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RINGS HAVING DOMINANT MODULES

TOYONORI KATO

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Recently the notion of dominant modules has been introduced in Kato [9] prompted by Tachikawa [17] and then studied further in Kato [10]. In this paper we shall be concerned with a class of rings which includes the class of left perfect rings as well as the class of left S-rings, namely, rings having dominant left modules.

Section 1 is devoted to illustrative examples of such rings, most of which are quoted from [9].

On the other hand, there appeared in Morita [13, 15] (cf. Jans [5]) the following condition on a ring R

$$\text{Hom}(\text{Ext}^1({}_RX, {}_RR)_R, E(R_R)) = 0 \tag{2}$$

for (finitely generated) ${}_RX \in {}_R\mathcal{M}$, where and throughout this paper, $E(\ )$ will denote the injective hull, and ${}_R\mathcal{M}$ the category of left R-modules.

For the class of rings having dominant left modules, this condition (2) characterizes left QF-3 rings[1]; the proof of this theorem is given in Section 2. The point of this theorem is that the converse of Morita [13, Theorem 4.1] holds.

It was Lambek [11] who pointed out for the first time that Utumi's maximal right quotient ring of a ring R (cf. Utumi [19]) is the bicommutator of $E(R_R)$. In what follows, let Q be Utumi-Lambek maximal right quotient ring of a ring R. If R has a dominant left module, so does Q (Example 8 in Section 3). This observation leads us to investigate the situation when Q has a dominant left module. The purpose of Section 3, the final section, is to examine this situation entirely based on Morita [14]. It is shown in Theorem 2 that Q has a dominant left module if and only if there exists a module ${}_RU$ such that

(i) ${}_RU$ is of type FP.

(ii) ${}_RU$ is faithful and flat.

(iii) U_S is lower distinguished, where $S = \text{End}({}_RU)$.

For an illustrative example of this situation, let $R = Z$ be the ring of integers and ${}_RU = {}_ZQ$ the rational number field. In this connection, if

[1] A ring R is called left QF-3 if $E({}_RR)$ is torsionless (cf. Colby and Rutter [4], Tachikawa [17] and Kato [6, 7]).

${}_QU$ is dominant, then

$$\mathrm{Hom}({}_RY, {}_RQ) \otimes {}_RU \approx \mathrm{Hom}({}_RY, {}_RU)$$

canonically for ${}_RY \in {}_R\mathscr{M}$, and

$$\mathrm{Hom}({}_RY, {}_RR) \otimes {}_RU \approx \mathrm{Hom}({}_RY, {}_RU)$$

canonically for finitely generated ${}_RY \in {}_R\mathscr{M}$, as is shown in Lemma 4. Theorem 3 discusses the situation when ${}_RU$ is injective for a dominant module ${}_QU$. Among other things it is shown that, if there exists a dominant module ${}_QU$ such that ${}_RU$ is injective, then the condition (2) above holds for all finitely generated modules ${}_RX$. Theorem 3 contains the converse part of Morita [15, Theorem 2] for the class of left Noetherian rings R for which Q has dominant left modules as well.

Throughout this paper, rings R will have unity element and modules will be unital. ${}_RX$ will signify the fact that X is a left R-module. As a matter of course, homomorphisms of modules will operate on the side opposite to the scalars.

1. Introduction to dominant modules. A faithful, finitely generated, projective module ${}_RU$ is called dominant if U_S is lower distinguished[2], where $S = \mathrm{End}({}_RU)$ is the endomorphism ring of ${}_RU$ (cf. Kato [9]). In this paper we are mainly concerned with rings having dominant modules, and so let us survey such rings by illustrative examples:

EXAMPLE 1. A progenerator ${}_RU$[3] is dominant if and only if R_R is lower distinguished.

This follows from the Morita equivalence $\mathscr{M}_S \sim \mathscr{M}_R$, $S = \mathrm{End}({}_RU)$.

The following example is an analogue of [9, Example 3] (cf. Morita [14, Theorem 8.2]).

EXAMPLE 2. R has a dominant left module and $E(R_R)$-domi. $\dim R_R \geqq 2$[4] if and only if R is the endomorphism ring of a lower distinguished generator for $\mathscr{M}_S$, where S is a ring.

EXAMPLE 3 (Kato [9, Example 4]). If R is a semi-perfect ring with the essential right socle, then R has a dominant left module. Thus left perfect rings as well as semi-primary rings have always dominant left modules.

EXAMPLE 4. The ring Z of integers has no dominant module.

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[2] U_S contains a copy of each simple right S-module (cf. Azumaya [1]).

[3] ${}_RU$ is a finitely generated projective generator for ${}_R\mathscr{M}$ (cf. Bass [2]).

[4] $E(R_R)/R \subsetneq \Pi E(R_R)$ (cf. Tachikawa [17, 18], Morita [14] and Kato [8]).

Azumaya's observation [1, Theorem 8] and Example 1 above will serve a verification of this example.

EXAMPLE 5. Let R be an infinite direct product of fields. Then R has no dominant module, and yet R is a commutative, self-injective, regular ring (cf. [9, Example 2]).

2. Characterization of QF-3 rings. In this section we are chiefly concerned with rings R having dominant left modules, and then give a characterization of left QF-3 rings in terms of the condition (2) mentioned in Introduction.

LEMMA 1. *Let ${}_RU$ be a dominant module. Then $E({}_RR)$ is torsionless if and only if ${}_RU$ is injective.*

PROOF. The "if" part follows directly from Kato [6, Proposition 1]. To show the "only if" part, suppose $E({}_RR)$ is torsionless. We observe first that $E({}_RU)$ is U-torsionless. Indeed, since ${}_RU \subsetneq \prod {}_RR \subsetneq \prod E({}_RR)$, $E({}_RR) \subsetneq \prod {}_RR$, and ${}_RR \subsetneq \prod {}_RU$ by assumption,

$$E({}_RU) \subsetneq \prod E({}_RR) \subsetneq \prod {}_RR \subsetneq \prod {}_RU .$$

Observe next that U_S is lower distinguished, where $S = \mathrm{End}({}_RU)$. Thus, according to Onodera [16, Lemma 4.4][5], ${}_RU$ is injective.

LEMMA 2 (Kato [9]). *Let ${}_RU$ be faithful, finitely generated projective and $S = \mathrm{End}({}_RU)$. Then*

$$\mathrm{Hom}(U_S, E(U_S))_R = E(R_R)^{6)} .$$

LEMMA 3 (Morita [15, Theorem 2′])[7]. *If R has a faithful, finitely generated projective, injective left module, then*

$$\mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR)_R, E(R_R)) = 0 \quad \text{for} \quad {}_RX \in {}_R\mathscr{M} .$$

REMARK. If R has a faithful, projective, injective left module, then

$$\mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR)_R, E(R_R)) = 0$$

for finitely generated ${}_RX \in {}_R\mathscr{M}$.

We shall sketch the proof. Given ${}_RU$ and ${}_RY$, there exists the canonical map

$$\alpha\colon \mathrm{Hom}({}_RY, {}_RR) \otimes {}_RU \longrightarrow \mathrm{Hom}({}_RY, {}_RU)$$

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5) By a slight modification of the proof of [6, Lemma 1], the author obtained this result independently.

6) The author is grateful to Dr. T. Onodera who showed him another simple proof (cf. forthcoming papers T. Onodera [Eine Bemerkung über Kogeneratoren] and T. Kato [U-distinguished modules]).

7) This has also been independently obtained by the author.

defined via

$$y((f \otimes u)\alpha) = (yf)u \quad \text{for} \quad y \in Y,\ f \in \mathrm{Hom}({}_RY, {}_RR),\ u \in U\,.$$

It is known that α is a monomorphism for ${}_RY \in {}_R\mathscr{M}$, if ${}_RU$ is projective. With this fact in mind, assume now that ${}_RU$ is faithful, projective, and injective. Then an exact sequence $0 \to {}_RY \to {}_RP \to {}_RX \to 0$ with ${}_RP$ finitely generated projective, gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}({}_RP, {}_RR) \otimes {}_RU & \longrightarrow & \mathrm{Hom}({}_RY, {}_RR) \otimes {}_RU & \longrightarrow & \mathrm{Ext}^1({}_RX, {}_RR) \otimes {}_RU & \longrightarrow & 0 \\
\wr & & \downarrow \alpha & & & & \\
\mathrm{Hom}({}_RP, {}_RU) & \longrightarrow & \mathrm{Hom}({}_RY, {}_RU) & \longrightarrow & 0\,. & &
\end{array}$$

Hence $\mathrm{Ext}^1({}_RX, {}_RR) \otimes {}_RU = 0$ since α is a monomorphism. On the other hand, since ${}_RU$ is faithful and projective,

$$E(R_R) \subsetneq \mathrm{Hom}(U_S, E(U_S))_R\,; \qquad S = \mathrm{End}({}_RU)\,.$$

It thus follows

$$\begin{aligned}
\mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR)_R, E(R_R)) &\subsetneq \mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR)_R, \mathrm{Hom}(U_S, E(U_S))_R) \\
&\approx \mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR) \otimes {}_RU_S, E(U_S)) = 0\,.
\end{aligned}$$

We are now ready for our main theorem.

THEOREM 1. *If R has a dominant left module, then the following conditions are equivalent:*

(1) *$E({}_RR)$ is torsionless.*

(2) $\mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR)_R, E(R_R)) = 0$ *for* ${}_RX \in {}_R\mathscr{M}$.

(2′) $\mathrm{Hom}(\mathrm{Ext}^1({}_RR, {}_RR)_R, E(R_R)) = 0$ *for finitely generated* ${}_RX \in {}_R\mathscr{M}$.

PROOF. (1) $\Rightarrow$ (2). Let ${}_RU$ be a dominant module. Since $E({}_RR)$ is torsionless, ${}_RU$ is injective by Lemma 1. Now, ${}_RU$ is faithful, finitely generated projective, and injective. Thus the condition (2) follows at once from Lemma 3.

(2) $\Rightarrow$ (2′) is trivial.

(2′) $\Rightarrow$ (1). It suffices to show that ${}_RU$ is injective, where ${}_RU$ is dominant, in view of Lemma 1. Let $0 \to {}_RY \to {}_RP \to {}_RX \to 0$ be an exact sequence with ${}_RP$ finitely generated projective. In the same manner as above, we have the following exact commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}({}_RP, {}_RR) \otimes {}_RU & \longrightarrow & \mathrm{Hom}({}_RY, {}_RR) \otimes {}_RU & \longrightarrow & \mathrm{Ext}^1({}_RX, {}_RR) \otimes {}_RU & \longrightarrow & 0 \\
\wr\alpha & & \wr\alpha & & & & \\
\mathrm{Hom}({}_RP, {}_RU) & \longrightarrow & \mathrm{Hom}({}_RY, {}_RU)\,, & & & &
\end{array}$$

where the vertical maps α are isomorphisms by the finitely generated projectivity of ${}_RU$ (cf. Morita [12, Lemma 7.1]). Here

$$\mathrm{Ext}^1({}_RX, {}_RR) \otimes {}_RU = 0 .$$

In fact,

$$\begin{aligned}\mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR) \otimes {}_RU_S, E(U_S)) &\approx \mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR)_R, \mathrm{Hom}(U_S, E(U_S))_R) \\ &\approx \mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR)_R, E(R_R)) = 0 ; \qquad S = \mathrm{End}({}_RU)\end{aligned}$$

making use of Lemma 2 and the condition (2′). However $E(U_S)$ is a cogenerator for $\mathscr{M}_S$ since ${}_RU$ is dominant. Therefore $\mathrm{Ext}^1({}_RX, {}_RR) \otimes {}_RU = 0$. It now follows from the above diagram that the induced map $\mathrm{Hom}({}_RP, {}_RU) \to \mathrm{Hom}({}_RY, {}_RU)$ is an epimorphism. We have thus established the injectivity of ${}_RU$.

REMARK. As we mentioned in Introduction, Theorem 1 is an improvement on Morita [13, Theorem 4.1], in view of Example 3 in Section 1.

The following two examples show that the "dominant" hypothesis is important in Theorem 1.

EXAMPLE 6. According to Morita [15, Theorem 2] (cf. Theorem 3), the ring Z of integers satisfies the condition (2′) above, whereas $E({}_ZZ)$ is not torsionless.

EXAMPLE 7[8]. As is stated just above, the ring Z fulfils the condition (2′), but not the condition (2). In fact, let

$${}_ZX = \bigoplus_{n=2}^{\infty} Z/nZ .$$

Then one verifies easily that

$$\mathrm{Ext}^1({}_ZX, {}_ZZ) \approx \prod_{n=2}^{\infty} \mathrm{Ext}^1(Z/nZ, {}_ZZ) \approx \prod_{n=2}^{\infty} Z/nZ .$$

Thus

$$\mathrm{Hom}(\mathrm{Ext}^1({}_ZX, {}_ZZ)_Z, E(Z_Z)) = \mathrm{Hom}(\prod_{n=2}^{\infty} Z/nZ, Q_Z) \neq 0 ,$$

where Q is the rational number field.

3. Dominant modules over maximal quotient rings. In what follows, let R be a ring and Q Utumi-Lambek maximal right quotient ring of R (cf. Lambek [11]). In this section we deal with rings R for which Q has a dominant left module.

EXAMPLE 8. If R has a dominant left module, so does Q.

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8) The author is indebted to Dr. K. Uchida for this example.

Indeed, let ${}_RU$ be dominant and $S = \mathrm{End}({}_RU)$. Then $Q = \mathrm{End}(U_S)$ is Utumi-Lambek maximal right quotient ring of R by Kato [10, Corollary 5]. Thus ${}_QU$ is dominant since U_S is a lower distinguished generator for $\mathcal{M}_S$ (cf. Example 2).

The following theorem is entirely based on Morita [14].

THEOREM 2. *Let R be a ring and Q Utumi-Lambek maximal right quotient ring of R. Then the following conditions are equivalent:*

(1) *Q has a dominant left module.*

(2) *There exists a module ${}_RU$ such that*

 (i) *${}_RU$ is of type FP*[9],

 (ii) *${}_RU$ is faithful and flat,*

 (iii) *U_S is lower distinguished, where $S = \mathrm{End}({}_RU)$.*

PROOF. (1) $\Rightarrow$ (2). Let ${}_QU$ be dominant and $S = \mathrm{End}({}_QU)$. We shall now show that ${}_RU$ satisfies (i), (ii), and (iii). By Lemma 2 and Lambek [11]

$$\mathrm{Hom}(U_S, E(U_S))_Q = E(Q_Q) = E(R_R) .$$

Hence ${}_RU$ is flat by Morita [14, Lemma 1.3], since $E(U_S)$ is an injective cogenerator for $\mathcal{M}_S$. On the other hand, since Q is Utumi-Lambek maximal right quotient ring of R,

$$\begin{aligned}\mathrm{Hom}(Q/R \otimes {}_RU_S, E(U_S)) &\approx \mathrm{Hom}(Q/R, \mathrm{Hom}(U_S, E(U_S))_R) \\ &\approx \mathrm{Hom}(Q/R, E(R_R)) = 0 .\end{aligned}$$

It follows that $Q/R \otimes {}_RU = 0$. Since ${}_RU$ is flat, the exact sequence $0 \to R_R \to Q_R \to Q/R \to 0$ induces an exact sequence

$$0 \longrightarrow R \otimes {}_RU \longrightarrow Q \otimes {}_RU \longrightarrow Q/R \otimes {}_RU = 0 .$$

Thus

$${}_QU_S \approx {}_QQ \otimes {}_RU_S .$$

Furthermore U_S is a generator for $\mathcal{M}_S$ and $Q = \mathrm{End}(U_S)$. Thus, applying Morita [14, Theorem 1,1] we conclude that ${}_RU$ is of type FP and $S = \mathrm{End}({}_RU)$.

(2) $\Rightarrow$ (1). Suppose ${}_RU$ satisfies (i), (ii), and (iii). Let $S = \mathrm{End}({}_RU)$ and $R' = \mathrm{End}(U_S)$. From the flatness of ${}_RU$, it follows that

$$E(R'_R) \subsetneq \mathrm{Hom}(U_S, E(U_S))_R ,$$

and hence

$$\begin{aligned}\mathrm{Hom}(R'/R, E(R'_R)) &\subsetneq \mathrm{Hom}\,(R'/R, \mathrm{Hom}(U_S, E(U_S))_R) \\ &\approx \mathrm{Hom}(R'/R \otimes {}_RU_S, E(U_S)) = 0 ,\end{aligned}$$

[9] For the definition, see Morita [14, §1].

for, ${}_RU$ is of type FP. This implies that R'_R is a rational extension of R_R. Moreover

$$E(R'_{R'})\text{-domi. dim } R'_{R'} \geqq 2 ,$$

since U_S is a lower distinguished generator for $\mathscr{M}_S$ (cf. Morita [14, Theorem 8.2]). Thus $R' = Q$ (cf. Tachikawa [18, Corollary 2]), and so ${}_QU$ is dominant.

REMARK. Q has a dominant left module if and only if, $\mathscr{L}(E(R_R))$, the full subcategory of $\mathscr{M}_R$ consisting of all modules having $E(R_R)$-dominant dimension $\geqq 2$, is equivalent to $\mathscr{M}_S$ for a ring S by Kato [10, Corollary 2] (cf. Morita [14], Tachikawa [17, 18], and Kato [7, 9]).

EXAMPLE 9. Let $R = Z$ be the ring of integers and Q the rational number field. Then there exists an equivalence

$$\mathscr{L}(E(Z_Z)) = \mathscr{L}(Q_Z) \sim \mathscr{M}_Q .$$

LEMMA 4. *Let R be a ring and Q Utumi-Lambek maximal right quotient ring of R. Suppose Q has a dominant module ${}_QU$. Then*

(1) $T \otimes {}_RU = 0 \Leftrightarrow \text{Hom}(T_R, E(R_R)) = 0$ *for* $T_R \in \mathscr{M}_R$.

(2) $\text{Hom}(\text{Hom}({}_RY, Q/R)_R, E(R_R)) = 0$ *for finitely generated* ${}_RY \in {}_R\mathscr{M}$.

(3) $\text{Hom}({}_RY, {}_RQ) \otimes {}_RU \approx \text{Hom}({}_RY, {}_RU)$ *canonically for* ${}_RY \in {}_R\mathscr{M}$.

(3′) $\text{Ext}^1({}_RX, {}_RQ) \otimes {}_RU \approx \text{Ext}^1({}_RX, {}_RU)$ *for* ${}_RX \in {}_R\mathscr{M}$.

(4) *The canonical map*

$$\alpha\colon \text{Hom}({}_RY, {}_RR) \otimes {}_RU \longrightarrow \text{Hom}({}_RY, {}_RU)$$

is a monomorphism (resp. an isomorphism) for ${}_RY \in {}_R\mathscr{M}$ *(resp. for finitely generated* ${}_RY \in {}_R\mathscr{M}$*).*

(4′) *There exists a monomorphism (resp. an epimorphism)*

$$\text{Ext}^1({}_RX, {}_RR) \otimes {}_RU \longrightarrow \text{Ext}^1({}_RX, {}_RU)$$

for finitely generated ${}_RX \in {}_R\mathscr{M}$ *(resp. for finitely related*[10] ${}_RX \in {}_R\mathscr{M}$*).*

PROOF. Let $S = \text{End}({}_QU)$. Then $Q = \text{End}(U_S)$ and $S = \text{End}({}_RU)$ as in the above proof.

(1) follows from the isomorphisms

$$\text{Hom}(T \otimes {}_RU_S, E(U_S)) \approx \text{Hom}(T_R, \text{Hom}(U_S, E(U_S))_R) \approx \text{Hom}(T_R, E(R_R))$$

and from the fact that $E(U_S)$ is a cogenerator for $\mathscr{M}_S$.

(2)

$$\text{Hom}({}_RY, Q/R) \otimes {}_RU \subsetneq \text{Hom}({}_RY, Q/R \otimes {}_RU) = 0 ,$$

10) ${}_RX$ is called finitely related if there exists an exact sequence $0 \to {}_RY \to {}_RP \to {}_RX \to 0$ with ${}_RP$ projective (not necessarily finitely generated) and ${}_RY$ finitely generated.

for, ${}_RY$ is finitely generated and ${}_RU$ is flat by Theorem 2. It follows that $\mathrm{Hom}({}_RY, Q/R)\otimes_R U=0$, or equivalently,

$$\mathrm{Hom}(\mathrm{Hom}({}_RY, Q/R)_R, E(R_R))=0$$

in view of (1).

(3)

$$\begin{aligned}\mathrm{Hom}({}_RY, {}_RQ)\otimes_R U_S &\approx \mathrm{Hom}({}_RY, {}_R\mathrm{Hom}(U_S, U_S))\otimes_R U_S\\ &\approx \mathrm{Hom}(U_S, \mathrm{Hom}({}_RY, {}_RU)_S)\otimes_R U_S \approx \mathrm{Hom}({}_RY, {}_RU)_S\end{aligned}$$

canonically for ${}_RY\in\mathscr{M}$, since ${}_RU$ is of type FP by Theorem 2 (cf. Morita [14, Theorem 1.1]).

(3′) An exact sequence $0\to{}_RY\to{}_RP\to{}_RX\to 0$ with ${}_RP$ projective yields an exact commutative diagram

$$\begin{array}{ccccccc}\mathrm{Hom}({}_RP, {}_RQ)\otimes_R U & \longrightarrow & \mathrm{Hom}({}_RY, {}_RQ)\otimes_R U & \longrightarrow & \mathrm{Ext}^1({}_RX, {}_RQ)\otimes_R U & \longrightarrow & 0\\ \wr & & \wr & & & & \\ \mathrm{Hom}({}_RP, {}_RU) & \longrightarrow & \mathrm{Hom}({}_RY, {}_RU) & \longrightarrow & \mathrm{Ext}^1({}_RX, {}_RU) & \longrightarrow & 0\end{array}$$

with vertical maps isomorphisms by (3). Thus

$$\mathrm{Ext}^1({}_RX, {}_RQ)\otimes_R U\approx\mathrm{Ext}^1({}_RX, {}_RU)\quad\text{for}\quad {}_RX\in{}_R\mathscr{M}\,.$$

(4) Since ${}_RU$ is flat, the exact sequence $0\to R\to Q\to Q/R\to 0$ induces the exact commutative diagram for ${}_RY\in{}_R\mathscr{M}$

$$\begin{array}{ccccccc}0 \longrightarrow & \mathrm{Hom}({}_RY, {}_RR)\otimes_R U & \longrightarrow & \mathrm{Hom}({}_RY, {}_RQ)\otimes_R U & \longrightarrow & \mathrm{Hom}({}_RY, Q/R)\otimes_R U\\ & \downarrow\alpha & & \wr & & \\ & \mathrm{Hom}({}_RY, {}_RU) & = & \mathrm{Hom}({}_RY, {}_RU) & & \end{array}$$

making use of (3). Hence α is a monomorphism for ${}_RY\in{}_R\mathscr{M}$ and an isomorphism for finitely generated ${}_RY\in{}_R\mathscr{M}$ by (1) and (2).

(4′) In the situation of (3′), consider the exact commutative diagram

$$\begin{array}{ccccccc}\mathrm{Hom}({}_RP, {}_RR)\otimes_R U & \longrightarrow & \mathrm{Hom}({}_RY, {}_RR)\otimes_R U & \longrightarrow & \mathrm{Ext}^1({}_RX, {}_RR)\otimes_R U & \longrightarrow & 0\\ \downarrow\alpha_P & & \downarrow\alpha_Y & & \downarrow & & \\ \mathrm{Hom}({}_RP, {}_RU) & \longrightarrow & \mathrm{Hom}({}_RY, {}_RU) & \longrightarrow & \mathrm{Ext}^1({}_RX, {}_RU) & \longrightarrow & 0\,.\end{array}$$

Each of the α's is a monomorphism and α_P (resp. α_Y) is an isomorphism if ${}_RP$ (resp. ${}_RY$) is finitely generated by (4). Thus (4′) follows from Five lemma.

REMARK. The statement (2) in Lemma 4 is still true without the assumption that Q has a dominant left module.

THEOREM 3. *Let R be a ring and Q Utumi-Lambek maximal right* *quotient ring of R. Assume Q has a dominant left module. Consider now the following conditions:*

(1) *If* ${}_QU$ *is dominant, then* ${}_RU$ *is injective.*

(1′) *There exists a dominant module* ${}_QU$ *such that* ${}_RU$ *is injective.*

(2) $\mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RQ)_R, E(R_R)) = 0$ *for* ${}_RX \in {}_R\mathcal{M}$.

(2′) $\mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RQ)_R, E(R_R)) = 0$ *for finitely generated* ${}_RX \in {}_R\mathcal{M}$.

(2″) $\mathrm{Hom}(\mathrm{Ext}^1({}_RX, {}_RR)_R, E(R_R)) = 0$ *for finitely generated* ${}_RX \in {}_R\mathcal{M}$.

(1″) *If* ${}_QU$ *is dominant, then* $\mathrm{Ext}^1({}_RX, {}_RU) = 0$ *for finitely presented* ${}_RX \in {}_R\mathcal{M}$.

(3) $E({}_RR)$ *is flat.*

Then (1) $\Leftrightarrow$ (1′) $\Leftrightarrow$ (2) $\Leftrightarrow$ (2′) $\Rightarrow$ (2″) $\Rightarrow$ (1″), *and if R is left Noetherian they all are equivalent.*

PROOF. (1) $\Leftrightarrow$ (1′) $\Leftrightarrow$ (2) $\Leftrightarrow$ (2′) $\Rightarrow$ (2″) $\Rightarrow$ (1″) by Lemma 4.

From now on, suppose R is left Noetherian. Then

(1″) $\Rightarrow$ (1) is well-known.

(1′) $\Rightarrow$ (3). Since ${}_RU$ is faithful and injective,

$$E({}_RR) \hookrightarrow \prod {}_RU .$$

Hence $E({}_RR)$ is flat by Theorem 2 and Cartan and Eilenberg [3, Exercise 4, p. 122].

(3) $\Rightarrow$ (2″) is due to Morita [15, Theorem 2].

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DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
TÔHOKU UNIVERSITY
KAWAUCHI, SENDAI, JAPAN