

CLASSIFICATION OF A CONFORMALLY FLAT K -SPACE

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(Received Dec. 11, 1971; Revised Feb. 21, 1972)

1. Introduction. Let (M, F, g) be an n -dimensional ($n \geq 4$) almost-Hermitian manifold with almost-Hermitian structure (F, g) , i.e. with an almost-complex structure F and a positive definite Riemannian metric g satisfying $F^2 = -I$, $g(FX, FY) = g(X, Y)$ for all tangent vectors X and Y , where I denotes the identity transformation. If an almost-Hermitian manifold (M, F, g) satisfies $(\nabla_X F)Y + (\nabla_Y F)X = 0$ for all tangent vectors X and Y , where ∇_X denotes the operation of the covariant differentiation with respect to the Riemannian connection, then the manifold is called a K -space (or Tachibana space, nearly Kähler manifold). Of course, a Kählerian space is a K -space, but in the sequel, by a proper K -space we shall mean a K -space which is not Kählerian. It is well known that there does not exist a 4-dimensional proper K -space [1], [6]. For any tensor T on M , by the components of T we shall mean the ones with respect to a local coordinate system $\{x^i\}$, $1 \leq k, j, i, \dots \leq n$.

In a conformally flat space, the curvature tensor has the following form:

$$(1.1) \quad (n-2)R_{kji h} = g_{kh}R_{ji} - g_{jh}R_{ki} + R_{kh}g_{ji} - R_{jh}g_{ki} \\ - \frac{R}{n-1}(g_{kh}g_{ji} - g_{jh}g_{ki}),$$

where $R_{kji}{}^h$, $R_{ji} = R_{sji}{}^s$ and $R = g^{ji}R_{ji}$ are the Riemannian curvature tensor, Ricci tensor and scalar curvature respectively.

For a conformally flat K -space, we have already known that there exists no conformally flat proper K -space of dimension $\neq 6, 8, 10$, a 6-dimensional conformally flat proper K -space is a space of constant curvature and a conformally flat K -space is locally symmetric ($n \geq 6$) [5], [7]. A conformally flat K -space of dimension $n \geq 12$ is Kählerian and it is necessarily locally flat.

The main purpose of the present paper is to prove that there exists no conformally flat proper K -space of dimension 10 and the following

THEOREM. *Let (M, F, g) be an n -dimensional connected conformally flat proper K -space. Then it is (I) a 6-dimensional K -space of positive*

constant curvature or (II) locally of the form $(M_1, F_1, g_1) \times (M_2, F_2, g_2)$, where the former is a 6-dimensional K -space of positive constant curvature C and the latter is a 2-dimensional Kählerian space of negative constant curvature $-C$ and, (F_1, g_1) and (F_2, g_2) are the restrictions of (F, g) to M_1 and M_2 respectively.

Recently, we received an information from S. Tanno, in which he proved following result [8], stating that a 4-dimensional conformally flat Kählerian manifold is either locally flat, or locally, a product space of 2-dimensional Kählerian manifolds of constant curvature K and $-K$, respectively.

Taking account of the above result, we would conclude that the classification of conformally flat K -space is completed.

Now, to prove Theorem, we need the following

LEMMA 1.1. (Sumitomo [3]) *Let M be an n -dimensional ($n > 3$) conformally flat Riemannian space satisfying*

$$(1.2) \quad \nabla_k \nabla_k R_{ji} - \nabla_k \nabla_j R_{ki} = 0,$$

then each characteristic root of Ricci tensor must be one of the roots of the following equation:

$$(1.3) \quad \lambda^2 - \frac{R}{n-1} \lambda - \frac{1}{n} \left(R_{rs} R^{rs} - \frac{R^2}{n-1} \right) = 0.$$

From (1.3), we have

$$(1.4) \quad \begin{aligned} \lambda &= \frac{1}{2} \left\{ \frac{R}{n-1} \pm \left(\frac{R^2}{(n-1)^2} + \frac{4}{n} \left(R_{rs} R^{rs} - \frac{R^2}{n-1} \right) \right)^{1/2} \right\} \\ &= \frac{1}{2} \left\{ \frac{R}{n-1} \pm \left(\frac{4}{n} R_{rs} R^{rs} - \frac{3n-4}{n(n-1)^2} R^2 \right)^{1/2} \right\}. \end{aligned}$$

Now, let λ_1, λ_2 be the roots of the equation (1.3) and m be the multiplicity of λ_1 , then the multiplicity of λ_2 is $n - m$ where $0 \leq m \leq n$. As the scalar curvature R is a trace of Ricci tensor, we have

$$(1.5) \quad R = m\lambda_1 + (n - m)\lambda_2.$$

LEMMA 1.2. (Sekigawa-Takagi [2]) *Let M be an n -dimensional ($n \geq 3$) connected conformally flat Riemannian space satisfying the condition (1.2). If the Ricci form is non-degenerate and indefinite of signature $2m - n$ at least at one point of M , then M is a locally product space of an m -dimensional space of constant curvature C and an $(n - m)$ -dimensional space of constant curvature $-C$, where $1 < m < n - 1$.*

In §2, we shall prepare some lemma and identities for a K -space and a conformally flat K -space. §3 will be devoted to the proof of the theorem.

2. Some lemma and identities. In a K -space, we have the following identities [4]:

$$(2.1) \quad R_{ji}^* = R_{ij}^*, F_j^i R_k^j = F_k^j R_j^i, F_j^i R_k^{*j} = F_k^j R_j^{*i},$$

where $R_{ji}^* = (1/2)F^{kh}R_{khsi}F_j^s$,

$$(2.2) \quad R - R^* = \nabla_j F_{ih}(\nabla^j F^{ih}) = \text{constant}$$

where $R^* = g^{ji}R_{ji}^*$.

Recently, Takamatsu [6] proved the following

LEMMA. *In a K -space, we have*

$$(2.3) \quad (R_{ji} - R_{ji}^*)(R^{ji} - 5R^{*ji}) = 0.$$

Next, let M be a conformally flat K -space, then from (1.1), by making use of (2.1), we have

$$(2.4) \quad 2R_{ji} - (n-2)R_{ji}^* = \frac{R}{n-1}g_{ji},$$

or

$$(2.5) \quad R_{ji}^* = \frac{2}{n-2}R_{ji} - \frac{R}{(n-1)(n-2)}g_{ji},$$

transvecting (2.5) with g^{ji} , we have

$$(2.6) \quad R = (n-1)R^*.$$

Taking account of (2.2) and (2.6), we get

$$(2.7) \quad \frac{n-2}{n-1}R = \nabla_j F_{ih}(\nabla^j F^{ih}).$$

From (2.3), we have the following

$$(2.8) \quad R_{ji}R^{*ji} = \frac{1}{6}(R_{ji}R^{ji} + 5R_{ji}^*R^{*ji})$$

and from the square of the both sides of (2.4), we get

$$(2.9) \quad 4R_{ji}R^{ji} - 4(n-2)R_{ji}R^{*ji} + (n-2)^2R_{ji}^*R^{*ji} = \frac{nR^2}{(n-1)^2}.$$

Similarly from (2.5), we get

$$(2.10) \quad R_{ji}^*R^{*ji} = \frac{4}{(n-2)^2}R_{ji}R^{ji} - \frac{3n-4}{(n-1)^2(n-2)^2}R^2.$$

Substituting (2.8) into (2.9), we have

$$(2.11) \quad 2(8-n)R_{ji}R^{ji} + (3n^2 - 22n + 32)R_{ji}^*R^{*ji} = \frac{3nR^2}{(n-1)^2}$$

and substituting (2.10) into (2.11), we have

$$(2.12) \quad (n-4)(n-12)R_{ji}R^{ji} + (6n^2 - 33n + 32)\frac{R^2}{(n-1)^2} = 0,$$

or

$$(2.13) \quad (n-4)(n-12)R_{ji}R^{ji} + 6\left(n - \frac{33 + \sqrt{321}}{12}\right) \\ \times \left(n - \frac{33 - \sqrt{321}}{12}\right)\frac{R^2}{(n-1)^2} = 0.$$

3. Proof of Theorem. By virtue of (2.13), we easily see that there exists no conformally flat proper K -space of dimension $\neq 6, 8, 10$ [5].

If we put $n = 6$ in (2.12), we have

$$R_{ji}R^{ji} = \frac{1}{6}R^2$$

or

$$\left(R_{ji} - \frac{R}{6}g_{ji}\right)\left(R^{ji} - \frac{R}{6}g^{ji}\right) = 0,$$

from which we have

$$R_{ji} = \frac{R}{6}g_{ji},$$

that is, M is an Einstein space. Hence (1.1) reduces to

$$(3.1) \quad R_{kji h} = \frac{R}{30}(g_{ji}g_{kh} - g_{ki}g_{jh})$$

and by (2.7), $R > 0$, that is, a 6-dimensional conformally flat proper K -space is a space of positive constant curvature.

Next, if we put $n = 10$ in (2.12), then we have

$$(3.2) \quad R_{rs}R^{rs} = \frac{151}{9^2 \times 6}R^2.$$

Therefore, from (1.4) we have

$$(3.3) \quad \lambda = \frac{1}{2}\left\{\frac{R}{9} \pm \left(\frac{2 \times 151}{5 \times 9^2 \times 6} - \frac{13}{5 \times 9^2}\right)^{1/2}|R|\right\} \\ = \frac{R}{18}\left(1 \pm 4\sqrt{\frac{7}{15}}\right).$$

Now, we put

$$(3.4) \quad \lambda_1 = \frac{R}{18} \left(1 + 4\sqrt{\frac{7}{15}} \right), \quad \lambda_2 = \frac{R}{18} \left(1 - 4\sqrt{\frac{7}{15}} \right).$$

Substituting (3.4) into (1.5), we obtain

$$\sqrt{\frac{7}{15}} m = 1 + 5\sqrt{\frac{7}{15}},$$

which contradicts that m is integer, therefore R must be zero. Consequently, we can see that any 10-dimensional conformally flat proper K -space does not exist.

Lastly, we shall consider an 8-dimensional conformally flat proper K -space. If we put $n = 8$ in (1.4), then we have

$$(3.5) \quad \lambda = \frac{1}{2} \left\{ \frac{R}{7} \pm \left(\frac{1}{2} R_{rs} R^{rs} - \frac{5}{2 \times 7^2} R^2 \right)^{1/2} \right\}.$$

On the other hand, if we put $n = 8$ in (2.12), then we have

$$(3.6) \quad R_{rs} R^{rs} = \frac{19}{7^2 \times 2} R^2.$$

Substituting (3.6) into (3.5), we get

$$(3.7) \quad \lambda = \frac{R}{28} (2 \pm 3),$$

because of $R > 0$.

In this place, if we put

$$(3.8) \quad \lambda_1 = \frac{5}{28} R, \quad \lambda_2 = -\frac{1}{28} R,$$

where $R > 0$ and substitute (3.8) into (1.5), then we have

$$(3.9) \quad m = 6.$$

Hence we can find that the Ricci form of an 8-dimensional conformally flat proper K -space is indefinite of signature $2m - n = 4$. Moreover, as our space is locally symmetric, i.e. $\nabla_i R_{kji}{}^h = 0$, it satisfies the condition (1.2).

Thus, by virtue of Lemma 1.2, we can conclude that an 8-dimensional connected conformally flat proper K -space is locally of the form $(M_1, g_1) \times (M_2, g_2)$ as a Riemannian manifold, where (M_1, g_1) and (M_2, g_2) are a 6-dimensional space of positive constant curvature $C = (\lambda_1/(m-1)) = (1/28)R$ and a 2-dimensional space of negative constant curvature $-C$ respectively.

Let $\{x^a\}$ and $\{x^p\}$ be local coordinate systems on M_1 and M_2 respectively, where $1 \leq a, b, \dots \leq 6, 7 \leq p, q, \dots \leq 8$. Then, with respect to this coordinate system $\{x^i\} = \{x^a, x^p\}$, we can put

$$(3.10) \quad (R_j^i) = \begin{pmatrix} R_b^a & 0 \\ 0 & R_q^p \end{pmatrix}$$

$$(3.11) \quad (F_j^i) = \begin{pmatrix} F_b^a & F_q^a \\ F_b^p & F_q^p \end{pmatrix}.$$

On the other hand, by (2.1) we know that

$$F_j^i R_k^j = F_k^j R_j^i$$

and therefore, from (3.10) and (3.11), after some calculation, we have

$$(3.12) \quad F_q^a = 0, \quad F_b^p = 0.$$

Furthermore, we get, taking account of (3.11)

$$\nabla_q F_b^a = -\nabla_b F_q^a = 0.$$

Similarly we have $\nabla_b F_q^p = 0$. These facts show that the space is compatible to the decomposition of the space.

Consequently, summarizing the above arguments, the proof of our theorem is completed.

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