N)$ and $\gamma_q'(0) \in T_q{}^{\perp}$ becomes a ray from $C(N)$ to ∞, we see that $F_q \cap F_{q_1} = \emptyset$ for every q, $q_1 \in C(N)$, $q_1 \neq q$.

We also see that the manifold M is the total space of a fibre bundle $(M, C(N), F)$.

5. Applications. Let M be a connected, complete and non-compact Riemannian manifold of class C^∞ whose sectional curvature is everywhere non-negative. Let N be a compact totally geodesic hypersurface of M. Making use of Theorem A, we obtain

PROPOSITION 13. *Suppose that N is an even dimensional real projective space of constant curvature* 1 *and M is orientable. Then, M must be isometric to $S_1^{n-1} \times R^1/f$, where $f: S_1^{n-1} \times R^1 \to S_1^{n-1} \times R^1$ is defined by $f(x, t) = (-x, -t)$, $x \in R_1^{n-1}$, $t \in R^1$. And if N is an odd dimensional real projective space and M is orientable, M is isometric to $PR^{n-1} \times R^1$ or otherwise N has a focal point.*

Making use of Theorem 2 and Theorem B, we obtain

PROPOSITION 14. *Suppose that there are two distinct compact totally geodesic submanifolds N and N^* in M. Then we have the following statements*:

(1) *If* $\dim N = \dim N^* = \dim M - 1$, *$N$ is either isometric to N^* or one of them is the double covering of the other. If N is the double covering of N^*, M is isometric to $N \times R^1/f$ where f is the isometric involution on Nd efined by the covering projection $\pi: N \to N^*$ such that $f(x_1, t) = (x_2, -t)$, $\pi(x_1) = \pi(x_2) \in N^*$, $t \in R^1$.*
(2) *If $F(N) = \emptyset$ and* $\dim N^* < \dim N = \dim M - 1$, *$N^*$ is contained in $C(N)$, or otherwise there exists an isometric immersion $\iota: N^* \to N$.*

If M is a locally symmetric space, some results are obtained in [8]. If a complete locally symmetric space M admits a compact totally geodesic hypersurface N, then the natural homomorphism of fundamental groups $\pi_1(N) \to \pi_1(M)$ is surjective or otherwise, the isometric structure of M is determined [8].

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