Tôhoku Math. Journ.
23(1971), 727–734.

A GENERALIZATION OF SCHATTEN-VON NEUMANN-DIXMIER′ THEOREM FOR TYPE I AW*-ALGEBRAS

CHÔICHIRÔ SUNOUCHI

(Received Sept. 2, 1971)

1. Introduction. The purpose of this paper is to establish a spectral decomposition for positive elements in the maximal CCR ideal of type I AW^*-algebra and the corresponding Schatten-von Neumann-Dixmier′ Theorem in a type I AW^*-algebra.

In [5], Kaplansky showed that any type I AW^*-algebra is *-isomorphic to the algebra of all bounded module endomorphisms on some AW^*-module over the center of the AW^*-algebra, and can be considered as the extension of type I factor. Moreover, in [5], a one-dimensional projection on a Hilbert space corresponds to an abelian projection in a type I AW^*-algebra. Therefore, the closed ideal (we shall in general mean two-sided ideal) generated by the abelian projections in a type I AW^*-algebra corresponds to the closed ideal of all completely continuous operators on the Hilbert space.

2. A spectral decomposition for positive elements in the maximal CCR ideal of a type I AW^*-algebra. Let A be a type I AW^*-algebra with the center $\mathcal{F}$. By the Kaplansky Theorem ([5]), we may identify A with the algebra $B(M)$ of all bounded module endomorphisms on the AW^*-module M over $\mathcal{F}$, where $M=Ae$, e being any abelian projection in A of central support 1, and the $\mathcal{F}$-valued inner product on M is denoted by $(x, y)_M = y^*x$ in $\mathcal{F}$ for $x, y \in M$; we shall identify the center, $\mathcal{F}$, of $B(M)$ with $\mathcal{F}e$. The closed ideal generated by the abelian projections of $B(M)$ is a maximal CCR ideal in $B(M)$ ([2]), which we denote by $C(M)$.

The following lemma is useful for the later discussions.

LEMMA 1. (Kaplansky [5]) *If ϕ is a bounded homomorphism on an AW^*-module H over an abelian AW^*-algebra $\mathfrak{A}$, there is a unique a_0 in H such that $\phi(a) = (a, a_0)_H$ for all $a \in H$ where $(\cdot, \cdot)_H$ denotes the $\mathfrak{A}$-valued* inner product on H.

Let Z be the spectrum of $\mathcal{F}$. For each ζ in Z, define $[\zeta]$ to be the closed ideal defined by

$$[\zeta] = \text{closure}\left\{\sum_{i=1}^{n} a_i b_i;\ a_i \in B(M),\ b_i \in \mathcal{F} \text{ and } b_i^{\wedge}(\zeta) = 0\right\}.$$

Let ψ_ζ be the natural map of $B(M)$ onto $B(M)/[\zeta]$. For a in $B(M)$, let $a(\zeta) = \psi_\zeta(a)$. Then $\|a\| = \sup\ \{\|a(\zeta)\|;\ \zeta \in Z\}$, and the function $\zeta \to \|a(\zeta)\|$ from Z into the positive numbers is known to be upper semi-continuous ([1]). However, we have the following lemma for the AW^*-algebra $B(M)$.

LEMMA 2. *The function $\zeta \to \|a(\zeta)\|$ of Z is continuous.*

PROOF. Since $\|a(\zeta)\| = \|a^*(\zeta)a(\zeta)\|^{1/2}$, it is sufficient to consider the case where a is positive. Let z be in $\mathcal{F}$, then $z(\zeta) = z^{\wedge}(\zeta)I(\zeta)$. If $z(\zeta) \geqq a(\zeta)$ for each ζ in Z, then $\varphi(z) \geqq \varphi(a)$ for any pure state φ of $B(M)$, hence $\psi(z) \geqq \psi(a)$ for any state ψ of $B(M)$, so $z \geqq a$. Let $\mathcal{F}(a) = \{z \in \mathcal{F};\ z \geqq a\}$. $\mathcal{F}(a)$ is a bounded net in $\mathcal{F}$ in the decreasing order. Then $b = \text{g.l.b.}\mathcal{F}(a)$ exists and lies in $\mathcal{F}$. Moreover, we have $b \geqq a$. For this, it is sufficient to show that, for every x in M, $(bx, x)_M \geqq (ax, x)_M$. For each x in M, let $c = (x, x)_M$, $d = (ax, x)_M$, then, for $z \in \mathcal{F}(a)$, $(zx, x)_M = zc$, $(bx, x)_M = bc$ and $zc \geqq d$. Hence, for all positive number ϵ, $z(c+\epsilon) \geqq d+\epsilon b$, $z \geqq (d+\epsilon b)(c+\epsilon)^{-1}$; the last inequality holds good for every $z \in \mathcal{F}(a)$, hence $b \geqq (d+\epsilon b)(c+\epsilon)^{-1}$, or $b(c+\epsilon) \geqq d+\epsilon b$, from which it follows $bc \geqq d$, i. e. $(bx, x)_M \geqq (ax, x)_M$. Since x is arbitrary, $b \geqq a$. Moreover, by [1], we have $b^{\wedge}(\zeta) = \|a(\zeta)\|$ and this completes the proof, since $b^{\wedge}(\zeta)$ is continuous function of ζ. Q. E. D.

Employing Lemma 2, we can now establish the spectral decomposition for positive elements in $C(M)$. This is an alternative form for AW^*-algebra of the one which was proved by H. Halpern.

THEOREM 1. *If a is a positive element in $C(M)$, then there is a monotone decreasing sequence $\{a_i\}$ of positive elements in $\mathcal{F}$ and a sequence of mutually orthogonal abelian projections $\{e_i\}$ uniquely such that*

$$\text{(1)}\qquad \text{closure } \{\zeta \in Z;\ a_i^{\wedge}(\zeta) \neq 0\} = \{\zeta \in Z;\ q_i^{\wedge}(\zeta) = 1\}.$$

where q_i is the central support of e_i;

$$\text{(2)}\qquad a_i \longrightarrow 0 \text{ as } i \longrightarrow \infty \text{ (in the uniform topology); and}$$

$$\text{(3)}\qquad a = \sum_{i=1}^{\infty} a_i e_i \text{ (in the uniform topology)}.$$

The proof goes just in the same way as Halpern ([3]), so we shall omit it.

3. Schatten-von Neumann-Dixmier′ Theorem in type I AW^*-algebras. Let P be the set of all positive elements a in $C(M)$ such that if $\sum_{i=1}^{\infty} a_i e_i$ is a spectral decomposition for a then the increasing sequence $\left\{\sum_{i=1}^{n} a_i\right\}_n$ is bounded above in $\mathcal{F}$.

DEFINITION. Let $S(M)$ be the set of a in $B(M)$ such that $a^*a \in P$, then $S(M)$ is an ideal in $B(M)$ ([3]). We define the Schmidt class of $B(M)$ to be the ideal $S(M)$ and the trace class of $B(M)$ to be the ideal $T(M) = S(M)^2$.

The set of positive elements of $T(M)$ is the set P. Let a be an element of P and let a have the spectral decomposition $\sum_{i=1}^{\infty} a_i e_i$. Define the trace $Tr(a)$ of a to be the least upper bound of the increasing sequence $\left\{\sum_{i=1}^{n} a_i\right\}_n$ in $\mathcal{F}$.

If a is an element of $T(M)$, then there exists the positive elements $\{a_n\}_{n=1}^{4}$ in $T(M)$ such that $a = a_1 - a_2 + i(a_3 - a_4)$. We can define uniquely $Tr(a) = Tr(a_1) - Tr(a_2) + i(Tr(a_3) - Tr(a_4))$ so that Tr is a $\mathcal{F}$-linear function of $T(M)$ into $\mathcal{F}$.

We show that the Schmidt class $S(M)$ of $B(M)$ is an AW^*-module over $\mathcal{F}$.

THEOREM 2. *The ideal $S(M)$ is an AW^*-module over $\mathcal{F}$.*

PROOF. Put $(a, b)_S = Tr(b^*a)$ for $a, b \in S(M)$, then this is the $\mathcal{F}$-valued inner product. Define, for $a \in S(T)$, $\|a\|_2 = \|(a, a)_S\|^{1/2}$, then $\|\cdot\|_2$ on $S(M)$ defines a norm.

At first, we show that $S(M)$ is complete in this norm. Let $\{a_n\}$ be a Cauchy sequence in $S(M)$. By the fact that $\|a\| \leqq \|a\|_2$ for each $a \in S(M)$ and that $C(M)$ is uniformly closed, there exists an element a in $C(M)$ such that $\|a_n - a\| \to 0$ as $n \to \infty$. We prove that $a \in S(M)$ and that $\|a_n - a\|_2 \to 0$ as $n \to \infty$. Let $\sum_{i=1}^{\infty} a_i e_i$ be the spectral decomposition of $|a|$. Put $\mu_n(a) = a_n$. By the fact that $\sum_{i=1}^{\infty} a_i^{\wedge}(\zeta) e_i(\zeta)$ is the spectral decomposition of the completely continuous operator $|a(\zeta)|$ ([3]) and [7], we have $\mu_n(a)^{\wedge}(\zeta) = \mu_n(a(\zeta))$ where $\mu_n(a(\zeta))$ is the n-th characteristic number of the operator $a(\zeta)$, and $\|\mu_k(a_n - a_m) - \mu_k(a_n - a)\| \to 0$ as $m \to \infty$. It follows that, for each ζ in Z and each positive number N,

$$\left\{\sum_{k=1}^{N} \mu_k(a_n - a)^2\right\}^{1/2\wedge}(\zeta) = \left\{\sum_{k=1}^{N} \mu_k(a_n(\zeta) - a(\zeta))^2\right\}^{1/2}$$

$$\leqq \limsup_{m\to\infty}\left\{\sum_{k=1}^{\infty}\mu_k(a_n(\zeta)-a_m(\zeta))^2\right\}^{1/2}$$

$$\leqq \limsup_{m\to\infty}\ \|a_n-a_m\|_2$$

for all n. Therefore, it follows that

$$\left\{\sum_{k=1}^{N}\mu_k(a_n-a)^2\right\}^{1/2}\leqq \limsup_{m\to\infty}\ \|a_n-a_m\|_2$$

for all n. Therefore, letting $N\to\infty$, we find

$$\|a_n-a\|_2\leqq \limsup_{m\to\infty}\ \|a_n-a_m\|_2,\ \text{so that}$$

$$\lim_{n\to\infty}\ \|a_n-a\|_2\leqq \lim_{m,n\to\infty}\ \|a_n-a_m\|_2=0\,.$$

Thus the completeness of $S(M)$ is proved.

Next, by the property of AW^*-algebra $\mathfrak{F}$, we can easily show the following: If $\{p_i\}$ is a set of mutually orthogonal projections in $\mathfrak{F}$ of l. u. b. p, and suppose a is an element of $S(M)$ with $p_ia=0$ for all i, then $pa=0$.

Finally, we show the following: If $\{p_i\}$ is a set of mutually orthoganal projections in $\mathfrak{F}$ of l. u. b. 1 and if $\{a_i\}$ is a bounded set in $S(M)$, there is a unique $a\in S(M)$ such that $p_ia=p_ia_i$ for all i. Let $\|a_i\|_2{}^2=\|Tr(a_i{}^*a_i)\|\leqq N^2$ for all i, then $\|a_i\|\leqq\|a_i\|_2\leqq N$. Therefore, there is an a in $B(M)$ such that $p_ia=p_ia_i$ for all i. Let e be an abelian projection in $B(M)$ and let p be the central support of e. The algebra $eB(M)e=\mathfrak{F}e$ is $*$-isomorphic to $\mathfrak{F}p$. Following [3], we let $\tau_e(a)$ denote the unique element in $\mathfrak{F}p$ such that $eae=\tau_e(a)e$. Let S be a set of mutually orthogonal projections and let $F(S)$ be the set of all finite subsets of S. For each J of $F(S)$ and for each i, we have

$$p_i\left(\sum_{f\in J}\tau_f(a^*a)\right)=p_i\left(\sum_{f\in J}\tau_f(a_i{}^*a_i)\right)\leqq Tr(a_i{}^*a_i)p_i\leqq Np_i\,.$$

This follows from the fact that $\underset{J\in F(S)}{\text{l. u. b.}}\sum_{f\in J}\tau_f(a_i{}^*a_i)=Tr(a_i{}^*a_i)$. The proof of this fact is essentially same as Halpern in [3], except for noting that $\tau_f(\cdot)$ is completely additive on projections ([4]). So $\sum_{f\in J}\tau_f(a^*a)\leqq N\cdot 1$. Thus, $\left\{\sum_{f\in J}\tau_f(a^*a)\right\}_{J\in F(S)}$ is bounded above. Therefore, by [3], $a^*a\in P$ and $a\in S(M)$. This completes the proof of this theorem. Q. E. D.

We now establish that Hom$(C(M),\ \mathfrak{F})$, the set of all uniformly bounded module homomorphisms on $C(M)$to $\mathcal{F}$, is identified with the trace class $T(M)$ of $B(M)$.

THEOREM 3. *If* ϕ *is an element of* Hom$(C(M), \mathcal{F})$, *there is a unique* a_ϕ *in* $T(M)$ *such that* $\phi(a) = Tr(aa_\phi)$ *for all* a *in* $C(M)$. *Moreover, for each* $a \in T(M)$ *let* $\|a\|_1 = \|Tr((a^*a)^{1/2})\|$. *Then the function* $\|\cdot\|_1$ *is a norm on* $T(M)$ *under which* $T(M)$ *is an involutive Banach algebra. The function* $\phi \rightarrow a_\phi$ *is an isometric isomorphism of* Hom$(C(M), \mathcal{F})$ *onto* $T(M)$ *(as Banach space).*

The proof of this theorem is same as that of Halpern in [3].

Since an abelian AW^*-algebra $\mathfrak{A}$ is $*$-isomorphic with the algebra of complex-valued continuous functions on a Stonean space which is the spectrum of $\mathfrak{A}$, we have in the followings to consider "order-convergence" of a net in $\mathfrak{A}$. The following criterion for the order-convergence in $\mathfrak{A}$ is useful.

LEMMA 3. (Widom [8]) *A net* $\{a_\lambda\}$ *of* $\mathfrak{A}$ *order-converges to* a *(written* $a_\lambda \rightarrow a$ (O)*) if and only if given a non-zero projection* e *in* $\mathfrak{A}$ *and a positive number* ϵ *there exists a non-zero projection* f *in* $\mathfrak{A}$ *with* $f \leqq e$ *and a* λ_0 *such that* $\lambda \geqq \lambda_0$ *implies* $\|f(a_\lambda - a)\| < \epsilon$.

Then, we have the following lemma, which is stated in the strong topology for a von Neumann algebra.

LEMMA 4. (Widom [8]) *The unit sphere of* $\mathfrak{A}$ *is complete relative to order-convergence.*

Now, we show that Hom$(T(M), \mathcal{F})$, the set of all bounded (with respect to $\|\cdot\|_1$) module homomorphisms on $T(M)$ to $\mathcal{F}$, is identified with $B(M)$.

THEOREM 4. *Let* $a_0 \in B(M)$. *The function* $\phi(a) = Tr(aa_0)$ *for* $a \in T(M)$ *is an element of* Hom$(T(M), \mathcal{F})$. *Conversely, if* ϕ *is an element of* Hom$(T(M), \mathcal{F})$, *there is a unique* $a_\phi \in B(M)$ *such that* $\phi(a) = Tr(aa_\phi)$ *for all* $a \in T(M)$. *The function* $\phi \rightarrow a_\phi$ *is an isometric isomorphism of* Hom$(T(M), \mathcal{F})$ *onto* $B(M)$ *(as Banach space).*

PROOF. Let $a_0 \in B(M)$ and let $a \in T(M)$. Then we have

$$\|Tr(aa_0)\| \leqq \|a_0\| \cdot \|Tr(|a|)\| = \|a_0\| \cdot \|a\|_1 .$$

Thus the function $\phi(a) = Tr(aa_0)$ for $a \in T(M)$ is an element of Hom$(T(M), \mathcal{F})$ and $\|\phi\| = \|a_0\|$ by [3].

Let ϕ be an element of Hom$(T(M), \mathcal{F})$. For every fixed b in $S(M)$, the function $a \to \phi(b^*a)$ is a bounded module homomorphism on $S(M)$. In fact, by [6],

$$\|\phi(b^*a)\| \leqq \|\phi\| \cdot \|Tr(|b^*a|)\| \leqq \|\phi\| \cdot \|b\|_2 \|a\|_2.$$

By Lemma 1 and Theorem 2, there is, for every $b \in S(M)$, a unique $\Phi(b)$ in $S(M)$ such that $\phi(b^*a) = (a, \Phi(b))_S$ for every a in $S(M)$. Let f be a projection in $C(M)$ and e be an abelian projection. Then

$$\begin{aligned}\|\tau_e(\Phi(f))\| &= \|\tau_e(\Phi(f))^*\| = \|Tr(\Phi(f)^*e)\| = \|(e, \Phi(f))_S\| \\ &= \|\phi(fe)\| \leqq \|\phi\| \cdot \|Tr(|fe|)\| \\ &= \|\phi\| \cdot \|Tr((\tau_e(f))^{1/2}e)\| = \|\phi\| \cdot \|\tau_e(f)^{1/2}\| \leqq \|\phi\| .\end{aligned}$$

Now, we have $\|\Phi(f)\| \leqq 2\|\phi\|$. Indeed, $\Phi(f)$ is in $S(M)$ and $\Phi(f) = \Phi(f)_1 - \Phi(f)_2 + i(\Phi(f)_3 - \Phi(f)_4)$ where $\Phi(f)_i$ in $S(M)^+$ for $i = 1, 2, 3, 4,$ hence

$$\begin{aligned}\|\Phi(f)\| &\leqq \|\Phi(f)_1 - \Phi(f)_2\| + \|\Phi(f)_3 - \Phi(f)_4\| \\ &= \max.\ (\|\Phi(f)_1\|, \|\Phi(f)_2\|) + \max.\ (\|\Phi(f)_3\|, \|\Phi(f)_4\|) .\end{aligned}$$

Suppose that $\max.(\|\Phi(f)_1\|, \|\Phi(f)_2\|) = \|\Phi(f)_1\|$, and let $\sum_{i=1}^{\infty} a_i e_i$ be the spectral decomposition of $\Phi(f)_1$ then

$$\begin{aligned}\|\Phi(f)_1\| &= \|a_1\| = \|a_1 e_1\| = \|e_1 a_1 e_1\| = \|e_1 \Phi(f)_1 e_1\| \\ &\leqq \|\tau_{e_1}(\Phi(f)_1)\| \leqq \|\phi\| .\end{aligned}$$

Similary, $\max.(\|\Phi(f)_3\|, \|\Phi(f)_4\|) \leqq \|\phi\|$. Therefore $\|\Phi(f)\| \leqq 2\ \|\phi\|$. Moreover, $\Phi(\cdot)$ has the following property: for $a \in S(M)$ and $b \in B(M)$, we have $b\Phi(a) = \Phi(ba)$. Indeed, for $c \in S(M)$,

$$(c, b\Phi(a))_S = (b^*c, \Phi(a))_S = \phi(a^*b^*c) = \phi((ba)^*c) = (c, \Phi(ba))_S .$$

The set $C(M)_p$ of all projections in $C(M)$ is an increasing directed set under the usual ordering for projections in $B(M)$ with l. u. b. 1. We shall show that when $g \in C(M)_p$ converges to 1 in $B(M)$, $\Phi(g)$ is strongly convergent to a fixed element a_ϕ in $B(M)$. If g varies over $C(M)_p$, for each $x \in M$, we have $(gx, x)_M$ order-converges to $(x, x)_M$ in $\mathcal{F}$, hence

$$((1-g)x, x)_M = |(1-g)x|^2 \longrightarrow 0\ (O) .$$

This means by Lemma 3, that, for each $x \in M$, and for a given non-zero projection $e \in \mathcal{F}$ and any positive number ϵ, there exists a non-zero projection $f \in \mathcal{F}$, $f \leqq e$ and a $g_0 \in C(M)_p$ such that $g \geqq g_0$, $g \in C(M)_p$ implies $\|f|(1-g)x|^2\| < \epsilon^2$. Now, for g, g' in $C(M)_p$ such that $g, g' \geqq g_0$, we have

$$\begin{aligned}(\Phi(g)x, x)_M &= (\Phi(g)x, g_0x + (1-g_0)x)_M \\ &= (g_0\Phi(g)x, x)_M + (\Phi(g)x, (1-g_0)x)_M \\ &= (\Phi(g_0)x, x)_M + (\Phi(g)x, (1-g_0)x)_M ;\end{aligned}$$

together with the similar relation by g', we have

$$\begin{aligned}&(\Phi(g)x, x)_M - (\Phi(g')x, x)_M \\ &\quad = (\Phi(g)x, (1-g_0)x)_M - (\Phi(g')x, (1-g_0)x)_M,\end{aligned}$$

so that

$$\begin{aligned}&\|f((\Phi(g)x, x)_M - (\Phi(g')x, x)_M)\| \\ &\leqq \|(\Phi(g)fx, f(1-g_0)x)_M\| + \|(\Phi(g')fx, f(1-g_0)x)_M\| \\ &\leqq \|\Phi(g)fx\| \cdot \|f|(1-g_0)x|^2\|^{1/2} + \|\Phi(g')fx\| \cdot \|f|(1-g_0)x|^2\|^{1/2} \\ &\leqq 4\,\epsilon\|\phi\| \cdot \|x\|,\end{aligned}$$

by $\|\Phi(g)\| \leqq 2\|\phi\|$ for all $g \in C(M)_p$ and, for b in $B(M)$, $b\Phi(a) = \Phi(ba)$ for every $a \in S(M)$. Hence, by Lemma 3 and Lemma 4, we see that $\Phi(g)$ strongly converges to an element a_ϕ in $B(M)$:

$$(\Phi(g)x, x)_M \longrightarrow (a_\phi x, x)_M \ (O) \text{ for every } x \in M.$$

Next, we have $\Phi(f) = fa_\phi$ for every $f \in C(M)_p$, because

$$\begin{aligned}(fa_\phi x, y)_M = (a_\phi x, fy)_M &= \operatorname*{order-lim}_{g \in C(M)_p} (\Phi(g)x, fy)_M \\ &= \operatorname*{order-lim}_{g \in C(M)_p} (f\Phi(g)x, y)_M = (\Phi(f)x, y)_M\end{aligned}$$

for every $x, y \in M$.

Let $a, b \in S(M)$, then

$$\|(a, \Phi(b))_S\| = \|\phi(b^*a)\| \leqq \|\phi\| \cdot \|b\|_2 \|a\|_2,$$

so we have $\|\Phi(b)\|_2 \leqq \|\phi\| \cdot \|b\|_2$. Therefore $\Phi(\cdot)$ is continuous in $S(M)$.

But, for a in $S(M)$, if $\sum_{i=1}^{\infty} a_i e_i$ is the spectral decomposition of $|a|$ and

$b_n = \sum_{i=1}^{n} a_i(ue_i)$ where u is the partial isometry such that $u|a|$ is polar decomposition of a, then $\|b_n - a\|_2 \to 0$ if $n \to \infty$([7]), hence

$$\Phi(a) = \lim_{n\to\infty} \Phi(b_n) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i u \Phi(e_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} a_i u e_i a_\phi = \lim_{n\to\infty} b_n a_\phi$$

$$\text{i. e. } \|\Phi(a) - b_n a_\phi\|_2 = \|\Phi(a) - \Phi(b_n)\|_2 \to 0 \text{ if } n \to \infty .$$

But $\|b_n a_\phi - a a_\phi\|_2 \leqq \|b_n - a\|_2 \|a_\phi\| \to 0$ if $n \to \infty$. Therefore $\Phi(a) = a a_\phi$ for every $a \in S(M)$.

Finally, let $b \in P$. Then $b^{1/2} \in S(M)$. We have

$$\phi(b) = \phi(b^{1/2}b^{1/2}) = (b^{1/2}, \Phi(b^{1/2}))_S$$

$$= (b^{1/2}, b^{1/2}a_\phi)_S = Tr(a_\phi{}^* b) .$$

Therefore, for every $b \in T(M)$, $\phi(b) = Tr(a_\phi{}^* b) = Tr(b a_\phi{}^*)$.

Therefore, if ϕ is an element of Hom$(T(M), \mathcal{F})$ there is one and only one a_ϕ in $B(M)$ such that $\phi(a) = Tr(a a_\phi)$ for all $a \in T(M)$. Since the function $\phi \to a_\phi$ is linear, the function is isometric isomorphism of the Banach space Hom$(T(M), \mathcal{F})$ onto $B(M)$. Q. E. D.

Finally, the author wishes to express his hearty thanks to Prof. M. Fukamiya in the presentation of this paper. The author also wishes to thanks to Mr. T. Okayasu, Mr. K. Saito, Mr. H. Takemoto and Mr. T. Kikuchi for several valuable comments.

REFERENCES

[1] J. GLIMM, A Stone-Weierstrass theorem for C^*-algebras, Ann. of Math., 72(1960), 216–244.
[2] H. HALPERN, The maximal GCR ideal in a type I AW^*-algebra, Proc. Amer. Math. Soc., 17(1966), 906–914.
[3] H. HALPERN, A spectral decomposition for self-adjoint elements in the maximal GCR ideal of a von Neumann algebra with applications to non-commutative integration theory, Trans. Amer. Math. Soc., 133(1968), 281–306.
[4] I. KAPLANSKY, Algebras of type I, Ann. of Math., 56(1952), 460–472.
[5] I. KAPLANSKY, Modules over operator algebras, Amer. J. Math., 75(1953), 839–858.
[6] R. SCHATTEN, Theory of cross spaces, Princeton Univ. Press. (1950).
[7] H. TAKEMOTO, On the C_p-classes in the maximal CCR ideal of a von Neumann algebra, Tôhoku Math. J., 23(1971), 301–311.
[8] H. WIDON, Embedding in algebras of trye I, Duke Math. J., 23(1956), 300–324.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN