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A NOTE ON SUBALGEBRAS OF A MEASURE ALGEBRA VANISHING ON NON-SYMMETRIC HOMOMORPHISMS

Dedicated to Professor Masanori Fukamiya on his 60th birthday

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The purpose of this paper is to prove a conjecture in [1] that "for any proper symmetric Raikov system $\mathfrak{F}$, $M(\mathfrak{F})$ is singular with respect to measures whose Gelfand transforms vanish on non-symmetric homomorphisms."

Throughout this paper, we shall follow [3] for our terminology. Let G be a non-discrete locally compact abelian group and we shall use additive notation for group operation in G. We denote by $M(G)$ the set of all bounded regular Borel measures on G. $M(G)$ becomes a commutative Banach $*$-algebra under the convolution product, the total variation norm and the usual involution in $M(G)$. Let $\Delta(G)$ be the maximal ideal space of $M(G)$ and Σ the subset of $\Delta(G)$ consisting of multiplicative linear functionals h on $M(G)$ symmetric in the sense that

$$h(\mu^*) = \overline{h(\mu)} \quad \text{for all} \quad \mu \in M(G) .$$

It is known that Σ is a proper closed subset of $\Delta(G)$ containing properly the closure of the dual of G ([2]).

A Raikov system is a collection $\mathfrak{F}$ of σ-compact subsets of G satisfying the following conditions:

(i) If $F \in \mathfrak{F}$ and E is a σ-compact subset of G with $E \subset F$ then $E \in \mathfrak{F}$.

(ii) If $F_1, F_2 \in \mathfrak{F}$ then $F_1 + F_2 \in \mathfrak{F}$,

(iii) If $F_i \in \mathfrak{F}$ for $i = 1, 2, \cdots$ then $\bigcup_{i=1}^{\infty} F_i \in \mathfrak{F}$,

(iv) If $F \in \mathfrak{F}$ and $x \in G$ then $F + x \in \mathfrak{F}$.

If the system also satisfies the following:

(v) If $F \in \mathfrak{F}$ then $-F \in \mathfrak{F}$,

we shall call it a symmetric Raikov system. A Raikov system will be called proper provided its system is contained properly in the Raikov system of all σ-compact subsets of G.

A closed subspace (subalgebra, ideal) M of $M(G)$ will be called an L-subspace (L-subalgebra, L-ideal) provided that if $\mu \in M$ and ν is absolutely continuous with respect to μ, then $\nu \in M$. For a proper Raikov system $\mathfrak{F}$, the measures in $M(G)$ concentrated on $\mathfrak{F}$ form a proper L-subalgebra $M(\mathfrak{F})$ of $M(G)$ ([4]).

We consider the subspace $B(\Sigma)$ of $M(G)$ consisting of all measures whose Gelfand transforms vanish outside Σ. $B(\Sigma)$ is a proper L-ideal in $M(G)$ containing the radical of the group algebra $L^1(G)$ ([5]). In [1] it was proved that $B(\Sigma)$ is singular with respect to $M(G_\tau)$, the measure algebra on G with a stronger locally compact group topology τ than the original topology of G. In our paper we will show that for any proper symmetric Raikov system $\mathfrak{F}$, $M(\mathfrak{F})$ and $B(\Sigma)$ are mutually singular.

For a given subset H of G, we shall call that a subset P of G is semi H-independent if any linear form

$$\sum_{r=1}^{N} n_r x_r \notin H ,$$

where $n_1, \cdots, n_N$ are integers satisfying $|n_{r_0}| = 1$ for some r_0 and $x_1, \cdots, x_N$ are distinct elements of P.

A subset of G will be called perfect if it is a compact non-empty set and has no isolated points.

LEMMA 1. [cf. 6: Proposition 1] *If G is metrizable and H is a σ-compact set in G with Haar measure zero then there exists a perfect semi H-independent subset of G.*

PROOF. Suppose that $H = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact with $K_1 \subset K_2 \subset \cdots$. At the first stage, choose two disjoint closed sets $F_1^{(1)}$ and $F_2^{(1)}$, of diameter not exceeding 1, such that $F_1^{(1)}$ and $F_2^{(1)}$ are perfect, and such that $F_1^{(1)} \times F_2^{(1)}$ does not meet $\{(x_1, x_2) \in G \times G;\ n_1x_1 + n_2x_2 \in K_1$ where $|n_k| \leqq 1$ and either $|n_1| = 1$ or $|n_2| = 1\}$.

At the j-th stage, if perfect closed sets $F_1^{(j-1)}, \cdots, F_{2j-1}^{(j-1)}$ are present, choose $F_1^{(j)}$ and $F_2^{(j)}$ in $F_1^{(j-1)}, \cdots, F_{2j-1}^{(j)}$ and $F_{2j}^{(j)}$ in $F_{2j-1}^{(j-1)}$ respectively, such that

(i) $F_r^{(j)}$ $(1 \leqq r \leqq 2^j)$ are disjoint,

(ii) each $F_r^{(j)}$ is perfect and of diameter not exceeding $1/j$,

(iii) $F_1^{(j)} \times \cdots \times F_{2j}^{(j)}$ does not intersect $\{(x_1, \cdots, x_{2j}) \in G^{2j};\ \sum_{r=1}^{2j} n_r x_r \in K_j$ where $|n_r| \leqq j$ $(1 \leqq r \leqq 2^j)$ and $|n_{r_0}| = 1$ for some $r_0\}$.

To show that this choice is always possible, let $s = (n_1, \cdots, n_{2j})$ with $|n_r| \leqq j$ for all r and $|n_{r_0}| = 1$ for r_0. Let the map of G^{2j} to G be defined by

$$f_s(x_1, \cdots, x_{2j}) = n_1x_1 + \cdots + n_{2j}x_{2j} .$$

Then $f_s^{-1}(K_j)$ is nowhere dense. For, if $f_s^{-1}(K_j)$ contains some neighborhood

$N_1 \times \cdots \times N_{2^j}$ of $(a_1, \cdots, a_{2^j})$ in G^{2^j}, the set

$$I = n_{r_0}K_j - n_{r_0} \sum_{r \neq r_0} n_r a_r$$

contains N_{r_0}. But, since a compact set of Haar measure zero must be nowhere dense, I is nowhere dense. This is a contradiction. This contradiction implies that $f_s^{-1}(K_j)$ is nowhere dense. Therefore, the finite union of the sets $f_s^{-1}(K_j)$, as s runs through the possible selections of the integers $n_1, \cdots, n_{2^j}$, is nowhere dense. It follows that $F_r^{(j)}$ can be chosen so as to satisfy (i)–(iii) above.

If we write

$$P^{(j)} = \bigcup_{r=1}^{2^j} F_r^{(j)} \quad \text{and} \quad P = \bigcap_{j=1}^{\infty} P^{(j)} ,$$

then it is clear that P is non-empty and perfect. To show that P is semi H-independent, we take finite distinct points. Then for j large enough, these points are in distinct sets $F_r^{(j)}$ since the lengths of $F_r^{(j)}$ tend to zero as j tends to infinity. Hence the form

$$\sum_{r=1}^{N} n_r x_r ,$$

where n_r integer $(1 \leqq r \leqq N)$ and $|n_{r_0}| = 1$ for some r_0, can not belong to H. This implies that P is a semi H-independent set.

LEMMA 2. [cf. 6: Proposition 2] *Let $\mathfrak{F}$ be a symmetric Raikov system generated by a σ-compact group H in G. Suppose that there exists a perfect semi H-independent subset P of G. Let μ be any non-negative continuous measure concentrated on $Q = P \cup (-P)$. If $\nu, \nu' \in M(\mathfrak{F})$ then $\nu * \mu^n \perp \nu' * \mu^m$ for distinct positive integers n, m.*

PROOF. Suppose first that ν and ν' are concentrated on H, then we wish to show that the measures $\nu * \mu^n * \delta_{z'}$ and $\nu' * \mu^m * \delta_z$ are mutually singular if $m \neq n$ and $z, z' \in G$. We may assume that $m < n$ and $z' = 0$. These measures are concentrated on $H + nQ$, $H + mQ + z$ respectively, where $mQ = \{x_1 + \cdots + x_m; x_i \in Q\}$. Evidently if these two sets are disjoint, the two measures are mutually singular. If the sets are not disjoint, we have

$$z = h' + \sum_{r=1}^{N} n'_r p'_r$$

for $h' \in H$ and $p'_r \in P$. Denote by S the set of points $(x_1, \cdots, x_n) \in Q^n$ such that

$$x_1 + \cdots + x_n \in H + mQ + z .$$

Let $x_1 + \cdots + x_n = h + y_1 + \cdots + y_m + z$; each x_i is of the form $\pm p_i$, with $p_i \in P$. Then, if $p_1, \cdots, p_n$ were all different, and different also from $p'_1, \cdots, p'_N$, there would be a linear form $\sum n_r q_r \in H$ such that $|n_{r_0}| = 1$ for some r_0 and $q_r \in P$ which is not possible. Thus S is contained in a finite union of sets of the form

$$\begin{aligned} &\{(x_1, \cdots, x_n) \colon x_i = x_j\} \qquad (i \neq j) \\ &\{(x_1, \cdots, x_n) \colon x_i = -x_j\} \qquad (i \neq j) \\ &\{(x_1, \cdots, x_n) \colon x_i = p'_j\} \qquad (\text{any } i, j) \\ &\{(x_1, \cdots, x_n) \colon x_i = -p'_j\} \qquad (\text{any } i, j) \end{aligned}$$

and these are all of $(\mu \times \cdots \times \mu)$-measure zero, since μ is continuous. It follows that $(\mu \times \cdots \times \mu)(S) = 0$. Therefore,

$$\begin{aligned} (\nu * \mu^n)(H + mQ + z) &= (\nu \times (\mu \times \cdots \times \mu))(H \times S) \\ &= \nu(H)(\mu \times \cdots \times \mu)(S) = 0 , \end{aligned}$$

and so $\nu * \mu^n$ and $\nu' * \mu^m * \delta_z$ are mutually singular.

We next relax the condition that the measures ν and ν' should be concentrated on H. Since H generates $\mathfrak{F}$, any ν, ν' in $M(\mathfrak{F})$ must be concentrated on countable unions of translates of H. Suppose that we have

$$\nu = \sum_{k=1}^{\infty} \nu_k \quad \text{and} \quad \nu' = \sum_{k=1}^{\infty} \nu'_k ,$$

where ν_k, ν'_k are concentrated on $H + z_k$, $H + z'_k$ respectively. Then $\nu_k * \mu^n$ and $\nu'_l * \mu^m$ are mutually singular for all k, l if $n \neq m$. It follows that the measures $\nu * \mu^n$ and $\nu' * \mu^m$ are mutually singular.

LEMMA 3. [cf. 1: Theorem 1] *Let $\mathfrak{F}$ be a symmetric Raikov system generated by a σ-compact group H in G. Suppose that there exists a perfect semi H-independent subset P of G. Then for each non-negative $\mu \in M(\mathfrak{F})$, there exists non-symmetric $h \in \Delta(G)$ with $h(\mu) \neq 0$.*

PROOF. Let ν_1 be a non-negative continuous measure concentrated on P, with $||\nu_1|| = 1$. If

$$\nu = \frac{1}{2}(\nu_1 + \nu_1^*)$$

then $\nu = \nu^*$, ν is concentrated on $Q = P \cup (-P)$, $\nu \geqq 0$ and ν is continuous with $||\nu|| = 1$. Similarly, for each non-negative $\mu \in M(\mathfrak{F})$ we can assume that $\mu = \mu^*$ and $||\mu|| = 1$. Put $\sigma = \mu^2 - \nu^2$. By Lemma 2 we obtain

$$||\sigma^n|| = \left\| \sum_{k=0}^{n} \binom{n}{k} (-1)^k \nu^{2k} \mu^{2(n-k)} \right\| = \sum_{k=0}^{n} \binom{n}{k} ||\nu^{2k} \mu^{2(n-k)}|| = 2^n \qquad (n = 1, 2, 3, \cdots) ,$$

so that the spectral norm $\|\sigma\|_{\mathrm{sp}} = 2$. Hence there is $h \in \Delta(G)$ such that $|h(\sigma)| = 2$. Since $|h(\mu^2)| \leqq 1$, $|h(\nu^2)| \leqq 1$,

$$|h(\mu^2) - h(\nu^2)| = |h(\sigma)| = 2$$

which is possible only if $-h(\nu^2) = h(\mu^2)$ and $|h(\nu^2)| = |h(\mu^2)| = 1$. This implies that h is non-symmetric with $h(\mu) \neq 0$.

LEMMA 4. *Let $\mathfrak{F}$ be a proper symmetric Raikov system with a single generator. Then $B(\Sigma) \perp M(\mathfrak{F})$.*

PROOF. Since $B(\Sigma)$ and $M(\mathfrak{F})$ are L-subspaces, it is sufficient to show that any positive $\mu \in M(\mathfrak{F})$ implies $\mu \notin B(\Sigma)$.

A single generator may be assumed to be a σ-compact subgroup H of G. Let $H = \bigcup_{i=1}^{\infty} K_i$, where each K_i is compact. We may assume without loss of generality that $0 \in K_1 \subset K_2 \subset \cdots$. Let λ be the Haar measure on G. From the fact that $\lambda(K_i) = 0$ for $i = 1, 2, \cdots$, we can choose compact neighborhoods $\{V_n\}_{n=1}$ of 0 in G such that

(i) $V_n = -V_n$

(ii) $V_n \supset V_{n+1} + V_{n+1}$

(iii) $\lambda(V_n + K_n) < 1/n$.

We put $H' = \bigcap_{n=1}^{\infty} V_n$. Then H' is a compact subgroup of G and G/H' is non-discrete and metrizable. Let α be the canonical map from G to G/H'. We can see that $\lambda(H' + H) = 0$. For,

$$H' + K_n \subset H' + K_m \subset V_m + K_m \quad \text{for} \quad m \geqq n .$$

Therefore

$$\lambda(H' + K_n) \leqq \lambda(V_m + K_m) < 1/m \quad \text{for all} \quad m \geqq n .$$

This implies $\lambda(H' + K_n) = 0$ for all n and $\lambda(H' + H) = \lambda(\bigcup_{n=1}^{\infty}(K_n + H')) = 0$.

From the fact above, $\alpha(H)$ is a σ-compact group in G/H' with Haar measure zero and $\alpha(\mathfrak{F}) = \{\alpha(F) : F \in \mathfrak{F}\}$ is a proper symmetric Raikov system generated by $\alpha(H)$ in G/H'. It is clear that any $\mu \in M(\mathfrak{F})$ implies $\alpha^*\mu \in M(\alpha(\mathfrak{F}))$, where α^* is the homomorphism, induced by α, from $M(G)$ onto $M(G/H')$. By Lemma 1 and Lemma 3, for a positive $\mu \in M(\mathfrak{F})$, there exists a non-symmetric $h \in \Delta(G/H')$ with $h(\alpha^*\mu) \neq 0$. But $h \circ \alpha^*$ belongs to $\Delta(G)$ and is non-symmetric. This implies $\mu \notin B(\Sigma)$. Thus the proof is complete.

THEOREM. *Let $\mathfrak{F}$ be a proper symmetric Raikov system. Then $B(\Sigma) \perp M(\mathfrak{F})$.*

PROOF. If $\mu \in M(\mathfrak{F})$ we can choose $H \in \mathfrak{F}$ on which μ is concentrated. The symmetric Raikov system $\mathfrak{F}_0$ generated by H is contained in $\mathfrak{F}$. By

Lemma 4, $M(\mathfrak{F}_0) \perp B(\Sigma)$. Especially, $\mu \perp B(\Sigma)$. Thus $M(\mathfrak{F}) \perp B(\Sigma)$.

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