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ON SUBALGEBRAS OF A CROSS PRODUCT VON NEUMANN ALGEBRA

Dedicated to Professor Masanori Fukamiya on his 60th birthday

YOSHINORI HAGA

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Introduction. Let $\mathscr{A}$ be a von Neumann algebra on a separable Hilbert space $\mathscr{H}$ and G a countable group of automorphisms acting freely on $\mathscr{A}$. Suppose that $\mathscr{A}$ is $[G]$-finite (see §1). In [7] established was the existence of the Dye correspondence between intermediate subalgebras of the cross product $G \otimes \mathscr{A}$ and full subgroups of the full group $[G]$. We continue the investigation on this correspondence in this paper.

In §2, we show that the notion of "determining functions", introduced by Dye [6], may be understood lucidly as the initial projections of the natural expectations of the unitaries $g \otimes I$ and we give the explicit form of intermediate subalgebras of $G \otimes \mathscr{A}$ by using this notion. Thus we get a one-to-one correspondence between intermediate subalgebras of $G \otimes \mathscr{A}$ and determining functions (Theorem 2.5).

Subsequently, in §3, we generalize the notion of abelian projections and of the types of von Neumann algebras according to M. Choda [2] and [3]. In §4, we discuss the conservation of types in the Dye correspondence (Theorems 4.9 and 4.10), generalizing the abelian case given by Dye [6].

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1. Preliminaries. We give a brief resume of the terminology and some results of [7]. Throughout this paper, $\mathscr{A}$ denotes a von Neumann algebra on a separable Hilbert space $\mathscr{H}$ with the identity I, and G denotes a discrete group of ($*$-)automorphisms of $\mathscr{A}$ with the unity e. $\mathscr{A}_p$ (resp. $\mathscr{A}_u$) is the set of all projections (resp. unitaries) of $\mathscr{A}$ and $\mathscr{Z}$ denotes the center of $\mathscr{A}$. An automorphism α of $\mathscr{A}$ is said to be freely acting on $\mathscr{A}$ when $AB = \alpha(B)A$ for all $B \in \mathscr{A}$ implies $A = 0$ ([8]). G is said to be freely acting on $\mathscr{A}$ if each $g \neq e$ in G is freely acting. An automorphism α is said to be locally inner on $P \in \mathscr{Z}_p$ if α is inner on $\mathscr{A}|_P$ as the restricted automorphism. For two automorphisms α and β, let $F(\alpha, \beta)$ denotes the maximal central projection on which $\alpha^{-1}\beta$ is locally inner. Then the full group $[G]$ determined by G is the set of all automorphisms α such that $\sup_{g \in G} F(\alpha, g) = I$, and G is called full if $[G] = G$. The full group $[G]$ consists of all automorphisms α which admits a representation

$$\alpha(A) = \sum_{g \in G} g\phi_V(Q_g A) \quad \text{for all} \quad A \in \mathscr{A} , \tag{1.1}$$

where ϕ_V denotes an inner automorphism induced by $V \in \mathscr{A}_u$ and $\{Q_g\}$ is a family of mutually orthogonal central projections with $\sum Q_g = I$ and $\sum g(Q_g) = I$. This representation is unique if G is freely acting on $\mathscr{A}$.

For a countable group G, let $G \otimes \mathscr{H}$ be the set of all formal sums $\sum_{a \in G} a \otimes \xi_a$ ($\xi_a \in \mathscr{H}$) for which $\sum_{a \in G} \|\xi_a\|^2 < \infty$. Then each element of $G \otimes \mathscr{H}$ is a $\mathscr{H}$-valued function defined on G and $G \otimes \mathscr{H}$ is nothing but the L^2-space of such functions. Define an operator $g \otimes A$ ($g \in G, A \in \mathscr{A}$) on $G \otimes \mathscr{H}$ by

$$(g \otimes A)\Big(\sum_a a \otimes \xi_a\Big) = \sum_a ag^{-1} \otimes a(A)\xi_a .$$

Then,

$$(g \otimes A)(h \otimes B) = gh \otimes h^{-1}(A)B$$

and

$$(g \otimes A)^* = g^{-1} \otimes g(A^*) .$$

The cross product $G \otimes \mathscr{A}$ is defined as the von Neumann algebra on $G \otimes \mathscr{H}$ generated by $\{g \otimes A \mid g \in G, A \in \mathscr{A}\}$. $e \otimes \mathscr{A} = \{e \otimes A \mid A \in \mathscr{A}\}$ is a von Neumann subalgebra of $G \otimes \mathscr{A}$ algebraically isomorphic to $\mathscr{A}$ so that we identify $e \otimes A \in e \otimes \mathscr{A}$ with $A \in \mathscr{A}$ when we have no fear to lead to confusion. According to [10], $\mathscr{A}$ is called to be *G-finite* when $\mathscr{A}$ has a faithful normal G-invariant state ρ. Then ρ can be extended to a faithful normal G-invariant state on $G \otimes \mathscr{A}$ and every element T of $G \otimes \mathscr{A}$ has a unique expansion

$$T = \sum_{g \in G} g \otimes T_g \qquad (T_g \in \mathscr{A})$$

converging in ρ-norm, where ρ-norm is defined by $\|T\|_\rho = \rho(T^*T)^{1/2}$. In this paper, we concern only with a finite von Neumann algebra with faithful normal G-invariant trace τ. This situation of $\mathscr{A}$, that is, finite and G-finite should be called to be *[G]-finite* since $[G]$ contains all the inner automorphisms of $\mathscr{A}$ (see [10; Remarks to Def. 1]). Then τ is extended to a faithful normal G-invariant trace of $G \otimes \mathscr{A}$ by $\tau(\sum g \otimes A_g) = \tau(A_e)$. Hence $G \otimes \mathscr{A}$ is also finite and, therefore, for an arbitrary von Neumann subalgebra $\mathscr{B}$ of $G \otimes \mathscr{A}$, there exists a uniquely determined element $\Phi(T) \in \mathscr{B}$ such that $\tau(TB) = \tau(\Phi(T)B)$ for every $T \in G \otimes \mathscr{A}$ and $B \in \mathscr{B}$. As well known, Φ is a faithful normal expectation of $G \otimes \mathscr{A}$ onto $\mathscr{B}$ which is uniquely determined by $\tau \circ \Phi = \tau$ ([13]). We shall call Φ the natural expectation of $G \otimes \mathscr{A}$ onto $\mathscr{B}$.

An intermediate subalgebra of $G \otimes \mathscr{A}$ means a von Neumann subalgebra of $G \otimes \mathscr{A}$ containing $\mathscr{A}$. Denote by $\mathscr{U}(G \otimes \mathscr{A}, \mathscr{A})$ the set of all unitaries of $G \otimes \mathscr{A}$ such that $\phi_U(\mathscr{A}) = U\mathscr{A}U^* = \mathscr{A}$. Then, one of the main results in [7] is the following theorem on the Dye correspondence.

THEOREM A ([7; Theorem 2]). *Let G be a countable group of automorphisms acting freely on $\mathscr{A}$. Suppose that $\mathscr{A}$ is $[G]$-finite. Then the lattice of all intermediate subalgebras $\mathscr{B}$ of $G \otimes \mathscr{A}$ and the lattice of all full subgroups K of $[G]$ are isomorphic by associating with each full subgroup K the intermediate subalgebra*

$$\mathscr{B}(K) = \mathscr{R}[U \mid \phi_U \in K]$$

and with each intermediate subalgebra $\mathscr{B}$ the full group

$$K(\mathscr{B}) = \{\phi_U \mid U \in \mathscr{U}(G \otimes \mathscr{A}, \mathscr{A}) \cap \mathscr{B}\} .$$

2. Determining functions and intermediate subalgebras. We begin with a discussion of the natural expectation of cross product $G \otimes \mathscr{A}$ on an intermediate subalgebra. Let G be a countable group of automorphisms acting freely on a von Neumann algebra $\mathscr{A}$. Suppose that $\mathscr{A}$ is $[G]$-finite.

LEMMA 2.1. *Let $\mathscr{B}$ be an intermediate subalgebra of $G \otimes \mathscr{A}$ and Φ the natural expectation of $G \otimes \mathscr{A}$ onto $\mathscr{B}$. Then, for any $g \in G$,*

$$\Phi(g \otimes I) = g \otimes E_g$$

where E_g is a central projection of $\mathscr{A}$.

PROOF. Put $V = \Phi(g \otimes I)$. According to [7; Lemma 5], V is a partial isometry whose initial projection V^*V and final projection $F = VV^*$ are both in $\mathscr{Z}$. For any $A \in \mathscr{A}$,

$$\begin{aligned}\Phi(g \otimes I)(e \otimes A) &= \Phi[(g \otimes I)(e \otimes A)]\\ &= \Phi[(e \otimes g(A))(g \otimes I)]\\ &= (e \otimes g(A))\Phi(g \otimes I)\end{aligned}$$

and hence,

$$VAV^* = g(A)F\,.$$

On the other hand, $W = g \otimes g^{-1}(F)$ is clearly a partial isometry in $G \otimes \mathscr{A}$ and for any $A \in \mathscr{A}$,

$$(g \otimes g^{-1}(F))(e \otimes A) = (e \otimes g(A))(g \otimes g^{-1}(F))\,.$$

Hence,

$$WAW^* = (e \otimes g(A))(g \otimes g^{-1}(F))(g^{-1} \otimes F) = g(A)F\,.$$

Therefore,

$$VAV^* = WAW^*\,.$$

It follows that W^*V commutes with any element of $\mathscr{A}$. Then

$$W^*V = (g^{-1} \otimes F)V = (g^{-1} \otimes I)V \in \mathscr{Z}$$

by [7; Lemma 4]. Hence $V \in (g \otimes I)\mathscr{Z}$ so that we can express V in the form $g \otimes E_g$ with $E_g \in \mathscr{Z}$.

It remains to show that E_g is a projection.

$$\begin{aligned} g \otimes E_g &= V = \Phi(g \otimes I) \\ &= \Phi[\Phi(g \otimes I)] = \Phi(g \otimes E_g) \\ &= \Phi[(g \otimes I)(e \otimes E_g)] = \Phi(g \otimes I)(e \otimes E_g) \\ &= (g \otimes E_g)(e \otimes E_g) = g \otimes E_g^2\,. \end{aligned}$$

Hence, $E_g = E_g^2$. On the other hand,

$$\|E_g\| = \|e \otimes E_g\| = \|g \otimes E_g\| = \|V\| = 1\,.$$

Therefore E_g is an idempotent contraction so that it is a projection.

LEMMA 2.2. *The projections E_g in Lemma 2.1 satisfy*

$$E_g g^{-1}(E_h) = E_g E_{hg} \quad \textit{for all} \quad g, h \in G \quad \textit{and} \quad E_e = I\,. \tag{2.1}$$

PROOF. $E_e = I$ is clear by $\Phi(e \otimes I) = e \otimes I$. Next,

$$\begin{aligned} \Phi[(h \otimes I)\Phi(g \otimes I)] &= \Phi[(h \otimes I)(g \otimes E_g)] \\ &= \Phi(hg \otimes E_g) = \Phi(hg \otimes I)(e \otimes E_g) \\ &= (hg \otimes E_{hg})(e \otimes E_g) = hg \otimes E_g E_{hg}\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Phi[(h \otimes I)\Phi(g \otimes I)] &= \Phi(h \otimes I)\Phi(g \otimes I) \\ &= (h \otimes E_h)(g \otimes E_g) = hg \otimes E_g g^{-1}(E_h)\,. \end{aligned}$$

Thus (2.1) is obtained.

DEFINITION 2.3. A mapping $g \rightarrow E_g$ of G into the set of central projections of $\mathscr{A}$ is called a determining function on G if it satisfies (2.1).

Any determining function satisfies

$$g(E_g) = E_{g^{-1}} \tag{2.2}$$

as we see by substituting $h = g^{-1}$ in (2.1) and using a symmetry argument. Further, direct computations show that E_g and $E_{g^{-1}}$ are the initial and final projections of $\Phi(g \otimes I)$ respectively.

REMARK. If we put $F_g = E_{g^{-1}}$, it is easily seen that F_g is a determining function defined by Dye [6] for abelian $\mathscr{A}$. In this paper, however, we use the initial projection E_g as a determining function as a matter of convenience. Hence Def. 2.3 is a slight modification of Dye's definition. If we follow the discussions of [6] on the connection between determining functions on G and full subgroups of $[G]$ using the notion of local innerness in place of absolute fixedness, we may get a correspondence between them analogous to a result of [6]. Combining this correspondence with the Dye correspondence cited as Theorem A in § 1, we may get a lattice isomorphism between the lattice of all intermediate subalgebras of $G \otimes \mathscr{A}$ and that of all determining functions on G. In this section, however, we discuss directly and clarify the connection between the determining functions and the intermediate subalgebras.

For a given intermediate subalgebra $\mathscr{B}$ of $G \otimes \mathscr{A}$, let E_g be the determeining function obtained by $\Phi(g \otimes I) = g \otimes E_g$. Then, by the preceding discussion, we get

$$\Phi(\sum g \otimes A_g) = \sum g \otimes E_g A_g \quad \text{for} \quad \sum g \otimes A_g \in G \otimes \mathscr{A} \tag{2.3}$$

and

$$\mathscr{B} = \{\sum g \otimes E_g A_g \mid \sum g \otimes A_g \in G \otimes \mathscr{A}\} . \tag{2.4}$$

Conversely,

LEMMA 2.4. *For a given determining function E_g on G, define $\mathscr{B}$ and Φ by* (2.4) *and* (2.3) *respectively. Then $\mathscr{B}$ is an intermediate subalgebra of $G \otimes \mathscr{A}$ and Φ is the natural expectation onto $\mathscr{B}$.*

PROOF. Clearly $\mathscr{B}$ is a linear subspace of $G \otimes \mathscr{A}$ and it is closed with respect to the weak operator topology since the g-components are always in $E_g\mathscr{A}$. For any two elements $\sum g \otimes E_g A_g$ and $\sum h \otimes E_h B_h$ in $\mathscr{B}$, we see using (2.1) that

$$\begin{aligned}\Big(\sum_g g\otimes E_gA_g\Big)\Big(\sum_h h\otimes E_hB_h\Big) &= \sum_{g,h} gh\otimes h^{-1}(E_gA_g)E_hB_h\\ &= \sum_{g,h} gh\otimes E_hE_{gh}h^{-1}(A_g)B_h\\ &= \sum_g g\otimes E_g\Big[\sum_h E_hh^{-1}(A_{gh^{-1}})B_h\Big]\in\mathscr{B}\,.\end{aligned}$$

Moreover, by (2.2)

$$\begin{aligned}(\textstyle\sum g\otimes E_gA_g)^* &= \textstyle\sum g^{-1}\otimes g(E_gA_g^*)\\ &= \textstyle\sum g^{-1}\otimes E_{g^{-1}}g(A_g^*)\\ &= \textstyle\sum g\otimes E_g[g^{-1}(A_{g^{-1}}^*)]\in\mathscr{B}\,.\end{aligned}$$

Therefore $\mathscr{B}$ is a subalgebra of $G\otimes\mathscr{A}$ and it contains $\mathscr{A}$ since $E_e=I$.

Next, according to a result of Tomiyama [12], an expectation onto $\mathscr{B}$ is characterized as an idempotent linear mapping of $G\otimes\mathscr{A}$ onto $\mathscr{B}$ having norm one, and which leaves the identity fixed. All of these properties are obvious for Φ defined by (2.3). In addition, for any $T=\sum g\otimes A_g\in G\otimes\mathscr{A}$,

$$\tau(\Phi(T))=\tau(\textstyle\sum g\otimes E_gA_g)=\tau(A_e)=\tau(T)\,.$$

Therefore Φ is the unique natural expectation of $G\otimes\mathscr{A}$ onto $\mathscr{B}$. Thus the proof is completed.

Now, for two determining functions E_g and F_q, we write $E_g\leqq F_g$ in case $E_g\leqq F_g$ as projections for every $g\in G$. Then we get the following main theorem in this section.

THEOREM 2.5. *Let G be a countable group of automorphisms acting freely on $\mathscr{A}$ and suppose that $\mathscr{A}$ is $[G]$-finite. Then, the lattice of all intermediate subalgebras of $G\otimes\mathscr{A}$ and that of all determining functions E_g on G are isomorphic by associating with each $\mathscr{B}$ the determining function*

$$E_g(\mathscr{B})=\Phi(g\otimes I)^*\Phi(g\otimes I)$$

and with each E_g the intermediate subalgebra

$$\mathscr{B}(E_g)=\{\textstyle\sum g\otimes E_gA_g\mid\sum g\otimes A_g\in G\otimes\mathscr{A}\}\,.$$

PROOF. $E_g(\mathscr{B})$ is actually the determining function in Lemma 2.2 associated with $\mathscr{B}$, and $\mathscr{B}(E_g)$ is an intermediate subalgebra as shown in Lemma 2.4. We see easily that the correspondence $\mathscr{B}\to E_g(\mathscr{B})\to\mathscr{B}(E_g(\mathscr{B}))$ between intermediate subalgebras of $G\otimes\mathscr{A}$ and determining functions on G is mutually one-to-one, and that it preserves the order so that the set of all E_g as well as the set of all $\mathscr{B}$ is a lattice. Consequently, the correspondence is lattice isomorphism. This completes the proof.

In particular, when $\mathscr{A}$ is a finite factor with a countable group G of outer automorphisms, any determining function E_g on G takes values only 0 or I. It is easily seen by (2.1) that $H = \{g \in G \mid E_g = I\}$ is a subgroup of G. Hence,

COROLLARY 2.6. ([11; Theorem 2]). *If $\mathscr{A}$ is a finite factor with a countable group G of outer automorphisms, then each intermediate subalgebra $\mathscr{B}$ of $G \otimes \mathscr{A}$ is the cross product $H \otimes \mathscr{A}$ of $\mathscr{A}$ by a subgroup H of G.*

COROLLARY 2.7. *If E_g and F_g are determining functions, then E_gF_g is a determining function and $\mathscr{B}(E_gF_g) = \mathscr{B}(E_g) \cap \mathscr{B}(F_g)$.*

PROOF. By direct computations and by Theorem 2.5.

Now, we proceed to discuss the connection between determining functions on G and full subgroups of $[G]$ to which we referred in the previous remark.

Let γ be an arbitrary automorphism of $\mathscr{A}$ and K a full group of automorphisms of $\mathscr{A}$. Then by [7; Lemma 7], there exists a unique maximal central projection $E(K, \gamma)$ and $\alpha \in K$ such that

$$E(K, \gamma)\gamma(A) = E(K, \gamma)\alpha(A) \quad \text{for all} \quad A \in \mathscr{A}$$

and it holds

$$E(K, \gamma) = \sup_{\beta \in K} \gamma[F(\gamma, \beta)] = \gamma[F(\gamma, \alpha)] \tag{2.5}$$

(for the notation $F(\gamma, \beta)$, see § 1). Recall that $K(\mathscr{B})$ denotes the full subgroup $[\phi_U \mid U \in \mathscr{U}(G \otimes \mathscr{A}, \mathscr{A}) \cap \mathscr{B}]$ of $[G]$ associated to an intermediate subalgebra $\mathscr{B}$ in the Dye correspondence.

LEMMA 2.8. *For every $g \in G$, there exists an automorphism $\alpha \in K$ such that*

$$E_g(\mathscr{B}) = F(g, \alpha) = \sup_{\beta \in K(\mathscr{B})} F(g, \beta) .$$

That is, $E_g(\mathscr{B})$ is the maximal central projection on which the action of g on $\mathscr{A}$ coincides with an automorphism in $K(\mathscr{B})$.

PROOF. Put $E_g = E_g(\mathscr{B})$ and $K = K(\mathscr{B})$. According to [7; Lemma 8], for every $g \in G$, there exists a $W \in \mathscr{U}(G \otimes \mathscr{A}, \mathscr{A}) \cap \mathscr{B}$ such that

$$\Phi(g \otimes I) = E(K, g)W .$$

It follows that

$$\begin{aligned} g(E_g) &= (g \otimes E_g)(g \otimes E_g)^* \\ &= \Phi(g \otimes I)\Phi(g \otimes I)^* \\ &= E(K, g)WW^*E(K, g) = E(K, g) . \end{aligned}$$

Hence by (2.5), the proof is completed.

Combining the correspondence in Theorem 2.5 with the Dye correspondence (Theorem A), we obtain the following theorem.

THEOREM 2.9. *Let G be a countable group of automorphisms acting freely on $\mathscr{A}$. Suppose that $\mathscr{A}$ is $[G]$-finite. Then the lattice of all determining functions E_g on G and that of all full subgroups K of $[G]$ are isomorphic by associating with each E_g the full subgroup*

$$K(E_g) = \{\alpha \in [G] \mid \alpha(A) = \sum_g g\phi_V(Q_g A), \quad Q_g \leqq E_g\}$$

where V and $\{Q_g\}$ is as in (1.1), *and with each K the determining function*

$$E_g(K) = g^{-1}[E(K, g)] = \sup_{\beta \in K} F(g, \beta) .$$

PROOF. According to Theorem A, any automorphism in $K(\mathscr{B})$ is induced by $U \in \mathscr{U}(G \otimes \mathscr{A}, \mathscr{A}) \cap \mathscr{B}$. By [7; Cor. 1 to Theorem 1], U has a representation $U = \sum g \otimes Q_g V$. In addition, $Q_g \leqq E_g(\mathscr{B})$ by Theorem 2.5. Now,

$$\begin{aligned} UAU^* &= \Big(\sum_g g \otimes Q_g V\Big)(e \otimes A)\Big(\sum_h h^{-1} \otimes h(Q_h V^*)\Big) \\ &= \sum_{g,h} gh^{-1} \otimes h(Q_g VA)h(Q_h V^*) \\ &= e \otimes \sum_g g\phi_V(Q_g A) . \end{aligned}$$

Therefore,

$$\begin{aligned} K(E_g) &= K(\mathscr{B}(E_g)) \\ &= \{\phi_U \mid U \in \mathscr{U}(G \otimes \mathscr{A}, \mathscr{A}) \cap \mathscr{B}(E_g)\} \\ &= \{\alpha \mid \alpha(A) = \sum_g g\phi_V(Q_g A), \quad Q_g \leqq E_g\} . \end{aligned}$$

Finally, by Lemma 2.8 and (2.5),

$$\begin{aligned} E_g(K) = E_g(\mathscr{B}(K)) &= \sup_{\beta \in K(\mathscr{B}(K))} F(g, \beta) \\ &= \sup_{\beta \in K} F(g, \beta) = g^{-1}[E(K, g)] . \end{aligned}$$

3. Abelian projections over a subalgebra. In a recent work [2], M. Choda has generalized the notion of abelian projections of von Neumann algebras and proved that some elementary properties of abelian projections are preserved under the generalization. This generalization naturally leads us to extend the definitions of types of von Neumann algebras and types of groups of automorphisms defined by Dye [5]. Then we can show the conservation of types in the Dye correspondence which will be discussed in the next section. The present section is devoted to the preparation.

DEFINITION 3.1 ([2]). Let $\mathscr{A}$ be a von Neumann algebra and $\mathscr{B}$ a von Neumann subalgebra of $\mathscr{A}$. Denote by $\mathscr{B}^c$ the relative commutant $\mathscr{B}' \cap \mathscr{A}$ of $\mathscr{B}$ in $\mathscr{A}$. A non-zero projection $P \in \mathscr{A}$ is called to be abelian over $\mathscr{B}$ if $P \in \mathscr{B}^c$ and, for any projection $Q \in \mathscr{A}$ with $Q \leqq P$, there exists a projection $C \in \mathscr{B}$ such that $Q = PC$.

If $\mathscr{B}$ is the center of $\mathscr{A}$, a projection abelian over $\mathscr{B}$ is abelian in the usual sense ([4; I.8.2]). When $\mathscr{A}$ is abelian, our definition reduces to that of Dye [5]. Analogously to these cases, a projection abelian over $\mathscr{B}$ is characterized as follows (cf. [3]).

LEMMA 3.2. *A non-zero projection $P \in \mathscr{A}$ is abelian over $\mathscr{B}$ if and only if $P \in \mathscr{B}^c$ and $P\mathscr{A}P = P\mathscr{B}$.*

PROOF. The "only if" part is obvious. Conversely, if $P \in \mathscr{B}^c$ and $P\mathscr{A}P = P\mathscr{B}$, then, for any projection Q in $\mathscr{A}$ dominated by P, there exists an element $B \in \mathscr{B}$ such that $Q = PB$. As easily seen, the range projection C of B is in $\mathscr{B}$ and satisfies the equality $Q = PC$.

COROLLARY 3.3. *A non-zero projection $P \in \mathscr{A}$ is abelian over $\mathscr{B}$ if and only if it is abelian over $\mathscr{A}'$ as a projection of $\mathscr{B}'$.*

For a given full group K of automorphisms of $\mathscr{A}$, Bures defined the notion of K-abelian projection as follows.

DEFINITION 3.4 ([1]). A non-zero projection P of $\mathscr{A}$ is called to be K-abelian if

$$\alpha(Q) \leqq Q + (I - P) \quad \text{for all} \quad \alpha \in K$$

and for all projections Q of $\mathscr{A}$ with $Q \leqq P$.

If $\mathscr{B}$ is the fixed subalgebra of $\mathscr{A}$ under K, that is, the subalgebra composed of all elements of $\mathscr{A}$ invariant with respect to each element of K, then $\mathscr{B}$ is contained in the center of $\mathscr{A}$, because K contains all the inner automorphisms of $\mathscr{A}$. In particular, if K is the full group of all the inner automorphisms of $\mathscr{A}$, $\mathscr{B}$ coincides with the center of $\mathscr{A}$ and a K-abelian projection is abelian in the usual sense. Incidentally, we show here that Bures' definition is a special case of Def. 3.1.

THEOREM 3.5. *Let $\mathscr{B}$ be a fixed subalgebra of $\mathscr{A}$ under a full group K. Then a projection $P \in \mathscr{A}$ is abelian over $\mathscr{B}$ if and only if it is K-abelian.*

PROOF. Let P be abelian over $\mathscr{B}$, Q a projection of $\mathscr{A}$ with $Q \leqq P$ and C a projection of $\mathscr{B}$ such that $Q = PC$. Then, for any $\alpha \in K$,

$$\begin{aligned}\alpha(Q) &= \alpha(PC) = \alpha(P)C \\ &\leqq C = PC + (I - P)C \\ &\leqq Q + (I - P) .\end{aligned}$$

Conversely, suppose that P is K-abelian and that $Q \leqq P$. Put

$$C = \sup_{\alpha \in K} \alpha(Q) .$$

Then clearly $C \in \mathscr{B}$, and it follows by $Q \leqq P$ that

$$PC = P \sup_{\alpha \in K} \alpha(Q) \leqq P[Q + (I - P)] = Q .$$

Evidently $Q \leqq PC$ and hence, $Q = PC$. Finally, $\mathscr{B}^c = \mathscr{B}' \cap \mathscr{A} \supset \mathscr{K}' \cap \mathscr{A} = \mathscr{A}$ and hence, $P \in \mathscr{A} \subset \mathscr{B}^c$.

Now, for the discussion of the next section, we give here the following definition. We assume the von Neumann algebra $\mathscr{A}$ to be finite for convenience' sake (cf. [3]).

DEFINITION 3.6. Let $\mathscr{B}$ be a von Neumann subalgebra of a finite von Neumann algebra $\mathscr{A}$. Then, $\mathscr{A}$ will be said to be of type I over $\mathscr{B}$ if each non-zero central projection in $\mathscr{A}$ dominates a projection abelian over $\mathscr{B}$. If $\mathscr{A}$ contains no projections abelian over $\mathscr{B}$, $\mathscr{A}$ will be said to be of type II over $\mathscr{B}$.

$\mathscr{A}$ is of tpye I (resp. type II) as a von Neumann algebra if and only if $\mathscr{A}$ is of type I (resp. type II) over the center of $\mathscr{A}$. Mixed type can occur, but then, by the usual argument we get the following decomposition.

LEMMA 3.7. *For any von Neumann subalgebra $\mathscr{B}$ of $\mathscr{A}$, there exists a unique central projection C of $\mathscr{A}$ such that $C\mathscr{A}$ is of type* I *over $C\mathscr{B}$ and $(I - C)\mathscr{A}$ is of type* II *over $(I - C)\mathscr{B}$.*

When $\mathscr{A}$ is abelian, the above definition of types coincides with that of [5]. In this case, if $\mathscr{A}$ is of type II over a subalgebra $\mathscr{B}$, $\mathscr{A}$ contains no minimal projections since a minimal projection is evidently abelian over $\mathscr{B}$. Therefore, when $\mathscr{A}$ is abelian and of type II over $\mathscr{B}$, we may use Maharam's lemma [5; p. 124]. That is, under the natural expectation Φ of $\mathscr{A}$ onto $\mathscr{B}$, the set of projections Q in $\mathscr{A}$ dominated by some fixed projection P in $\mathscr{A}$ maps onto the set $\{B \in \mathscr{B} \mid 0 \leqq B \leqq \Phi(P)\}$. See [3] for some generalizations of Maharam's lemma.

LEMMA 3.8. *Let $\mathscr{A}$ be a finite von Neumann algebra of type* I *and $\mathscr{B}$ an abelian von Neumann subalgebra of $\mathscr{A}$ containing the center $\mathscr{Z}$ of $\mathscr{A}$. Then $\mathscr{B}$ is of type* I *over $\mathscr{Z}$.*

PROOF. Suppose the contrary and let C be the supremum of all projections in $\mathscr{B}$ abelian over Z. Then clearly $D = I - C$ is a non-zero projection in $\mathscr{Z}$ such that $D\mathscr{B}$ is of type II over $D\mathscr{Z}$. Since $\mathscr{A}$ is of type I, we can choose an abelian projection P in $\mathscr{A}$ with central support D. By Maharam's lemma, there exists a projection $Q \in \mathscr{B}$ such that $\Psi(Q) = \Psi(P)$ where Ψ is the natural expectation of $\mathscr{A}$ onto $\mathscr{Z}$. This means that P and Q have the same center-valued trace in $\mathscr{A}$. Therefore, P and Q are equivalent projections in $\mathscr{A}$ so that Q is also an abelian projection and it is dominated by D. Hence $D\mathscr{B}$ contains a projection abelian over $D\mathscr{Z}$, a contradiction.

4. Conservation of types in the Dye correspondence.

In this section, we apply the results of the preceding section to the cross product of von Neumann algebras. Let G be a countable group of automorphisms acting freely on a von Neumann algebra $\mathscr{A}$ and suppose that $\mathscr{A}$ is $[G]$-finite.

LEMMA 4.1. *Let $\mathscr{B}$ be an intermediate subalgebra of $G \otimes \mathscr{A}$. Then the relative commutant of $\mathscr{A}$ in $\mathscr{B}$ coincides with the center of $\mathscr{A}$. That is,*

$$(e \otimes \mathscr{A})' \cap \mathscr{B} = e \otimes \mathscr{Z} .$$

PROOF.
$$\begin{aligned} e \otimes \mathscr{Z} &\subset (e \otimes \mathscr{A})' \cap \mathscr{B} \\ &\subset (e \otimes \mathscr{A})' \cap (G \otimes \mathscr{A}) = e \otimes \mathscr{Z} , \end{aligned}$$

where the last equality follows from [7; Lemma 4].

This lemma shows that any projection of $\mathscr{B}$ abelian over $e \otimes \mathscr{A}$ is in $\mathscr{Z}$. Thus, when $\mathscr{A}$ is a factor, there exists no abelian projections, that is, any intermediate subalgebra $\mathscr{B}$ is of type II over $\mathscr{A}$ except the trivial case.

LEMMA 4.2. *Let $\mathscr{B}$ and $\mathscr{C}$ be intermediate subalgebras such that $\mathscr{B} \subset \mathscr{C}$. Then the relative commutant of $\mathscr{B}$ in $\mathscr{C}$ coincides with the fixed subalgebra $\mathscr{Z}^K$ of $\mathscr{Z}$ under $K = K(\mathscr{B})$;*

$$\mathscr{B}' \cap \mathscr{C} = e \otimes \mathscr{Z}^K .$$

PROOF. By the above lemma, $\mathscr{B}' \cap \mathscr{C} \subset (e \otimes \mathscr{A})' \cap \mathscr{C} = e \otimes \mathscr{Z}$. In addition, any element B' in $\mathscr{B}' \cap \mathscr{C}$ commutes with all elements of

$\mathscr{B} = \mathscr{R}[U|\phi_U \in K]$ so that it is fixed under any automorphisms $\phi_U \in K$. Therefore $\mathscr{B}' \cap \mathscr{C} \subset e \otimes \mathscr{Z}^K$. Conversely, any element C in $\mathscr{Z}^K$ is invariant under all $\phi_U \in K$. Hence C commutes with all elements of $\mathscr{B}$. Since $C \in \mathscr{C}$ is obvious, we see $e \otimes \mathscr{Z}^K \subset \mathscr{B}' \cap \mathscr{C}$ completing the proof.

COROLLARY 4.3. *The center of $\mathscr{B}$ is $\mathscr{Z}^K$.*

LEMMA 4.4. *Let $\mathscr{B}$ and $\mathscr{C}$ be intermediate subalgebras such that $\mathscr{B} \subset \mathscr{C}$. Let $K = K(\mathscr{B})$ and $L = K(\mathscr{C})$. Then, if a projection P in $\mathscr{C}$ is abelian over $\mathscr{B}$, it is abelian over $\mathscr{Z}^L$ as a projection of $\mathscr{Z}^K$.*

PROOF. Since P is in $\mathscr{B}' \cap \mathscr{C}$, it is in $\mathscr{Z}^K$ which is clearly the relative commutant of $\mathscr{Z}^L$ in $\mathscr{Z}^K$. From $P\mathscr{C}P = P\mathscr{B}$ and Cor. 4.3 follows easily $P\mathscr{Z}^K P = P\mathscr{Z}^L$.

Henceforth, we discuss under the more restricted assumption that G is freely acting on the center $\mathscr{Z}$ of $\mathscr{A}$. G is then freely acting on $\mathscr{A}$ itself ([8; Prop. 1.8]). The converse holds if $\mathscr{A}$ is of type I;

LEMMA 4.5. *Let $\mathscr{A}$ be of type* I. *Then, if G is freely acting on $\mathscr{A}$, it is freely acting on $\mathscr{Z}$.*

PROOF. If not, there exist a $g \in G$, $g \neq e$ and a non-zero central projection C such that g leaves the center of $C\mathscr{A}$ element-wise fixed. Hence it is inner on $C\mathscr{A}$ ([9; Theorem 3]), contradicting [7; Cor. to Lemma 2].

LEMMA 4.6. *Suppose that G is freely acting on $\mathscr{Z}$. Let $\mathscr{B}$ be an intermediate subalgebra, and let $E_g = E_g(\mathscr{B})$ and $K = K(\mathscr{B})$. Then, for any projection P in $\mathscr{Z}$ abelian over $\mathscr{Z}^K$,*

$$E_g P g^{-1}(P) = 0 \quad \textit{for every} \quad g \neq e \textit{ in } G\,.$$

PROOF. Suppose to the contrary that $Q = E_g P g^{-1}(P) \neq 0$ for some $g \neq e$ in G. By Lemma 2.8, there exists an $\alpha \in K$ such that $E_g = F(g, \alpha)$. Then, for all central projection $R \leqq Q$, we see that $R \leqq P$ and $\alpha(R) = g(R) \leqq g(Q) \leqq P$. Now, since P is abelian over $\mathscr{Z}^K$, there exist projections C and D in $\mathscr{Z}^K$ such that $R = PC$ and $\alpha(R) = PD$. Then,

$$\alpha(R - RD) = \alpha(R)(I - D) = PD(I - D) = 0$$

so that $R = RD$. Likewise $R = RC$. Therefore,

$$\alpha(R) = \alpha(RC) = PDC = RD = R\,,$$

contrary to the free action of g on E_g.

LEMMA 4.7. *Let G, $\mathscr{B}$ and K be as above. Then, if P is a projection in $\mathscr{Z}$ abelian over $\mathscr{Z}^K$, it is abelian over $\mathscr{A}$ as a projection of $\mathscr{B}$.*

PROOF. It suffices to show that $P\mathscr{B}P = P\mathscr{A}$. Let $E_g = E_g(\mathscr{B})$. For any element $\sum g \otimes E_g A_g \in \mathscr{B}$, we see by the above lemma that

$$\begin{aligned}(e \otimes P)(\textstyle\sum g \otimes E_g A_g)(e \otimes P) \\ = \textstyle\sum g \otimes E_g P g^{-1}(P) A_g \\ = e \otimes P A_e \in P\mathscr{A}\ ,\end{aligned}$$

completing the proof.

REMARK. In the assumption of the above lemma, the free action of G on $\mathscr{Z}$ can not be replaced by the free action on $\mathscr{A}$. For example, let Γ be a II_1-factor with a countable group G of outer automorphisms, and $\mathscr{M}$ a multiplication algebra of a finite measure space. Construct the tensor product $\Gamma \otimes \mathscr{M}$ as $\mathscr{A}$. Then each $g \in G$ may be extended to an automorphism on $\mathscr{A} = \Gamma \otimes \mathscr{M}$ defining $g(A \otimes x) = g(A) \otimes x$ for $A \in \Gamma$, $x \in \mathscr{M}$. Then the extended $g \neq e$ is freely acting on $\mathscr{A}$ by [8; Cor. 1.12]. On the other hand, however, g can not be freely acting on the center $\mathscr{Z} = e \otimes \mathscr{M}$ since each central projection is fixed by every g. Now any non-zero central projection P is abelian over $\mathscr{Z}^G$ since $\mathscr{Z}^G = \mathscr{Z}$. But, as a projection of $G \otimes \mathscr{A}$, it can not be abelian over $e \otimes \mathscr{A}$ because $P(G \otimes \mathscr{A})P \neq P\mathscr{A}$ by $Pg(P) = P \neq 0$.

DEFINITION 4.8. (cf. [5] and [1; § 8]). Let K be a full subgroup of $[G]$. K is said to be of type I (resp. type II) on $\mathscr{Z}$ if the center $\mathscr{Z}$ is of type I (resp. type II) over $\mathscr{Z}^K$.

If K is of mixed types, then K can be divided into purely type I and type II parts. That is, there exists a unique projection C in $\mathscr{Z}^K$ such that K is the direct sum $K_C + K_{(I-C)}$ of two groups, the first a type I group on $C\mathscr{Z}$, the second a type II group on $(I - C)\mathscr{Z}$, where $K_C = \{\alpha \in K \mid \alpha = e \text{ on } I - C\}$. For details, see [5; p. 124].

THEOREM 4.9. *Suppose that G is freely acting on the center $\mathscr{Z}$ of $\mathscr{A}$ and that $\mathscr{A}$ is $[G]$-finite. Then the Dye correspondence conserves type in the following sense. Namely, $\mathscr{B}$ is an intermediate subalgebra of type* I (*resp. type* II) *over $\mathscr{A}$ if and only if the corresponding full subgroup $K = K(\mathscr{B})$ is of type* I (*resp. type* II) *on $\mathscr{Z}$.*

PROOF. Suppose that $\mathscr{B}$ is of type I over $\mathscr{A}$. If K is not of type I, there exists a non-zero projection C in $\mathscr{Z}^K$ such that K_C is of tpye II on $C\mathscr{Z}$. C contains a projection D of $\mathscr{B}$ abelian over $\mathscr{A}$, and D is abelian over $\mathscr{Z}^K$ as a projection of $\mathscr{Z}$ by Lemma 4.4, a contradiction. Next, suppose that $\mathscr{B}$ is of type II over $\mathscr{A}$. If K is not of type II, $\mathscr{Z}$ contains a projection abelian over $\mathscr{Z}^K$. This contradicts Lemma 4.7.

Finally, we show a conservation of types in another sense which is a generalization of [6; Prop. 6.1].

THEOREM 4.10. *Suppose that* $\mathscr{A}$ *is a finite von Neumann algebra of type* I, *that* G *is freely acting on* $\mathscr{A}$ *and that* $\mathscr{A}$ *is* $[G[$*-finite. Then the Dye correspondence conserves type;* $\mathscr{B}$ *is an intermediate subalgebra of type* I (*resp. type* II) *as a von Neumann algebra if and only if the corresponding subgroup* $K = K(\mathscr{B})$ *is of type* I (*resp. type* II) *on* $\mathscr{X}$.

PROOF. Suppose that $\mathscr{B}$ is of type I as a von Neumann algebra, namely that each non-zero projection of $\mathscr{B}$ dominates an abelian projection. Then applying Lemma 3.8 to $\mathscr{B}$ with center $\mathscr{X}^K$ and its abelian subalgebra $\mathscr{X}$, we see that $\mathscr{X}$ is of type I over $\mathscr{X}^K$. Hence K is of type I on $\mathscr{X}$. Next, suppose that $\mathscr{B}$ is of type II and that K is not of type II. Then $\mathscr{X}$ contains a projection P abelian over $\mathscr{X}^K$. Hence $P\mathscr{X} = P\mathscr{X}^K$ and $P\mathscr{B}P = P\mathscr{A}$ by Lemmas 4.5 and 4.7. Since $\mathscr{A}$ is of type I, P dominates an abelian projection Q as a projection of $\mathscr{A}$. Hence $Q\mathscr{A}Q = Q\mathscr{X}$. It follows that

$$Q\mathscr{B}Q = QP\mathscr{B}PQ = QP\mathscr{A}Q = QP\mathscr{X} = QP\mathscr{X}^K = Q\mathscr{X}^K .$$

Therefore Q is an abelian projection in $\mathscr{B}$, a contradiction.

COROLLARY 4.11 (cf. [6; Prop. 6.1]). *Suppose that* $\mathscr{A}$ *is abelian, that* G *is freely acting on* $\mathscr{A}$ *and that* $\mathscr{A}$ *is* G*-finite. Then the Dye correspondence conserves type.*

For a von Neumann algebra $\mathscr{A}$ of type II, the conservation of type in Theorem 4.10 does not hold. In fact, it is easily seen that each intermediate subalgebra $\mathscr{B}$ is then of type II since the center of $\mathscr{B}$ is contained in $\mathscr{X}$.

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IBARAKI UNIVERSITY
FACULTY OF ENGINEERING
HITACHI, IBARAKI, JAPAN