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TORSION ASSOCIATED WITH DUALITY

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In this note we shall examine certain characteristic submodules that arise from the study of the following two conditions.

(A) The class of torsionless left R-module is closed under extension,

(B) The class of left R-modules with zero duals is closed under taking submodules.

By the dual of an R-module M we mean $M^* = \mathrm{Hom}_R(M, R)$. A module M is torsionless if $0 = \bigcap_{f \in M^*} \mathrm{Ker}\, f$. In [9] it was shown, that for a ring with minimum condition, the ring has conditions (A) and (B) if and only if it is a QF-3 ring. Recently, several generalizations of QF-3 rings have been considered [1, 4, 5, 7, 8]. Colby and Rutter [1] give the following:

DEFINITION. The ring R is left QF-3$'$ if the minimal injective $Q({}_RR)$ of ${}_RR$ is torsionless.

Kato [4] observed that the proof in [9] showed that a ring is left QF-3$'$ if and only if it satisfies conditions (A) and (B). No chain conditions are needed in this case.

We wish to relate conditions (A) and (B) to torsion theories in the sense of Dickson [2]. Such a torsion theory is a pair $(\mathfrak{T}, \mathfrak{F})$ of classes of R-modules such that:

1. $\mathfrak{T}$ and $\mathfrak{F}$ have only zero in common
2. $\mathfrak{T}$ is closed under taking factors and $\mathfrak{F}$ is closed under taking submodules
3. For each R-module M there is a unique submodule $T(M)$ such that

$$0 \to T(M) \to M \to M/T(M) \to 0$$

is exact with $T(M) \in \mathfrak{T}$ $M/T(M) \in \mathfrak{F}$.

$T(M)$ is the torsion submodule of M for the particular torsion theory $(\mathfrak{T}, \mathfrak{F})$. It is not hard to see that T is idempotent in the sense that $T(T(M)) = T(M)$ for all M. Also one can show that each class $\mathfrak{T}$ and $\mathfrak{F}$ is closed under taking extensions.

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Now consider

$$K(M)=\bigcap_{f\in M^*}\operatorname{Ker} f\,.$$

Note that K is not the torsion functor for a torsion theory because it is not, in general, idempotent.

We can get a torsion theory from $K(M)$ if we define a chain of submodules inductively as follows: Let $K^0(M)=M$ and for an ordinal α, let $K^{\alpha+1}(M)=K(K^\alpha(M))$. If β is a limit ordinal let $K^\beta(M)=\bigcap_{\alpha<\beta}K^\alpha(M)$.

The usual cardinality argument shows that there is an ordinal α such that $K^\alpha(M)=K^{\alpha+1}(M)$ (and $K^\alpha(M)=K^\gamma(M)$ for all $\gamma\geqq\alpha$ also).

We let $I(M)=K^\alpha(M)$ for that α and we give some of the properties of $I(M)$ in the following theorem.

THEOREM 1. *$I(M)$ is the unique largest submodule of M with zero dual and containing all submodules of M having zero dual.*

PROOF. From the definition of $K^{\alpha+1}(M)$ it is clear that $K^\alpha(M)=K^{\alpha+1}(M)$ if and only if $K^\alpha(M)^*=0$, so $I(M)^*=0$.

Let S be a submodule of M such that $S^*=0$.

We show by induction that $S\subseteqq K^\beta(M)$ for all β. We assume that $S\subseteqq K^\alpha(M)$ for all $\alpha<\beta$. If $\beta=\alpha+1$ and $S\nsubseteqq K^{\alpha+1}(M)=K(K^\alpha(M))$ then there is an element $s\in S$ and $f\in(K^\alpha(M))^*$ such that $f(s)\neq 0$. But f restricted to S is nonzero and this contradicts the assumption that $S^*=0$. If β is a limit ordinal, this definition of $K^\beta(M)$ implies $S\subseteqq K^\beta(M)$.

It follows that $I(M)$ contains all submodules of M having zero duals and is therefore the unique largest such submodule.

We observe that I is a torsion functor in the sense of Dickson.

The following theorem relates K, I and some other properties.

THEOREM 2. *The following are equivalent*:

(1) *R satisfies condition* (A).

(2) *K is a torsion functor for a torsion theory.*

(3) *$K(M)=I(M)$ for all R-modules M.*

PROOF. If the class of torsionless modules is closed under extension then, for each M, we have the following exact sequence of torsionless

[1] The referee observed that, if $\mathfrak{T}=\{{}_RT\mid \operatorname{Hom}({}_RT,{}_RR)=0\}$ and $\mathfrak{F}=\{{}_RF\mid \operatorname{Hom}({}_RF',{}_RR)\neq 0$ for all nonzero submodules ${}_RF'$ of ${}_RF\}$, then $(\mathfrak{T},\mathfrak{F})$ is a torsion theory, R satisfies (A) if and only if $\mathfrak{F}=\{{}_RF\mid{}_RF$ is torsionless$\}$, and R satisfies (B) if and only if $\mathfrak{T}=\{{}_RT\mid \operatorname{Hom}({}_RT, Q({}_RR))=0\}$.

modules

$$0 \to K(M)/K^2(M) \to M/K^2(M) \to M/K(M) \to 0 \,.$$

But if $M/K^2(M)$ is torsionless then $K(M) \subseteq K^2(M)$ and we have equality $K(M) = K^2(M) = I(M)$. Thus (1) implies (3).

Since I is a torsion functor for a torsion theory it is clear that (3) implies (2).

Finally, if K is a torsion functor the torsionless modules are torsion free for the associated torsion theory and are closed under extension.

We consider another torsion theory. Let $T_0(M) = \bigcap_{f \in \mathrm{Hom}(M, Q(R))} \mathrm{Ker} f$ where $Q(R)$ is the minimal injective of R considered as a left module. This torsion theory has been studied in [3, 6]. The following theorem relates $T_0(M)$ and $I(M)$.

THEOREM 3. *$I(M) \supseteq T_0(M)$ for all M.*

PROOF. We show $T_0(M)^* = 0$ and apply Theorem 1. Suppose the contrary, then there exists $f \in T_0(M)^*$ and $x \in T_0(M)$ such that $f(x) \neq 0$. This gives a diagram

$$\begin{tikzcd}
0 \arrow[r] & T_0(M) \arrow[r] \arrow[d, "f"'] & M \arrow[ddl, dashed, "g"] \\
& R \arrow[d] & \\
& Q(R) &
\end{tikzcd}$$

and since $Q(R)$ is injective, there is a g making it commutative. But then $g(x) = f(x) \neq 0$ contradicting $x \in T_0(M) = \bigcap_{g \in \mathrm{Hom}(M, Q(R))} \mathrm{Ker}\, g$.

For the equality of $I(M)$ and $T_0(M)$ we have the following:

THEOREM 4. *The following are equivalent:*

(1) *R satisfies condition* (B).

(2) *$I(M) = T_0(M)$ for all R-modules M.*

PROOF. We show that (1) implies (2) by a contrapositive proof. Suppose (1) and $I(M) \neq T_0(M)$. Choose $x \notin T_0(M)$ such that $x \in I(M)$, then there is $f \in \mathrm{Home}\,(M, Q(R))$ such that $f(x) \neq 0$. Since $f(I(M)) \neq 0$, $f(I(M)) \cap R \neq 0$ because R is essential in $Q(R)$. This implies that $f^{-1}(f(I(M)) \cap R) \cap I(M) = M'$ is a submodule of $I(M)$ and $M'^* \neq 0$ because $f \mid M' \neq 0$ and $f \mid M' \in M'^*$. However, this contradicts (1).

The implication $(2) \to (1)$ follows from the fact that the class of torsion modules for the T_0 torsion theory are always closed under taking submodules and a module M has a zero dual if and only if $M = I(M)$.

COROLLARY. *The following are equivalent:*
(1) *R is left QF-3′.*
(2) *$K(M) = T_0(M)$ for all R-modules M.*

PROOF. We have in general that $K(M) \supseteqq I(M) \supseteqq T_0(M)$. By Theorem 2 $K(M) = I(M)$ is equivalent to condition (A) by Theorem 4, $I(M) = T_0(M)$ is equivalent to condition (B). The Corollary then follows from the theorem cited at the beginning.

REMARK 1. It might be of interest to study the ordinals α for which $K^{\alpha}(M) = I(M)$. From Theorem 2 it is clear that $\alpha \leqq 1$ for exactly those rings satisfying condition (A).

If R is a semiprimary ring such that for all $M \neq 0$, $M^* \neq 0$ (i.e. only zero has a zero dual), it can be shown that $K^{n+1}(M) = K^n(M) = 0$ for all R-modules M where n is the power of the radical that is zero.

REMARK 2. It is not hard to show that the submodules $K(M)$, $K^{\alpha}(M)$, $I(M)$ and $T_0(M)$ are also submodules for the R endomorphism ring acting on M.

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