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ON CLOSED GEODESICS OF LENS SPACES[1]

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We shall consider a generalized lens space $L(q; p_2, \cdots, p_n)$ which is defined as follows. Let S^{2n-1} be a unit hypersphere in R^{2n}, and $G=\{T^k\}_{0\leqq k\leqq q-1}$ be a group of isometries of R^{2n} with

$$T=\begin{pmatrix} \cos\frac{2\pi p_1}{q} & -\sin\frac{2\pi p_1}{q} & & & \\ \sin\frac{2\pi p_1}{q} & \cos\frac{2\pi p_1}{q} & & 0 & \\ & & \ddots & & \\ & & & \cos\frac{2\pi p_n}{q} & -\sin\frac{2\pi p_n}{q} \\ & 0 & & \sin\frac{2\pi p_n}{q} & \cos\frac{2\pi p_n}{q} \end{pmatrix}.$$

We assume that $p_1=1$ and p_i's$(2\leqq i\leqq n)$ are relatively prime to $q(>2)$. Then we define $M=L(q; p_2, \cdots, p_n)=S^{2n-1}/G$ which is a compact Riemannian manifold of constant curvature 1 with $\pi_1(M)=G\approx Z_q$. In this note we are concerned with the cut locus of a point and closed geodesics of $L(q; p_2, \cdots, p_n)$. K.Shiohama ([1]) has studied the cut locus of $L(q; 1)$. The methods are completely elementary.

1. Let φ: $S^{2n-1}\to M$ be a covering projection. Because the diameter of M is not greater than $\pi/2$, no cut point of $p\in M$ along any geodesic through p can be conjugate to p. Thus a point q belongs to the cut locus $C(p)$ of p if and only if the following holds.

(*) If we put $\tilde{p}\in\varphi^{-1}(p)$, $\tilde{q}_1\in\varphi^{-1}(q)$, then there exist $\tilde{q}_2\in\varphi^{-1}(q)$ with $\tilde{q}_1\neq\tilde{q}_2$, and $\tilde{\tau}_1\in |||\tilde{p}, \tilde{q}_1|||, \tilde{\tau}_2\in |||\tilde{p}, \tilde{q}_2|||$ such that length $\tilde{\tau}_1=$ length $\tilde{\tau}_2=d(p, q)$, where $|||\tilde{p}, \tilde{q}|||$ denotes the set of minimizing geodesics between $\tilde{p}$ and $\tilde{q}$.

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In this section we shall consider the (tangent) cut locus of $m=\varphi\ (1, 0\cdots, 0)$. This loses no generality when $\cos\frac{2\pi p_i}{q}=\cos\frac{2\pi}{q}\ (i=1, \cdots, n)$ hold, since in this case and only in this case M is homogeneous. (Wolf([2]) The same method is applicable to more general situation. See the Remark 2 after Theorem 1.

Now the equation of geodesic of S^{2n-1} with initial point $m=(1, 0, \cdots, 0)$ and initial direction $y=(0, y_2, \cdots, y_{2n})$ is given by

$$\sigma\colon t\to(\cos t,\ y_2\sin t, \cdots, y_{2n}\sin t)\,. \tag{1.1}$$

Then the condition (*) is equivalent to the following: $\varphi\tilde{\sigma}(\theta)$ is a cut point of m along $\sigma=\varphi\tilde{\sigma}$ if and only if

(**) there exisists a unit vector $z=(0, z_2, \cdots, z_{2n})$ which is different from y and such that

$$\begin{pmatrix} \cos\frac{2\pi p_1}{q}k & -\sin\frac{2\pi p_1}{q}k & & & \\ \sin\frac{2\pi p_1}{q}k & \cos\frac{2\pi p_1}{q}k & & 0 & \\ & & \ddots & & \\ & & & \cos\frac{2\pi p_n}{q}k & -\sin\frac{2\pi p_n}{q}k \\ & 0 & & \sin\frac{2\pi p_n}{q}k & \cos\frac{2\pi p_n}{q}k \end{pmatrix} \begin{pmatrix} \cos t \\ y_2\sin t \\ \vdots \\ y_{2n-1}\sin t \\ y_{2n}\sin t \end{pmatrix} = \begin{pmatrix} \cos t \\ z_2\sin t \\ \vdots \\ z_{2n-1}\sin t \\ z_{2n}\sin t \end{pmatrix} \tag{1.2}$$

holds for some $1\leqq k\leqq q-1$, and $0<\theta\leqq\pi/2$ is the minimum value of t which satisfies (1.2).

Now this t is determined by

$$\cos\frac{2\pi k}{q}\ \cos t - y_2\ \sin\frac{2\pi k}{q}\sin t = \cos t\,, \tag{1.3}$$

and z is determined by

$$(1.4)\qquad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \vdots \\ z_{2n-1} \\ z_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ -y_2 \\ y_3 \cos\frac{2\pi p_2}{q}k - y_4 \sin\frac{2\pi p_2}{q}k \\ y_3 \sin\frac{2\pi p_2}{q}k + y_4 \cos\frac{2\pi p_2}{q}k \\ \vdots \\ y_{2n-1} \cos\frac{2\pi p_n}{q}k - y_{2n} \sin\frac{2\pi p_n}{q}k \\ y_{2n-1} \sin\frac{2\pi p_n}{q}k + y_{2n} \cos\frac{2\pi p_n}{q}k \end{pmatrix}.$$

By (1.3) we have $\cot t = -y_2 \cot\frac{\pi}{q}k$. Since $\cot t$ is monotone decreasing for $0 < t \leqq \pi/2$, θ in search is determined as follows:

If $y_2 = 0$: then $\theta = \pi/2$ and possible k's are $1, 2, \cdots, q-1 \pmod q$.
If $y_2 > 0$: then $k = q-1 \pmod q$, and $\theta = \cot^{-1}\left(y_2 \cot\frac{\pi}{q}\right)$.
If $y_2 < 0$: then $k = 1 \pmod q$, and $\theta = \cot^{-1}\left(-y_2 \cot\frac{\pi}{q}\right)$.

Thus we get,

THEOREM 1. *Let $M = L(q; p_2, \cdots, p_n)$ be a lens space which we have considered as a compact Riemannian manifold of constant sectional curvature 1. Then the cut point c_y of $m = \varphi(1, 0, \cdots, 0)$ along a geodesic σ_y with the initial direction $y = (0, y_2, \cdots, y_{2n})$ is given as follows:*

(i) *If $y_2 = 0$, then $c_y = \sigma_y(\pi/2)$ and this point coincides with $c_{z_k} = \sigma_{z_k}(\pi/2)$ with*

$$z_k = \left(0, 0, y_3 \cos\frac{2\pi p_2}{q}k - y_4 \sin\frac{2\pi p_2}{q}k, \cdots, y_{2n} \cos\frac{2\pi p_n}{q}k + y_{2n-1} \sin\frac{2\pi p_n}{q}k\right),$$

$k = 1, \cdots, q-1.$

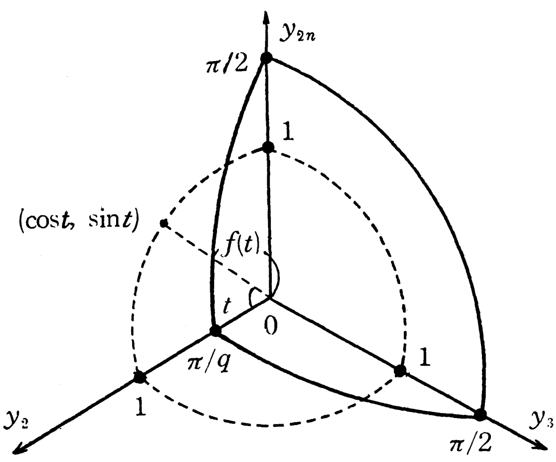
(ii) *If* $(1\geqq)\ y_2>0$, *then* $c_y=\sigma_y\left(\cot^{-1}\left(y_2\cot\frac{\pi}{q}\right)\right)$ *and this point coincides with* $c_z=\sigma_z\left(\cot^{-1}\left(y_2\cot\frac{\pi}{q}\right)\right)$ *with*

$$z_k=\left(0,-y_2,y_3\cos\frac{2\pi p_2}{q}+y_4\sin\frac{2\pi p_2}{q},\cdots,y_{2n}\cos\frac{2\pi p_n}{q}-y_{2n-1}\sin\frac{2\pi p_n}{q}\right).$$

(iii) *If* $(-1\leqq)\ y_2<0$, *then* $c_y=\sigma_y\left(\cot^{-1}\left(-y_2\cot\frac{\pi}{q}\right)\right)$ *and this point coincides with* $c_z=\sigma_z\left(\cot^{-1}\left(-y_2\cot\frac{\pi}{q}\right)\right)$ *with*

$$z=\left(0,-y_2,y_3\cos\frac{2\pi p_2}{q}-y_4\sin\frac{2\pi p_2}{q},\cdots,y_{2n}\cos\frac{2\pi p_n}{q}+y_{2n-1}\sin\frac{2\pi p_n}{q}\right).$$

REMARK 1. The tangent cut locus of $m=\varphi(1,0,\cdots,0)$ is given in Figure 1.



$f(t)=\cot^{-1}(\cos t\cot\pi/q)$

Figure 1.

REMARK 2. Let $\{i_1,\cdots,i_{k_1};i_{k_1+1},\cdots,i_{k_2};\cdots;i_{k_{a-1}+1},\cdots,i_{k_a}=i_n\}$ be a partition of $\{1,\cdots,n\}$ such that

$$\cos\frac{2\pi p_{i_1}}{q}=\cdots=\cos\frac{2\pi p_{i_{k_1}}}{q}=c^{(1)},$$

$$\cos\frac{2\pi p_{i_{k_1+1}}}{q}=\cdots=\cos\frac{2\pi p_{i_{k_2}}}{q}=c^{(2)},$$

$$\vdots$$

$$\cos\frac{2\pi p_{i_{k_{a-1}+1}}}{q}=\cdots=\cos\frac{2\pi p_{i_{k_a}}}{q}=c^{(a)},$$

and $c^{(1)},\cdots,c^{(a)}$ are all distinct. Now fix any $s\in\{1,\cdots,a\}$, then for every point $\varphi(x_1,\cdots,x_{2n})$ with $x_{2j-1}=x_{2j}=0$ for $j\in\{1,2,\cdots,n\}-\{i_{k_{s-1}+1},\cdots,i_{k_s}\}$, cut locus of this point may be determined by the same way.

Next we consider the angle between the geodesics when they meet at their cut points.

THEOREM 2. *Let $\sigma_y(\theta)$ be the cut point of $m=\varphi(1,0,\cdots,0)$ along the geodesic σ_y with the initial direction $y=(0,y_2,\cdots,y_{2n})$. Then $\sigma_y(\theta)=\sigma_z(\theta)$ holds where z has been determined in Theorem 1. Now the angle α between $\dot\sigma_y(\theta)$ and $\dot\sigma_z(\theta)$ is determined as follows.*

(i) *If $y_2=0$, then the angle α_k between $\dot\sigma_y(\pi/2)$and $\dot\sigma_{z_k}(\pi/2)$ is given by*

$$\alpha_k=\begin{cases}2\pi k/q, & \text{if } k/q\leqq 1/2,\\ 2\pi(q-k)/q, & \text{if } k/q>1/2.\end{cases}\tag{1.5}$$

(ii) *If $y_2\neq 0$, then we have*

$$\cos\alpha=\cos 2\pi/q-(1+\cos 2\pi/q)y_2^2.\tag{1.6}$$

PROOF. Since $\varphi_*\dot\sigma_y(\theta)=(\varphi\circ T^k)_*\dot\sigma_y(\theta)=\varphi_*(T^k{}_*\dot\sigma_y(\theta))$, we have

$$\cos\alpha=<\varphi_*\dot\sigma_y(\theta),\varphi_*\dot\sigma_z(\theta)>=<T^k{}_*\dot\sigma_y(\theta),\dot\sigma_z(\theta)>.$$

In case $y_2>0$, we get

$\cos\,\alpha =$

$$\begin{pmatrix} \cos\frac{2\pi p_1}{q} & \sin\frac{2\pi p_1}{q} & & & \\ -\sin\frac{2\pi p_1}{q} & \cos\frac{2\pi p_1}{q} & & 0 & \\ & & \ddots & & \\ & & & \cos\frac{2\pi p_n}{q} & \sin\frac{2\pi p_n}{q} \\ & 0 & & -\sin\frac{2\pi p_n}{q} & \cos\frac{2\pi p_n}{q} \end{pmatrix} \begin{pmatrix} -\sin\theta \\ y_2\cos\theta \\ \vdots \\ y_{2n-1}\cos\theta \\ y_{2n}\cos\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ -y_2\cos\theta \\ \vdots \\ (y_{2n-1}c_n+y_{2n}s_n)\cos\theta \\ (y_{2n}c_n-y_{2n-1}s_n)\cos\theta \end{pmatrix}$$

$$= c_1\sin^2\theta - 2y_2s_1\sin\theta\cos\theta - y_2^2c_1\cos^2\theta$$
$$+\sum_{k=2}^{n}\{(y_{2k-1}c_k\cos\theta + y_{2k}s_k\cos\theta)^2 + (-y_{2k-1}s_k\cos\theta - y_{2k}c_k\cos\theta)^2\}$$
$$= c_1\sin^2\theta - 2y_2s_1\sin\theta\cos\theta - y_2^2c_1\cos^2\theta + \cos^2\theta(1-y_2^2)$$
$$= c_1 + (\cos 2\theta+1)/2\{(1-y_2^2-c_1(1+y_2^2)\} - y_2s_1\sin 2\theta = c_1-(1+c_1)y_2^2,$$

where we have put $c_k = \cos\frac{2\pi p_k}{q}$, $s_k = \sin\frac{2\pi p_k}{q}$ for the sake of simplicity. In case of $y_2 < 0$ or $y_2 = 0$, the same calculation is valid. q. e. d.

REMARK 3. If $\sigma_y(\theta) = \sigma_z(\theta)$ be the cut point of m along geodesics σ_y and σ_z respectively, then the initial directions y and z are at an angle β with $\cos\beta = \langle y, z\rangle = -y_2^2 + \sum_{a=?}^{n}(y_{2a-1}^2 + y_{2a}^2)\cos\frac{2\pi p_n k}{q}$. In particular let the initial directions be given by $(0, \pm 1, 0, \cdots, 0)$, then $\langle \dot\sigma_y(\pi/q), \dot\sigma_z(\pi/q)\rangle = -1$ holds, and we have a simple closed geodesic of length $2\pi/q$.

2. Closed geodesics. Next we shall consider closed geodesics of $M = L(q; p_2, \cdots, p_n)$. First, note that every geodesic loop in M of length θ may be obtained as follows. Let $p \in M$ and $\tilde{p} \in \varphi^{-1}(p)$, then the geodesic with initial point $p = (x_1, \cdots, x_{2n})$ and initial direction $y = (y_1, \cdots, y_{2n})$ is given as follows.

$$\text{(2.1)} \qquad \tilde\sigma : t \longrightarrow \begin{pmatrix} x_1\cos t + y_1\sin t \\ \vdots \\ x_{2n}\cos t + y_{2n}\sin t \end{pmatrix}.$$

Now $\sigma = \varphi\tilde\sigma$ is the geodesic loop of length θ with base point p if and only if there exists some integer k determined by modulo q such that

$$(2.2)\qquad \begin{pmatrix} c'_1 & -s'_1 & & & & 0 \\ s'_1 & c'_1 & & & & \\ & & \ddots & & & \\ & & & & c'_n & -s'_n \\ 0 & & & & s'_n & c'_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{pmatrix} = \begin{pmatrix} x_1\cos\theta + y_1\sin\theta \\ x_2\cos\theta + y_2\sin\theta \\ \vdots \\ x_{2n-1}\cos\theta + y_{2n-1}\sin\theta \\ x_{2n}\cos\theta + y_{2n}\sin\theta \end{pmatrix}$$

holds where we put $c'_i = \cos\dfrac{2\pi p_i}{q}k$ and $s'_i = \sin\dfrac{2\pi p_i}{q}k$. Since $< x, y > = 0$, and $\|x\| = 1$ holds, we have easily from (2. 2)

$$(2.3)\qquad \cos\theta = (x_1^2 + x_2^2)c'_1 + \cdots + (x_{2n-1}^2 + x_{2n}^2)c'_n .$$

Secondly we calculate the angle α between $\dot\sigma(0) = y$ and $\dot\sigma(\theta)$. By the same way as the proof of Theorem 2 (§1), we get

$$\cos\alpha = \begin{pmatrix} c'_1 & -s'_1 & & & & 0 \\ s'_1 & c'_1 & & & & \\ & & \ddots & & & \\ & & & & c'_n & -s'_n \\ 0 & & & & s'_n & c'_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2n-1} \\ y_{2n} \end{pmatrix} \cdot \begin{pmatrix} -x_1\sin\theta + y_1\cos\theta \\ -x_2\sin\theta + y_2\sin\theta \\ \vdots \\ -x_{2n-1}\sin\theta + y_{2n-1}\cos\theta \\ -x_{2n}\sin\theta + y_{2n}\cos\theta \end{pmatrix}$$

$$= \sum_{j=1}^{n} (c'_j y_{2j-1} - s'_j y_{2j})(-x_{2j-1}\sin\theta + y_{2j-1}\cos\theta)$$
$$+ (s'_j y_{2j-1} + c'_j y_{2j})(-x_{2j}\sin\theta + y_{2j}\cos\theta)$$
$$= \sum_{j=1}^{n} \left\{ A_j(s'^2_j - c'^2_j) + A_j c'_j\cos\theta + \frac{A_j c'_j(1 - 2c'_j\cos\theta + \cos^2\theta)\cos\theta}{\sin^2\theta} \right\}$$
$$= \frac{1}{1 - \left(\sum_{j=1}^{n} A_j c'_j\right)^2} \left\{ 1 - 2\sum_{j=1}^{n} A_j c'^2_j + \left(\sum_{j=1}^{n} A_j c'_j\right)^2 \right\}$$

where we put $A_j = x_{2j-1}^2 + x_{2j}^2$. Note that $\sum_{j=1}^{n} A_j = 1, 0 \leqq A_j \leqq 1$, and $\cos\theta = \sum_{j=1}^{n} A_j c'_j$ hold by virtue of (2. 3). In particular, for $\alpha = 0$ it is necessary and sufficient that

$$(2.4)\qquad \left(\sum_{j=1}^{n} A_j c'_j\right)^2 = \sum_{j=1}^{n} A_j c'^2_j$$

holds. On the other hand by Cauchy-Schwarz ineqality we have

$$\left(\sum_{j=1}^{n}\sqrt{A_j}\cdot\sqrt{A_j}\ c_j'\right)^2 \leqq \sum_{j=1}^{n} A_j c_j'^2$$

where eqality holds if and only if $(\sqrt{A_j}c_j')/\sqrt{A_j}=$ constant. $j=1,\cdots,n$. And in this case we have $\cos\theta=\sum_{j=1}^{n}A_jc_j'=\cos\dfrac{2\pi p_0}{q}k$, where p_0 is one of p_j.

Thus closed geodesics on M of length $<2\pi$ and multiplicity 1 are given as follows:

1) Case of $p_0=1$. If q is odd, then $k\equiv 1$ or $q-1$(mod. q) and $\theta=2\pi/q$ holds. If q is even, then $k\equiv 1$, or $q/2$ (mod. q), and $\theta=2\pi/q$ (corresponding to the case $k\equiv 1$ or $q-1$) or $\theta=\pi$ (corresponding to the case $k\equiv q/2$) holds.
2) Generally, since p_0 is relatively prime to q, there exists an integer s_0 (mod. q) such that $p_0s_0+qt=1$ holds for some integer t. Thus $\cos\theta=\cos\dfrac{2\pi p_0}{q}s_0$ $=\cos\dfrac{2\pi}{q}$ holds. So we get the following: if q is odd, $k\equiv s_0$ or $q-s_0$ (mod. q) and we have $\theta=2\pi/q$. If q is even, then $k\equiv s_0$, $q-s_0$, or $qs_0/2$ (mod. q) and we have $\theta=2\pi/q$ (corresponding to the case $k\equiv s_0$ or $q-s_0$) or $\theta=\pi$ (corresponding to the case $k\equiv qs_0/2$).

Now we introduce the following equivalence relation $\sim$ in $\{p_1,\cdots,p_n\}$. Let s_i(mod. q) be an integer such that $p_is_i+qt_i=1$ holds for some integer $t_i(i=1,\cdots,n)$. Then we define $p_i\sim p_j$ if and only if $\cos\dfrac{2\pi p_j}{q}s_i=\cos\dfrac{2\pi p_i}{q}s_i=\cos\dfrac{2\pi}{p}$ holds. Let $\{p_1=p_{j_1},\cdots,p_{j_{m_1}};\cdots,p_{j_{m_{b-1}+1}},\cdots,p_{j_{m_b}}=p_{j_n}\}$ be a partition of $\{p_1,\cdots,p_n\}$, with respect to this equivalence relation. Then we have

THEOREM 3. (i) *Case of odd q. Fix any $s\in\{m_1,\cdots,m_b\}$, then through every point $\varphi(x_1,\cdots,x_{2n})$ with $x_{2j-1}=x_{2j}=0$ for $p_j\in\{p_1,\cdots,p_n\}-\{p_{j_{m_{s-1}+1}},\cdots,p_{j_{m_s}}\}$, there exists a unique simple closed geodesic with initial direction $\varphi_*(x_2,-x_1,\cdots,x_{2n}-x_{2n-1})$ and of length $2\pi/q$. Another geodesics are closed geodesics of length 2π and of multiplicity* 1.

(ii) *Case of even q. Fix any $s\in\{m_1,\cdots,m_b\}$, then through every point $\varphi(x_1,\cdots,x_{2n})$ with $x_{2j-1}=x_{2j}=0$ for $p_j\in\{p_1,\cdots,p_n\}-\{p_{j_{m_{s-1}+1}},\cdots,p_{j_{m_s}}\}$, there exists a unique simple closed geodesic with initial direction $\varphi_*(x_2,-x_1,\cdots,x_{2n},-x_{2n-1})$ and of length $2\pi/q$. Another geodesics are closed geodesics of length π and of multiplicity* 1.

Finally we shall treat some special cases. We put $c_i=\cos\dfrac{2\pi p_i}{q}(i=1,\cdots,n)$.

COROLLARY 1. *If* $c_1 = c_2 = \cdots = c_n$ *holds, that is, if* $p_i \equiv 1$ or $q-1(mod.\ q)$, *then through every point of M, there exists a unique simple closed geodesic of length* $2\pi/q$. *If q is odd (respectively even,) another geodesics are closed geodesics of length* 2π *(respectively* π*) and of multiplicity* 1.

COROLLARY 2. *Let q be a prime, and* $\{i_1, \cdots, i_{k_1}; \cdots; i_{k_{a-1}+1}. \cdots, i_{k_a} = i_n\}$ *be a partition of* $\{1, \cdots, n\}$ *such that* $c_{i_1} = \cdots = c_{i_{k_1}} = c^{(1)}; \cdots; c_{i_{k_{a-1}+1}}, \cdots = c_{i_{k_a}} = c^{(a)}$ *holds and* $c^{(1)}, \cdots, c^{(a)}$ *are all distinct Fix any* $s \in \{1, \cdots, a\}$, *then through every point* $\varphi(x_1, \cdots, x_{2n})$ *with* $x_{2j-1} = x_{2j} = 0$ *for* $j \in \{1, \cdots, n\} - \{i_{k_{s-1}+1}, \cdots, i_{k_s}\}$, *there exists a unique simple closed geodesic of length* $2\pi/q$. *Another geodesics are closed geodesics of langth* 2π *and muitiplicity* 1,

REMARK 4. (i) $c_1 = \cdots = c_n$ *holds if and only if M is homogeneous.*

(ii) $\{(x_1, \cdots, x_{2n}) \mid x_{2j-1} = x_{2j} = 0$ *for* $j \in \{1, \cdots, n\} - \{i_{k_{s-1}+1}, \cdots, i_{k_s}\}\}$ *is a homogeneous totally geodesic submanifold of M which is isometric to* $L(q; \underbrace{1, \cdots, 1}_{k_s - k_{s-1}})$ *of dimension* $2(k_s - k_{s-1}) - 1$.

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