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RELATIVE HOMOTOPY OPERATIONS

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Introduction. In [1], J.W.Rutter defines a relative cohomology operation to be a natural transformation between two cohomology functors, each of which prescribes to a given pair of spaces (X, A) a product of assorted cohomology groups of the spaces A and X, and of the pair (X, A). A result [1; Theorem 2] of that paper states that all such cohomology operations are generated by certain elementary ones. Below we define the dual notion of relative homotopy operation and show that these are also generated by elementary operations. These include the mixed Whitehead products defined in [2; §6].

1. Definitions. Given three sequences $\{P_i\}$, $\{Q_i\}$ and $\{R_i\}$ of compact pointed spaces having the base point preserving homotopy type of a pointed CW-complex, we may define a functor Γ, from the category of base point preserving maps and commutative squares to the category of pointed sets, by

$$\Gamma(f\colon A\to X) = \Pi_i[SP_i, X] \times \Pi_j[SQ_j, P_f] \times \Pi_k[SR_k, A],$$

where P_f denotes the induced fibre space of f. We call such a functor a *relative homotopy functor* and define a *relative homotopy operation* to be a natural transformation $T\colon \Gamma\to\Theta$ between relative homotopy functors.

Corresponding to [1; §5], we define below the elementary operations.

(a) *Natural set functions.* The diagonal, addition, identity and zero functions.

(b) *One-variable operations.* The induced functions
(i) $\omega^*: [SQ, X]\longrightarrow[SP, X]$
(ii) $\omega^*: [SQ, P_f]\longrightarrow[SP, P_f]$
(iii) $\omega^*: [SQ, A]\longrightarrow[SP, A]$
where $\omega: SP\to SQ$ is a map.

(c) *Homotopy sequence homomorphisms.*

$$[SP, X]\xleftarrow{f_*}[SP, A]\xleftarrow{p_*}[SP, P_f]\xleftarrow{\partial}[S^2P, X].$$

(d) *Mixed Whitehead products.*

$$[\ ,\ ]_M : [SP, P_f] \times [SQ, A] \longrightarrow [S(P \wedge Q), P_f]\,.$$

2. Composites and cartesian products of relative homotopy operations are defined in the obvious way. We now state our main result.

THEOREM 2.1. *All relative homotopy operations are generated under composition and cartesian product by elementary homotopy operations.*

Following Rutter, we consider homotopy operations T_1, T_2 and T_3 of types (1), (2) and (3) respectively:

$$T_1 : \Gamma(f : A \to X) \longrightarrow [SK, X]\,, \tag{1}$$

$$T_2 : \Gamma(f : A \to X) \longrightarrow [SK, P_f]\,, \tag{2}$$

$$T_3 : \Gamma(f : A \to X) \longrightarrow [SK, A]\,, \tag{3}$$

and in §3 we show that operations of types (1) and (3) are generated by the elementary operations (a), (b) and (c). In §4 we deal with operations of type (2) and, invoking the relative Milnor-Hilton theorem [2; Theorem 6.4], we bring the mixed Whitehead products into the picture.

3. Operations of types (1) and (3). By directly dualising the proof of [1; Lemma 7.1] we can easily show that operations of types (1) and (3) may be obtained from natural set functions, homotopy sequence homomorphisms and a homotopy operation of one of the following forms:

(a) $T : \Pi_i[SP_i, X] \longrightarrow \Pi_j[SQ_j, X]\,,$

(b) $T' : \Pi_i[SP_i, A] \longrightarrow \Pi_j[SQ_j, A]\,.$

It is elementary to show that both T and T' are generated by addition and one-variable operations and hence we have the following result.

PROPOSITION 3.1. *Operations of types (1) and (3) are generated by elementary operations.*

4. Operations of type (2). For the sake of brevity, we consider only operations of the form $T: \Gamma \to \Theta$ where

$$\Gamma(f) = [SP, X] \times [SQ, P_f] \times [SR, A] \text{ and } \Theta(f) = [SK, P_f].$$

The corresponding argument for the general case will be immediate. Let h: $SQ \vee SR \longrightarrow SP \vee CSQ \vee SR$ be the obvious inclusion and denote by $\mathrm{Hom}(\Gamma, \Theta)$ the set of natural transformations $\Gamma \rightarrow \Theta$, then it is elementary to establish a bijection

$$\mathrm{Hom}(\Gamma, \Theta) \longleftrightarrow [SK, P_h]\,. \qquad (*)$$

Using standard homotopy sequence homomorphisms we may obtain a split short exact sequence

$$[S^2K,\ SP \vee SR] \longrightarrow [SK, P_h] \longrightarrow [SK, P_g]\,, \qquad (**)$$

where $g: SQ \vee SR \longrightarrow CSQ \vee SR$ is the inclusion.

As a direct consequence of $(*)$ and $(**)$ we now have the following result.

PROPOSITION 4.1. *Let*

$$T: \Pi_i[SP_i, X] \times \Pi_j[SQ_j, P_f] \times \Pi_k[SR_k, A] \longrightarrow [SK, P_f]$$

be a relative homotopy operation of type (2), *then*

$$T = T'(0 \times 1 \times 1) + \partial T''(1 \times 0 \times f_*)$$

where

$$T': \Pi_j[SQ_j, P_f] \times \Pi_k[SR_k, A] \longrightarrow [SK, P_f]$$

and

$$T'': \Pi_i[SP_i, X] \times \Pi_k[SR_k, X] \longrightarrow [SK, X]$$

are homotopy operations.

Since T'' in the above proposition is an operation of type (1), to show operations of type (2) are generated by elementary ones it only remains to deal with the mixed operation T'. Moreover, applying the relative Milnor-Hilton theorem [2; Theorem 6.4] we have the following result.

PROPOSITION 4.2. *Let* T: $[SQ, P_f] \times [SR, A] \longrightarrow [SK, P_f]$ *be a homotopy operation, then*

$$T = \sum_{\mu \neq 1} \omega_\mu^* T_\mu^R$$

where $T_{\mu}^{R}: [SQ, P_f] \times [SR, A] \longrightarrow [SA_{\mu}, P_f]$ *is the* μ*-th basic relative Whitehead product.*

Now the relative Whitehead products, by definition, are generated by the absolute product $W: [SP, P_f] \times [SQ, P_f] \longrightarrow [S(P \vee Q), P_f]$ and the mixed Whitehead product. However $W = [\ ,\ ](p_* \times 1)$ and hence, by Propositions 4.1 and 4.2, we have the following corollary completing the objective of this section.

COROLLARY 4.3. *Homotopy operations of type* (2) *are generated by elementary operations.*

5. Since any homotopy operation

$$\Gamma(f) \longrightarrow \Pi_i[SP_i, X] \times \Pi_j[SQ_j, P_f] \times \Pi_k[SR_k, A]$$

is determined by the projections onto each of its components, all relative homotopy operations are generated by operations of types (1), (2), and (3) and, thus, gathering up the results of §§3 and 4, Theorem 2.1 is proved.

REFERENCES

[1] J. W. RUTTER, Relative cohomology operations, Quart. J. Math., 15(1964), 77-88.
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