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DIMENSION OF COMPACT GROUPS AND THEIR REPRESENTATIONS

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In Pontrjagin's duality theory of compact abelian groups, it is well known that:

Let G be a compact abelian group, G^* the dual group of G, dim G the topological dimension in the sense of Lebesgue. Then the dimension of G is equal to the rank of discrete group G^*.

By S. Takahashi in [6], two analogies for non-commutative cases are formulated as follows:

THEOREM A. *Let G be a compact group, $\hat{G}$ the aggregate of finite dimensional continuous representations of G, $R(\hat{G})$ the algebra over the complex number field C generated by the coefficients of representations in G, i. e., the representative ring in the sense of C. Chevalley in* [1]. *Then the dimension of G is equal to the transcendental degree of $R(\hat{G})$ over C.*

THEOREM B. *Let G be a compact group, $\overline{G}$ the space consisting of conjugacy classes of G, $R[G^*]$ the algebra over C genarated by characters of elements of $\hat{G}$, i. e., the character ring of G. Then the dimension of $\overline{G}$ is equal to the transcendental degree of $R[G^*]$ over C.*

Theorem A was solved affirmatively, but Theorem B was solved merely for compact connected Lie groups, (See S. Takahashi [6]).

In [5] the author has refered to Theorem B in the general case, but has not been able to solve it completely. Recently in [4], G. Segal gives several concepts and theorems which are useful for solving Theorem B, i. e., the concept of a Cartan subgroup of a compact Lie group, and its properties, (See § 2).

Now in this paper, we shall give an affirmative complete solution of Theorem B.

1. **On the dimension**. Let X be a normal space, $\{U_i\}$ a finite open covering

of X, then the order of $\{U_i\}$ is the largest integer n such that there are n members of the covering $\{U_i\}$ which have a non-empty intersection. A finite open covering $\{V_j\}$ is called a refinement of a covering $\{U_i\}$ if each member of $\{V_j\}$ is contained in some member of $\{U_i\}$. We recall that a space X has the dimension$\leqq n$, i. e., dim $X \leqq n$ if for any finite open covering $\{U_i\}$ of X, there is a refinement $\{V_j\}$ with the order$\leqq n+1$ of $\{U_i\}$.

Let X be a compact Hausdorff space, G a compact group and (G, X) a transformation group. Then it is well known that the orbit space X/G of X by G is a compact Hausdorff space, (cf. D. Montgomery, L. Zippin [3], p. 61).

LEMMA 1. *Let X be a compact Hausdorff space, G a finite group, (G, X) a transformation group, X/G the orbit space of X by G. Then we have* dim $X/G \geqq$ dim X. *Especially if X has the second countability axiom, then it follows equality.*

PROOF. 1) We shall prove the first part of the lemma. Let $\{U_i\}$ be any finite open covering of X, $x \in X$. Then there is an open set V_x of X, containing x such that:

a) for any $a, b \in G$, it follows $aV_x \cap bV_x = \phi$ or $aV_x = bV_x$

b) for any $a \in G$, there is a member U_i of $\{U_i\}$ such that $aV_x \subset U_i$.

Because, if we put $H_x = \{a \in G | ax = x\}$ and $G = a_1H_x + a_2H_x + \cdots + a_rH_x$ as the coset decomposition of G by H_x, then there is an open set V_x, containing x, such that $hV_x = V_x$ for $h \in H_x$, $a_iV_x \cap a_jV_x = \phi$ for $i \neq j$ and $a_iV_x \subset U_i$ for some member U_i of $\{U_i\}$.

Since $\{V_x\}_{x\in X}$ is an open covering of X, and X is compact, we can take a finite open covering $\{V_i\}$ of X such that $V_i \in \{V_x\}$ $x \in X$. If Φ is the natural mapping: $X \to X/G$ and we put $\Phi(V_i) = V'_i$, then $\{V'_i\}$ is a finite open covering of X/G.

Now assume dim $X/G \leqq n$, then there is a finite open covering $\{W'_k\}_{k\in \Lambda}$ of X/G which is a refinement of $\{V'_i\}$ and has the order$\leqq n+1$. If we take some V'_k for W'_k such that $W'_k \subset V'_k$, and put $\Phi^{-1}(W'_k) \cap V_k = W_k$, then it follows $aW_k = bW_k$ or $aW_k \cap bW_k = \phi$ for any a, $b \in G$, and the finite open covering $\{aW_k\}_{a\in G, k\in \Lambda}$ of X is a refinement of $\{U_i\}$ with the order$\leqq n+1$. Therefore it follows dim $X \leqq$ dim X/G.

2) If X has the second countability axiom, then X is a separable metric space, and we shall prove dim $X/G \leqq$ dim X, using the induction with respect to the order of G.

i) Assume that G has no non-trivial subgroups of G.
$X^G = \{x \in X | ax = x$ for all $a \in G\}$ is closed in X/G, and each element of G operates on $Y = X - X^G$, thus Y/G is an open subset of X/G.

For any $y \in Y$, there is an open set U of Y containing y, which is homeomorphic to its image $\bar{U}$ by the canonical mapping $Y \to Y/G$.

Therefore we have $\dim Y/G = \dim Y \leqq \dim X$. Since by the assumption of G, it follows $X/G = X^G \cup Y/G$, we have $\dim X/G \leqq \dim X$.

ii) When G has non-trivial normal subgroup S, then each element of $\Gamma = G/S$ operates on X/S and we have $X/G = X/S/\Gamma$. By the induction it follows $\dim X/G \leqq \dim X$.

iii) Assume that G has a non-trivial non-normal subgroup S.
Each element of the normalizer $N(S)$ of S in G operates on X^S. If we put $X'_S = \bigcup_{g \in G} X^{gSg^{-1}}$, then it is closed in X, and each element of G operates in X'_S, thus X'_S/G is closed in X/G. Since $X^S/N(S)$ is compact, and the natural mapping $X^S/N(S) \rightarrow X'_S/G$ is continuous and bijective, this mapping is homeomorphic, thus we have $\dim X^S/N(S) = \dim X'_S/G$. Since $N(S) \neq G$, it follows $\dim X^S \geqq \dim X^S/N(S)$ by the induction, thus $\dim X \geqq \dim X'_S/G$. If we put $F = X^G \cup \left\{ \bigcup_{S \subset G} X'_S \right\}$ where S ranges over non-trivial subgroups of G, then $Y = X - F$ is open and as i) we have $\dim Y/G \leqq \dim X$. Since X/G is equal to $F/G \cup Y/G$, it follows $\dim X/G \leqq \dim X$.

LEMMA 2. *Let X be a compact Hausdorff space, G a compact group with* $\dim G = 0$, (G, X) *a tranformation group. Then* $\dim X \leqq \dim X/G$.

PROOF. Assume $\dim X \geqq n$, then there is a finite open covering $\{U_i\}$ of X such that all refinements of $\{U_i\}$ have the order $\geqq n+1$. When for any $x \in X$, we select a member U_i of $\{U_i\}$ containing x, then there is an open neighborhood 0_x of unit element in G, and V_x of x in X such that $0_x \cdot V_x \subset U_i$.
Since G is 0-dimensional, there is an open closed normal subgroup H_x of G such that $H_x \subset 0_x$. Thus $\{H_x \cdot V_x\}_{x \in X}$ is an open covering of X, therefore we can select a finite open covering $\{H_i \cdot V_i\}_{i \in A}$ of X from members of $\{H_x \cdot V_x\}$ which is a refinement of $\{U_i\}$. Since $H = \bigcap_{i \in \Lambda} H_i$ is an open normal subgroup of G, $\Gamma = G/H$ is a finite group and we have $X/G = X/H/\Gamma$. By Lemma 1, it follows $\dim X/G \geqq \dim X/H$.
Thus it is sufficient to prove $\dim X/H \geqq \dim X$. Let φ be the canonical mapping of X to X/H, $V'_i = \varphi(H_i x_i)$, then $\{V'_i\}$ is a finite open covering of X/H, and we have $\varphi^{-1}(V'_i) = H_i V_i$. Now if $\{W'_j\}$ is any refinement of $\{V'_i\}$ with the order $\leqq n$, and we put $\varphi^{-1}(W'_j) = W_j$, then $\{W_j\}$ is a finite open covering of X with the order $\leqq n$, and a refinement of $\{U_i\}$. It contradicts to the assumption of $\{U_i\}$. Thus we have $\dim X/H \geqq n$, that is, $\dim X/H \geqq \dim X$.

2. On the case of compact Lie groups. In this section, we shall assume that G is a compact Lie group. In the beginning we shall give some notions introduced by G. Segal in [3].

Let G^0 be the identity connected component of a compact Lie group G, then $G/G^0=\Gamma$ is a finite group. C is called a cyclic group of G if C is the closure of a subgroup generated by a non-unit element of G. Then a cyclic group C is the product of a finite cyclic group and a torus.

DEFINITION. A subgroup S of G is a Cartan subgroup if it is cyclic and of finite index in its normalizer $N(S)$. The finite group $N(S)/S$ is called the Weyl group of S, and is denoted by W_S.

PROPOSITION 1. *Each element g of G is contained in a Cartan subgroup S.*

PROPOSITION 2. *The projection*

$$\{\textit{Cartan subgroups of } G\} \to \{\textit{cyclic subgroups of } \Gamma\}$$

induces a bijection of conjugacy classe.

Let S be a Cartan subgroup of G and S^* the set consisting of elements $g \in S$ such that gS^0 generates S/S^0, S^*/W_S the orbit space of S^* by W_S. Then the following holds:

PROPOSITION 3. *If $\overline{G}$ is the space consisting of conjugacy classes of G, and $\overline{\Gamma}$ is that of Γ, then in the projection $\overline{G}\to\overline{\Gamma}$, the inverese image of a conjugacy class γ is homeomorphic to S^*/W_S, where S is a Cartan subgroup of G with a generator in γ.*

See G. Segal [3] for proofs of Propositions 1, 2, and 3,

PROPOSITION 4. *Let $\overline{G}$ be as above, T a maximal torus of G. Then we have* $\dim \overline{G}=\dim T$.

PROOF. We have $\overline{G}=\bigcup_S S^*/W_S$ where S ranges over non-conjugate Cartan subgroups of G. By Lemma 1, it follows $\dim S^*/W_S=\dim S^*=\dim S^0$, thus $\dim \overline{G}$ is equal to the largest of $\dim S^0$, that is, $\dim \overline{G}=\dim T$.

Now we shall refer to the character ring of a compact Lie group G. Firstly let G be an n-dimensional torus. Then the character ring $R[G^*]$ of G is isomorphic with $C[X_1, X_2, \cdots, X_n, (X_1X_2\cdots X_n)^{-1}]$, therefore $\dim \overline{G}$ is equal to the trascendental degree of $R[G^*]$ over C.

PROPOSITION 5. *Let $R[G^*]$ be the character ring of a compact Lie* *group G, $<R[G^*]; C>$ the transcendental degree of $R[G^*]$ over C, T a maximal torus of G. Then we have* $<R[G^*]; C> = \dim T$.

PROOF. Let φ be the canonical mapping of $R[G^*]$ to the direct product ring $\prod_S R[S^*]$ where S ranges over non-conjugate Cartan subgroups of G. By Proposition 1, φ is injective, therefore we have $<R[G^*]; C> \leqq <\prod_S R[S^*]; C> = \max <R[S^*]; C> = <R[T^*]; C> = \dim T$. Now we shall prove $<R[G^*]; C> \geqq \dim T$. Let χ be a character element of $R[G^*]$, D a representation of G with degree n associated with χ, where the restricition D_T of D to T is diagonal. If $M = C^n$ is the standard G-module, then the k-exterior power $\Lambda^k M$ is a G-module. If $\xi_1, \xi_2, \cdots, \xi_n$ are characteristic roots of D_T, then the character s_k of the representation of T by $\Lambda^k M$, is the k-th symmetric function of $\xi_1, \xi_2, \cdots, \xi_n$, thus, $\xi_1, \xi_2, \cdots, \xi_n$ are roots of the following equation;

$$X^n - s_1 X^{n-1} + \cdots + (-1)^n s_n = 0 .$$

Let $R[G^*]_T$ be the algebra consisting of restrictions of elements of $R[G^*]$ to T, then $s_1, s_2, \cdots, s_n$ belong to $R[G^*]_T$.

Thus, since $R[T^*]$ is an integral domain and any character element of $R[T^*]$ is prolonged to a character element of $R[G^*]$, $R[T^*]$ is algebraic over $R[G^*]_T$. Therefore we have $<R[G^*]; C> \geqq <R[G^*]_T; C> = <R[T^*]; C> = \dim T$.

Thus,we obtain the main theorem for a compact Lie group.

THEOREM 1. *Let G be a compact Lie group. Then we have* $\dim \overline{G} = <R[G^*]; C>$.

3. On the case of compact topological groups. In this section, let G be a compact topological group. Then it is well known that G is the projective limit of compact Lie groups;

$$G = \varprojlim G_\alpha, \ \alpha \in \Lambda . \qquad \text{(a)}$$

where G_α are compact Lie groups, the index set Λ is an ordered set, and for any finite elements $\alpha_1, \alpha_2, \cdots, \alpha_n$ of Λ there is an element $\alpha \in \Lambda$ such that $\alpha_i \leqq \alpha$ for $i = 1, 2, \cdots, n$. Let $\overline{G}$, $\overline{G}_\alpha$ be the spaces consisting of conjugacy classes of G, G_α, respectively. Then it is clear that $\overline{G}$ is the projective limit of $\overline{G}_\alpha$ induced by (a);

$$\overline{G} = \varprojlim \overline{G}_\alpha, \ \alpha \in \Lambda . \qquad \text{(b)}$$

Let $\bar{f}_\alpha$; $\overline{G} \to \overline{G}_\alpha$ be the continuous mapping induced by the homomorphism f_α; $G \to G_\alpha$ and put $\overline{V} = \{\bar{x} \in \overline{G} | \bar{f}_\alpha(\bar{x}) \in \overline{V}_\alpha$ for some open set $\overline{V}_\alpha$ of $\overline{G}_\alpha\}$, then the aggregate $\{\overline{V}\}$ is an open bases of $\overline{G}$.

LEMMA 3. *Let G be a compact group and $\overline{G}=\varprojlim \overline{G}_\alpha$, $\alpha\in\Lambda$. Then we obtain* $\dim \overline{G}\leqq\max_{\alpha\in\Lambda} \dim \overline{G}_\alpha$.

PROOF. Let $\{\overline{V}_i\}$ be a finite covering cosisting of members of open bases of $\overline{G}$. Then there is an element β of Λ such that $\bar{x}$ belongs to $\overline{V}_i$ if and only if $\bar{x}_\beta$ belongs to $\overline{V}_{\beta,i}$ where $\bar{x}_\beta$, $\overline{V}_{\beta,i}$ are images of $\bar{x}$, $\overline{V}_i$ by $\bar{f}_\beta$ respectively. Now assume $\max_{\alpha\in\Lambda} \dim \overline{G}_\alpha\leqq p$, then there is a finite open covering $\{\overline{U}_j'\}$ with the order $\leqq p+1$ which is a refinemet of $\{\overline{V}_{\beta,i}\}$. Thus, putting $\bar{f}_\beta^{-1}(\overline{U}_j')=\overline{U}_j$, $\{\overline{U}_j\}$ is a refinement of $\{\overline{V}_i\}$ with the order$\leqq p+1$. Therefore we obtain $\dim \overline{G}\leqq p$, i. e., $\dim G\leqq\max_{\alpha\in\Lambda} \dim \overline{G}_\alpha$.

LEMMA 4. *Let G, G_α, $\alpha\in\Lambda$ be as in Lemma 3. Then we have* $\dim \overline{G}\geqq\dim \overline{G}_\alpha$.

PROOF. Let G^0 be the identity connected component of G. Then each element of a finite group $\Gamma=G/G^0$ operates on the space $\overline{G}^0$ consisting of conjugacy classes of G^0, and the orbit space $\overline{G}^0/\Gamma$ of $\overline{G}^0$ by Γ is a closed subset of $\overline{G}$. By Lemma 2, we have the following;

$$\dim \overline{G}\geqq \dim \overline{G}^0/\Gamma\geqq \dim \overline{G}^0\,. \qquad \text{(a)}$$

On the other hand, it is well known that G^0 is isomorphic with $(S\times H)/Z$ where S is a simply connected semi-simple Lie group, H a connected abelian group, $S\times H$ the direct product, Z a finite subgroup of the center of $S\times H$, and $H\cap Z=\{e\}$. Thus $\overline{G}^0$ is homemorphic with the orbit space $(\overline{S\times H})/Z$ of $\overline{S\times H}$ by Z. If T is a maximal torus of S, and $W=N_S(T)/T$ the Weyl group of S with respect to T, then each element w of W operates on $T\times H$ as $w(t, h)=(wt, h)$ for each element $(t, h)\in T\times H$. Thus $\overline{S\times H}$ is homemorphic with the orbit space $(T\times H)/W$ of $T\times H$ by W. Using Lemma 2, we get the following;

$$\dim \overline{G}^0\geqq \dim(T\times H)\,. \qquad \text{(b)}$$

If f, f_α are homomorphisms such that $f: S\times H\rightarrow G^0, f_\alpha: G\rightarrow G_\alpha$, then $T_\alpha=f_\alpha(T\times H)$ is a maximal torus of G_α, and we get $\dim (T\times H)\geqq\dim T_\alpha$. Thus by Proposition 3, we have the following:

$$\dim\ (T\times H)\geqq \dim \overline{G}_\alpha\,. \qquad \text{(c)}$$

Therefore by (a), (b), (c), it follows

$$\dim \overline{G} \geqq \dim \overline{G}_\alpha, \text{ for } \alpha \in \Lambda.$$

On the other, hand we see easily that the character ring $R[G^*]$ is isomorphic with the injective limit of the character rings $R[G_\alpha^*]$, $\alpha \in \Lambda$, thus we get $<R[G^*]; C> = \max_{\alpha \in \Lambda} <R[G_\alpha^*]; C>$. Therefore using Theorem 1, and Lemma 4, we obtain the following main theorem for compact groups:

THEOREM 2. *Let G be a compact group, $\overline{G}$ the space consisting of conjugacy classes of G. Then the dimension of $\overline{G}$ is equal to the transcendental degree of the character ring $R[G^*]$ of G over C.*

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NOTES.

1) Let X be compact Hausdorff space, G a compact group with $\dim G = 0$, (G, X) a transformation group, then the canonical map $X \to X/G$ is closed and open. Therefore the first assertion of Lemma 1 and Lemma 2 follow from the Hurwicz-Wallman's theorem: Let f be a closed mapping of a normal space X onto a non-empty paracompact Hausdorff space Y, then $\dim X \leqq \sup_{y \in Y} \dim f^{-1}(y) + \operatorname{Ind} Y$, where $\operatorname{Ind} Y$ is the large inductive dimension of Y. Especially with an additional condition $\sup_{y \in Y} \dim f^{-1}(y) = 0$, $\dim X \leqq \dim Y$ holds (cf. K. Nagami [2], p. 123-130).

Furthermore, the second assertion of Lemma 1 is already known in K. Nagami [2] p. 97.

2) As for Lemma 3, see K.Nagami [2] p. 162.

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