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THE DEFICIENCIES OF A COMPOSITE FUNCTION OF INTEGRAL FUNCTIONS

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1. Let $f(z)$ and $g(z)$ be two integral functoins. It is assumed that the reader is familiar with the following symbols of frequent use in Nevanlinna's theory:

$m(r, f)$, $n(r, f)$, $N(r, f)$, $T(r, f)$, $\delta(a, f)$, etc..

Valiron [4] proved the following theorem.

If $f(z)$ is a meromorphic function of finite order μ and of lower order λ, and if $\mu-\lambda<1$, then all deficiencies of $f(z)$ are invariant under a change of origin.

Gol'dberg and Belinsky [3] showed that the condition $\mu-\lambda<1$ in the theorem can not be dropped.

These theorems can be considered as results on relation between deficiencies of a composite function $f(g(z))$ and those of the integral function $f(z)$, where $g(z)$ is a polynomial of degree 1.

From this point of view we consider the following two problems:

(A) Is $f(a_0)$ also a deficient value of $f(g(z))$, when a_0 is a deficient value of $g(z)$?

(B) Is w_0 also a deficient value of $f(g(z))$, when w_0 is a deficient value of $f(z)$?

2. We first deal with the problem (A).

THEOREM 1. *Let $g(z)$ be a transcendental integral function and $f(z)$ a polynomial of degree n. Then, for any w_0, it holds that*

$$\delta(w_0, \ f(g(z))) \geqq \frac{1}{n} \sum_{i=0}^{n-1} \delta(\gamma_i, \ g(z)), \tag{1}$$

where $\{\gamma_i\}_{i=0}^{n-1}$ are zeros of $f(z)-w_0$.

PROOF. Clearly we see

$$N\left(r, \frac{1}{f(g(z))-w_0}\right) = \sum_{i=0}^{n-1} N\left(r, \frac{1}{g(z)-\gamma_i}\right).$$

It is known that if $g(z)$ is a non-constant integral function and if $f(z)$ a polynomial of degree n, then

$$\frac{T(r, f(g(z)))}{T(r, g(z))} \longrightarrow n \quad \text{as} \quad r \longrightarrow \infty,$$

(see Hayman [2], p. 54). Thus we have

$$\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f(g)-w_0}\right)}{T(r, f(g))} \leqq \limsup_{r\to\infty} \frac{\sum_{i=0}^{n-1} N\left(r, \frac{1}{g-\gamma_i}\right)}{\{1-o(1)\}n\cdot T(r, g)}$$
$$\leqq \frac{1}{n}\sum_{i=0}^{n-1}(1-\delta(\gamma_i, g)),$$

so

$$\delta(w_0, f(g(z))) \geqq \frac{1}{n}\sum_{i=0}^{n-1}\delta(\gamma_i, g(z)),$$

which proves the theorem.

REMARK. Given any transcendental integral function $g(z)$, there exists a polynomial $f(z)$ such that the equality in (1) holds.

From Theorem 1 the following is easily obtained.

COROLLARY 1. *Let $f(z)$ and $g(z)$ be as in Theorem 1. If a_0 is a deficient value of $g(z)$, then $f(a_0)$ is also a deficient value of $f(g(z))$.*

3. We shall next show that if $f(z)$ and $g(z)$ are both transcendental integral functions, then the conclusion of Corollary 1 does not always hold.

THEOREM 2. *Let $g(z)$ be a transcendental integral function of finite order μ and of a positive lower order λ. Then, for any $a_0 (\neq \infty)$, there exists a transcendental integral function $f(z)$ such that $f(a_0)$ is not a deficient value of $f(g(z))$.*

PROOF. We can choose a sequence $\{a_i\}_{i=1}^{\infty}$ of complex numbers and two sequences $\{r_i\}_{i=1}^{\infty}$, $\{K_i\}_{i=1}^{\infty}$ of positive numbers which satisfy the following conditions (i) and (ii):

$$\text{(i)} \quad \sum_{i=1}^{\infty}\frac{1}{|a_i|} < +\infty, \sum_{i=1}^{\infty}\frac{1}{r_i^{\lambda/2}} < +\infty, \; r_i \longrightarrow +\infty (i \longrightarrow +\infty)$$

and

$$\sum_{i=1}^{\infty} \log \frac{1}{1-\dfrac{1}{K_i}} < +\infty, \ K_i > 1,$$

(ii) $T(r_{i+1}, g) \geqq r_{i+1}^{\lambda/2} \log^+ |a_i|,$

$$|a_{i+1}| > K_{i+1} \cdot M(r_{i+1}, g)$$

and

$$\lim_{r\to\infty} \frac{h(r)\cdot \log r}{T(r, g)} = 0,$$

where $M(r, g) = \max\limits_{|z|=r} |g(z)|$ and $h(r)$ denotes the number of a_k such that $|a_k| < K_i \cdot M(r_i, g)$ for $r_i = \min\limits_{1\leqq n\leqq\infty} \{r_n \in \{r_i\}_{i=1}^{\infty}: r \leqq r_n\}$.

In fact, we fix $r_1 > 1$ and choose a_1 such that $|a_1| > M(r_1, g)$. We next take r_2 $(> r_1)$ such that

$$T(r_2, g) \geqq r_2^{\lambda/2} \log^+ |a_1| \quad \text{and} \quad r_2 \geqq 2^{3/\lambda}.$$

Further, for this r_2, we choose a_2 such that

$$|a_2| > K_2 \cdot M(r_2, g),$$

where

$$K_2 = \frac{1}{1-e^{-1/2^2}}.$$

If a_i has already been determined such that $|a_i| > K_i \cdot M(r_i, g)$, then we determine $r_{i+1} (> r_i)$ satisfying

$$T(r_{i+1}, g) \geqq r_{i+1}^{\lambda/2} \log^+ |a_i| \quad \text{and} \quad r_{i+1} \geqq (i+1)^{3/\lambda}$$

and we choose a_{i+1} such that $|a_{i+1}| > K_{i+1} \cdot M(r_{i+1}, g)$ where

$$K_{i+1} = \frac{1}{1-e^{-1/(i+1)^2}}.$$

By constructing sequences $\{a_i\}_{i=1}^{\infty}$, $\{r_i\}_{i=1}^{\infty}$ and $\{K_i\}_{i=1}^{\infty}$ in this way, we see easily that these sequences satisfy the conditions (i) and (ii). We now consider the infinite product

$$f(z) = \begin{cases} \prod_{i=0}^{\infty}\left(1-\frac{z}{a_i}\right), & a_0 \neq 0 \\ \prod_{i=1}^{\infty} z\cdot\left(1-\frac{z}{a_i}\right), & a_0 = 0\,. \end{cases}$$

Then $f(z)$ is an integral function with simple zeros at a_i. Hence we have

$$\begin{aligned} m\left(r, \frac{1}{f(g)}\right) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \prod_{i=0}^{\infty}\left|\frac{1}{1-\frac{g(re^{i\theta})}{a_i}}\right| d\theta + O(\log r) \\ &\leqq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \prod_{i=0}^{h(r)-1}\left|\frac{1}{1-\frac{g(re^{i\theta})}{a_i}}\right| d\theta + \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \prod_{i=h(r)}^{\infty}\left|\frac{1}{1-\frac{g(re^{i\theta})}{a_i}}\right| d\theta + O(\log r) \end{aligned}$$

and

$$\prod_{i=h(r)}^{\infty}\left|\frac{1}{1-\frac{g(re^{i\theta})}{a_i}}\right| \leqq \prod_{i=h(r)}^{\infty}\frac{1}{1-\left|\frac{g(re^{i\theta})}{a_i}\right|} \leqq \prod_{i=h(r)}^{\infty}\frac{1}{1-\frac{M(r, g)}{K_iM(r, g)}} = \prod_{i=h(r)}^{\infty}\frac{1}{1-\frac{1}{K_i}},$$

so we see

$$\log^+ \prod_{i=h(r)}^{\infty}\left|\frac{1}{1-\frac{g(re^{i\theta})}{a_i}}\right| \leqq \log^+ \prod_{i=h(r)}^{\infty}\frac{1}{1-\frac{1}{K_i}} = \sum_{i=h(r)}^{\infty}\log\frac{1}{1-\frac{1}{K_i}} < 2\,.$$

Thus it follows that

$$\begin{aligned} m\left(r, \frac{1}{f(g)}\right) &\leqq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \prod_{i=0}^{h(r)-1}\left|\frac{a_i}{g(re^{i\theta})-a_i}\right| d\theta + O(\log r) \\ &\leqq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \prod_{i=0}^{h(r)-1}\left|\frac{1}{g(re^{i\theta})-a_i}\right| d\theta + \log^+ \prod_{i=0}^{h(r)-1} |a_i| + O(\log r) \\ &\leqq \sum_{i=0}^{h(r)-1} m\left(r, \frac{1}{g-a_i}\right) + \sum_{i=0}^{h(r)-1} \log^+ |a_i| + O(\log r)\,. \end{aligned}$$

By the second fundamental theorem, we obtain

$$\sum_{i=0}^{h(r)-1} m\left(r, \frac{1}{g-a_i}\right) \leqq 2\, T(r, g) + m\left(r, \frac{g'}{g}\right) + m\left(r, \sum_{i=0}^{h(r)-1} \frac{g'}{g-a_i}\right) + O(\log r)\,.$$

It is easy to see that

$$m\left(r, \frac{g'}{g}\right) = O(\log r)$$

and

$$m\left(r, \sum_{i=0}^{h(r)-1} \frac{g'}{g-a_i}\right) = O(h(r)\cdot\log r).$$

On the other hand, a result of Clunie (see Hayman [2] p. 54) implies that if $f(z)$ and $g(z)$ are two transcendental integral functions, then

$$(2) \qquad \frac{T(r, f(g))}{T(r, g)} \longrightarrow \infty \text{ as } r \longrightarrow \infty .$$

Hence, for the sequence $\{r_n\}_{n=1}^{\infty}$ chosen above, we have

$$\frac{m\left(r_n, \dfrac{1}{f(g)}\right)}{T(r_n, f(g))} \leqq \frac{2\,T(r_n, g) + \displaystyle\sum_{i=0}^{h(r_n)-1} \frac{T(r_i, g)}{r_i^{\lambda/2}} + O(h(r_n)\ \log r_n)}{T(r_n, g)} \cdot \frac{T(r_n, g)}{T(r_n, f(g))}$$

which gives us

$$\liminf_{r\to\infty} \frac{m\left(r, \dfrac{1}{f(g)}\right)}{T(r, f(g))} = 0,$$

by the conditions (i), (ii) and (2). Thus we obtain

$$\delta(0, f(g)) = \liminf_{r\to\infty} \frac{m\left(r, \dfrac{1}{f(g)}\right)}{T(r, f(g))} = 0.$$

This proves Theorem 2.

COROLLARY 2. *Let $f(z)$ and $g(z)$ be transcendental integral functions. Even if $a_0(\neq\infty)$ is a deficient value of $g(z)$, $f(a_0)$ is not always a deficient value of $f(g(z))$.*

For, by a result of Edrei and Fuchs [1], the function $g(z)$ constructed in the proof of Theorem 2 is of positive lower order, since $g(z)$ has a finite deficient value. Hence Theorem 2 implies this corollary.

4. Concerning the problem (B), we can prove the following theorem.

THEOREM 3. *Let $g(z)$ be a polynomial of degree n and $f(z)$ a transcen-*

dental meromorphic function of order μ_f *and of lower order* λ_f. *Assume that* $\mu_f-\lambda_f<1/n$. *Then, for any* w_0, *it holds that*

$$\delta(w_0, f(g))=\delta(w_0, f).$$

PROOF. Using the method due to Valiron [4], we obtain

$$(1-\varepsilon)\, N\left(s', \frac{1}{f-w_0}\right)-O(\log r)\leqq N\left(r, \frac{1}{f(g)-w_0}\right)$$
$$\leqq(1+\varepsilon)\, N\left(s, \frac{1}{f-w_0}\right)+O(\log r)$$

and

$$(1-\varepsilon)T(s', f)-O(\log\ r)\leqq T(r, f(g))\leqq(1+\varepsilon)T(s, f)+O(\log r)$$

for any $\varepsilon>0$ and for all sufficiently large values of r and we have

$$\lim_{r\to\infty}\frac{T(s, f)}{T(s', f)}=1$$

provided that $\mu_f-\lambda_f<1/n$, where $g(z)=a_0z^n+\cdots+a_n$, $|z|=r$, $|a_i|=\alpha_i$ $(i=0, 1, 2, \cdots, n)$, $s=\alpha_0r^n+\cdots+\alpha_n$ and $s'=\alpha_0r^n-\alpha_1r^{n-1}-\cdots-\alpha_n$. Hence

$$\limsup_{r\to\infty}\frac{N\left(r, \dfrac{1}{f(g)-w_0}\right)}{T(r, f(g))}\leqq\limsup_{r\to\infty}\frac{(1+\varepsilon)\, N\left(s, \dfrac{1}{f-w_0}\right)}{(1-\varepsilon)T(s, f)}\cdot\frac{T(s, f)}{T(s', f)}$$
$$=1-\delta(w_0, f).$$

Similarly we have

$$\limsup_{r\to\infty}\frac{N\left(r, \dfrac{1}{f(g)-w_0}\right)}{T(r, f(g))}\geqq 1-\delta(w_0, f).$$

Thus we obtain

$$\delta(w_0, f(g))=\delta(w_0, f).$$

As a direct consequence of Theorem 3, we have

COROLLARY 3. *Under the same condition as in Theorem 3, if* w_0 *is a deficient value of* $f(z)$, *then* w_0 *is also a deficient value of* $f(g(z))$.

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