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ON A STRUCTURE SIMILAR TO SASAKIAN 3-STRUCTURE

Didicated to Professor Shigeo Sasaki on his 60th birthday

ISUKE SATÔ

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Recently, Kuo [4] has introduced the notion of Riemannian space with almost contact 3-structure and since then Kashiwada [3], Sasaki [6], Tachibana and Yu [8] and Tanno [9] have studied several interesting subjects concerning this structure or Sasakian 3-structure. As is well known, an almost contact space is odd dimensional and a space with the structure above stated is of dimensionality $4n + 3$ (n: non-negative integer). So, the case where the dimensionality is $4n + 1$ remains on our discussion. The main purpose of this paper is to discuss a structure similar to almost contact 3-structure or Sasakian 3-structure for the case of dimension $4n + 1$.

In § 1, we shall define an almost contact 3-structure of the second kind similar to the one defined by Kuo [4]. In § 2, it is proved that the structure group of the tangent bundle of this space is reduced to $\mathrm{Sp}(n) \times 1$, and the converse is also true. In § 3, we introduce a 3-structure which is similar to Sasakian 3-structure. One of the three structures is a Sasakian structure and the remaining two are almost contact structures such that their fundamental 2-forms are Killing tensors. In § 4, we deal with the case where these Killing tensors satisfy a certain condition. In this case we prove that our space is an Einstein space. Finally in § 5, we give some examples.

1. Almost contact 3-structure of the second kind. Let M be an m-dimensional differentiable manifold, which admits two almost contact structures (φ, ξ, η) and (ψ, ξ, η) such that

$$\varphi\psi + \psi\varphi = 0 .$$

Define tensor field θ by

$$\theta = \varphi\psi , \tag{1.1}$$

then we can easily prove that (θ, ξ, η) is an almost contact structure too and that the following identities are valid:

$$
(1.2) \quad
\begin{aligned}
&\varphi^2 = \psi^2 = \theta^2 = -E + \eta \otimes \xi\,, \\
&\theta = \varphi\psi = -\psi\varphi\,, \quad \varphi = \psi\theta = -\theta\psi\,, \quad \psi = \theta\varphi = -\varphi\theta\,, \\
&\varphi\xi = \psi\xi = \theta\xi = 0\,, \quad \eta\varphi = \eta\psi = \eta\theta = 0\,.
\end{aligned}
$$

PROPOSITION 1.1. *The dimension of a space with the structure above stated is always* $4n+1$, *n being an integer* $\geqq 1$.

PROOF. Let M_p be a tangent space at p of M and put

$$V_p = \{X \in M_p \mid \eta(X) = 0\}\,.$$

Then we can easily see from (1.2) that V_p admits an almost quaternion structure. Hence $\dim V_p = 4n$ and so $\dim M_p = 4n + 1$.

A space with the structure above mentioned is nothing but an almost quaternion contact space by Hashimoto [2]. We shall say that this space has an almost contact 3-structure of the second kind. It is known that if M has an almost contact 3-structure of the second kind, there exists a positive definite Riemannian metric g such that

$$
(1.3) \quad
\begin{aligned}
&\eta(X) = g(X, \xi)\,, \\
&g(\varphi X, \varphi Y) = g(\psi X, \psi Y) = g(\theta X, \theta Y) = g(X, Y) - \eta(X)\eta(Y)
\end{aligned}
$$

hold good for any vector fields X, Y on M. This metric g is called an associated metric of the structure.

LEMMA [8]. *Let M be a differentiable manifold with an almost quaternion structure* $\Phi_{(a)}$ $(a = 1, 2, 3)$, *i.e., three almost complex structures satisfying*

$$\Phi_{(1)}\Phi_{(2)} = -\Phi_{(2)}\Phi_{(1)} = \Phi_{(3)}\,,$$

then there does not exist an almost complex structure $\Phi_{(4)}$ *such that*

$$\Phi_{(a)}\Phi_{(4)} = -\Phi_{(4)}\Phi_{(a)}\,.$$

PROPOSITION 1.2. *There does not exist an almost contact structure* (ρ, ξ, η) *satisfying*

$$\varphi\rho = -\rho\varphi, \quad \psi\rho = -\rho\psi, \quad \theta\rho = -\rho\theta$$

for an almost contact 3-structure of the second kind $(\varphi, \psi, \theta, \xi, \eta)$.

PROOF. Take the vector space V_p appeared in the proof of Proposition 1.1. Applying the above Lemma for V_p, it follows that our assertion is true.

2. Structure group of the tangent bundle. In a space M with an almost contact 3-structure of the second kind, let g be an associated metric of the almost contact 3-structure of the second kind and let $\{U_\alpha\}$

be an open covering of M by coordinate neighborhoods. Let X_1 be a unit vector field over U_α, orthogonal to ξ with respect to g. Then ξ, X_1, φX_1, ψX_1 and θX_1 are orthonormal. If $n > 1$, we may take a unit vector field X_2 over U_α, orthogonal to ξ, X_1, φX_1, ψX_1 and θX_1. Then these vector fields and X_2, φX_2, ψX_2, θX_2 are orthogonal. Proceeding similarly further, in every U_α, we finally can choose $4n+1$ orthonormal vector fields

$$X_\lambda,\ \varphi X_\lambda,\ \psi X_\lambda,\ \theta X_\lambda,\ \xi \qquad (\lambda = 1, 2, \cdots, n)\,.$$

We call this an adapted frame and denote it by (u).

Then with respect to the adapted frame, the structure tensors g, φ, ψ and ξ have components:

$$g = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \qquad \varphi = \left(\begin{array}{cccc:c} 0 & I_n & 0 & 0 & \\ -I_n & 0 & 0 & 0 & \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & -I_n & 0 & \\ \hdashline & & 0 & & 0 \end{array}\right),$$

(2.1)

$$\psi = \left(\begin{array}{cccc:c} 0 & 0 & I_n & 0 & \\ 0 & 0 & 0 & -I_n & \\ -I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & \\ \hdashline & & 0 & & 0 \end{array}\right), \qquad \xi = {}^t(0, 0, \cdots, 0, 1)\,,$$

where I_n denotes $n \times n$ unit matrix.

Now take another adapted frame $(\bar{u})$, then we have

$$\bar{u} = \gamma u$$

where γ is an orthogonal matrix such that

$$\gamma = \begin{pmatrix} A_{4n} & 0 \\ 0 & 1 \end{pmatrix},$$

As the tensors g, φ and ψ have same components as (2.1) with respect to $(\bar{u})$, we can easily see that A_{4n} must have the form

$$A_{4n} = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

where a, b, c and d denote $n \times n$ matrices. Thus the group of the tangent bundle of M can be reduced to $\mathrm{Sp}(n) \times 1$.

Conversely, suppose that M be a $(4n+1)$-dimensional differentiable manifold such that the group of its tangent bundle reduces to $\mathrm{Sp}(n) \times 1$. Let $\{U_\alpha\}$ be an open covering of M by coordinate neighborhoods. By assumption, we can take frames over every U_α so that, if $U_\alpha \cap U_\beta$ is not empty, the transformation of the same vector with respect to frames of U_α and U_β is given by a matrix of $\mathrm{Sp}(n) \times 1$. In each U_α, take the tensor field g of type (0.2), tensor fields φ, ψ of type (1.1) and the contravariant vector field ξ having (2.1) as components and covariant vector field η with components $(0, 0, \cdots, 0, 1)$ with respect to these frames. As the components of g satisfy

$$g = \gamma g^t \gamma, \quad \text{for} \quad \gamma \in \mathrm{Sp}(n) \times 1 ,$$

all such tensor fields over U_α's constitute a single positive definite tensor field g over M. The same is true for φ, ψ, ξ, η. It is easily seen that (1.2) and (1.3) hold good with respect to these frames. Since these equations are all tensor equations, they hold for every natural frame too. Thus we have

THEOREM 2.1. *A necessary and sufficient condition for a $(4n+1)$-dimensional space to admit an almost contact 3-structure of the second kind is that the group of the tangent bundle of the space is reducible to the group* $\mathrm{Sp}(n) \times 1$. (This theorem owes to T. Kashiwada.)

3. Sasaki-Killing structure. In this section we shall define a structure similar to Sasakian 3-structure. A $(4n+1)$-dimensional Sasakian space M (or normal contact metric space) is by definition a Riemannian space which admits a unit Killing vector field ξ such that

$$\begin{aligned} R(X, \xi)Y &= -g(X, Y)\xi + g(\xi, Y)X \qquad \text{or} \\ \nabla_X(\nabla\xi)Y &= -g(X, Y)\xi + g(\xi, Y)X , \end{aligned} \tag{3.1}$$

where X and Y are vector fields on M and R and ∇ are the Riemannian curvature tensor and the Riemannian connection. If we put $\varphi = \nabla\xi$, then (φ, ξ, η) $(\eta(X) = g(\xi, X))$ gives an almost contact structure on M and g is its associated metric. Furthermore suppose that the Sasakian space M admits another almost contact metric structure (ψ, ξ, η, g) having the following properties:

(i) the 2-form $\tilde{\psi}$ defined by $\tilde{\psi}(X, Y) = g(\psi X, Y)$ is a Killing form, i.e.,

$$(\nabla_X \tilde{\psi})X = 0 . \tag{3.2}$$

(ii) for the two tensors φ, ψ,

$$\varphi\psi + \psi\varphi = 0 \tag{3.3}$$

holds good. Then M is said to have a Sasaki–Killing structure, for brevity, an SK-structure and a space with such a structure is called an SK-space.

Define a tensor field θ by

$$\theta = \varphi\psi\ , \tag{3.4}$$

then (θ, ξ, η, g) is also an almost contact metric structure (see § 1).

Hereafter, in order to simplify the statement, we sometimes use classical tensor notation and replace ξ by η. Define a 2-form $\tilde{\theta}$ by $\tilde{\theta}(X, Y) = g(\theta X, Y)$. Then we have

PROPOSITION 3.1. *In an SK-space 2-form $\tilde{\theta}$ is also a Killing form.*

PROOF. Let $\eta^h, \psi_{rh} = \psi_r{}^s g_{sh}$ and $\theta_{rh} = \theta_r{}^s g_{sh}$ $(\theta_r{}^s = \varphi_t{}^s \psi_r{}^t)$ are the local components of ξ, $\tilde{\psi}$ and $\tilde{\theta}$ respectively. Applying ∇_i to $\eta^r \psi_{rh} = 0$, we have

$$0 = \varphi_i{}^r \psi_{rh} + \eta^r \nabla_i \psi_{rh} = -\theta_{ih} - \eta^r \nabla_r \psi_{ih}\ .$$

Operating ∇_j to the last equation, we get

$$\nabla_j \theta_{ih} = -\varphi_j{}^r \nabla_r \psi_{ih} + \eta^r \nabla_j \nabla_i \psi_{rh}\ . \tag{3.5}$$

Since ψ_{ih} is Killing tensor,

$$\nabla_j \nabla_i \psi_{rh} = (1/2)(R_{sjir}\psi_h{}^s + R_{sjrh}\psi_i{}^s + R_{sjhi}\psi_r{}^s)$$

holds good [7]. Making use of the last equation, we have

$$\eta^r \nabla_j \nabla_i \psi_{rh} = \eta_j \psi_{ih}\ .$$

Substituting this into (3.5) we get

$$\nabla_j \theta_{ih} = -\varphi_j{}^r \nabla_r \psi_{ih} + \eta_j \psi_{ih}\ . \tag{3.5$'$}$$

And operating ∇_i to $\theta_{jh} = -\varphi_j{}^r \psi_{rh}$, we have

$$\nabla_i \theta_{jh} = \varphi_j{}^r \nabla_r \psi_{ih} - \eta_j \psi_{ih}\ . \tag{3.6}$$

From (3.5)$'$ and (3.6), we have

$$\nabla_j \theta_{ih} + \nabla_i \theta_{jh} = 0\ ,$$

i.e., θ_{ih} is a Killing tensor.

Next, by Ricci identity

$$\nabla_k \nabla_j \theta_i{}^h - \nabla_j \nabla_k \theta_i{}^h = R_{kjr}{}^h \theta_i{}^r - R_{kji}{}^r \theta_r{}^h\ . \tag{3.7}$$

Contracting with respect to h and k in the last equation, we have by virtue of Bianchi identity and $\nabla_r \theta_i{}^r = 0$ the relation

(3.8) $$\nabla_r \nabla_j \theta_i^{\ r} = R_{jr}\theta_i^{\ r} - (1/2)\theta^{sr}R_{srji}\ ,$$

from which we get

(3.9) $$R_{jr}\theta_i^{\ r} + R_{ir}\theta_j^{\ r} = 0\ .$$

Transvecting (3.8) with $\theta_h^{\ i}$ we have

(3.10) $$\nabla_r \nabla_j \theta_i^{\ r} \cdot \theta_h^{\ i} = -R_{jh} + 4n\eta_j\eta_h + R^*{}_{hj}(\theta)\ ,$$

where we have put

$$R^*{}_{jh}(\theta) = (1/2)\theta^{sr}R_{srth}\theta_j^{\ t}\ .$$

Applying $\nabla_k \nabla_j$ to $\theta_i^{\ r}\theta_h^{\ i} = -\delta_h^{\ r} + \eta_h\eta^r$ and then contracting with respect to r and k we have

(3.11) $$\nabla_r \nabla_j \theta_i^{\ r} \cdot \theta_h^{\ i} + \nabla_j \theta_{ir} \cdot \nabla_h \theta^{ir} - \theta^{ri}\nabla_r \nabla_i \theta_{jh} = -g_{jh} + (4n+1)\eta_j\eta_h\ .$$

Transvecting (3.7) with θ^{kj} we have

$$\theta^{rs}\nabla_r \nabla_s \theta_{ih} = R^*{}_{ih}(\theta) - R^*{}_{hi}(\theta)\ .$$

As $R^*{}_{ih}(\theta)$ is symmetric with respect to i and h [5], the last equation reduces to

(3.12) $$\theta^{rs}\nabla_r \nabla_s \theta_{ih} = 0\ .$$

Making use of (3.10), (3.11) and (3.12), we have

(3.13) $$R_{jh} - R^*{}_{jh}(\theta) = g_{jh} - \eta_j\eta_h + \nabla_j \theta_{sr} \cdot \nabla_h \theta^{sr}\ ,$$

from which we get

(3.14) $$R - R^*(\theta) = 4n + \nabla_j \theta_{sr} \cdot \nabla^j \theta^{sr}\ ,$$

where we have put

$$R^*(\theta) = R^*{}_{jh}(\theta)g^{jh}\ .$$

PROPOSITION 3.2. *In an SK-space we have*

(3.15) $$R^*{}_{kj}(\theta) = g_{kj} - \eta_k\eta_j\ .$$

PROOF. Operating ∇_k to (3.1) we find

$$\nabla_k \nabla_j \varphi_{ih} = \varphi_{ki}g_{jh} - \varphi_{kh}g_{ji}\ .$$

From this and Ricci identity for φ_{ih}, we have

$$(\varphi_{ki}g_{jh} - \varphi_{kh}g_{ji}) - (\varphi_{ji}g_{kh} - \varphi_{jh}g_{ki}) = -\varphi_{rh}R_{kji}^{\ \ \ r} - \varphi_{ir}R_{kjh}^{\ \ \ r}\ .$$

Transvecting the last equation with ψ^{ih}, we get

$$R_{kji}^{\ \ \ r}\theta_r^{\ i} = 2\theta_{kj}\ ,$$

from which we can easily see that (3.15) holds good.

4. Special Sasaki-Killing structure. When, in an SK-space M, Killing tensor ψ_{ih} is special, that is, ψ_{ih} satisfies

$$\nabla_k\nabla_j\psi_{ih} = -c(g_{kj}\psi_{ih} + g_{ki}\psi_{hj} + g_{kh}\psi_{ji}) \,, \tag{4.1}$$

where c is a constant, then such an SK-space is called a special SK-space. Substituting (4.1) into (3.5) we have

$$\nabla_j\theta_{ih} = -\varphi_j{}^r\nabla_r\psi_{ih} + c\eta_j\psi_{ih} \,. \tag{4.2}$$

From this and (3.5)′ we find $c = 1$.

PROPOSITION 4.1. *In a special SK-space, the Killing tensor θ_{ih} is also special.*

PROOF. Applying ∇_k to (4.2) substituted 1 for c, we have

$$\begin{aligned}\nabla_k\nabla_j\theta_{ih} &= -(\eta_j\delta_k^r - \eta^r g_{kj})\nabla_r\psi_{ih} + \varphi_j{}^r(g_{kr}\psi_{ih} + g_{ki}\psi_{hr} + g_{kh}\psi_{ri}) \\ &\quad + \varphi_{kj}\psi_{ih} + \eta_j\nabla_k\psi_{ih} = g_{kj}\eta^r\nabla_r\psi_{ih} + g_{ki}\theta_{jh} - g_{kh}\theta_{ji} \\ &= -(g_{kj}\theta_{ih} + g_{ki}\theta_{hj} + g_{kh}\theta_{ji}) \,.\end{aligned}$$

This shows that θ_{ih} is special.

PROPOSITION 4.2. *In a special SK-space we have*

$$R_{kj} - R^*{}_{kj}(\theta) = (4n-1)g_{kj} + \eta_k\eta_j \,, \qquad R - R^*(\theta) = 16n^2 \,.$$

PROOF. From $\theta_{sr}\theta^{sr} = 4n$, we have

$$\nabla_j\theta_{sr}\cdot\nabla_h\theta^{sr} = -\theta^{sr}\nabla_h\nabla_j\theta_{sr} = (4n-2)g_{jh} + 2\eta_j\eta_h \,,$$

from which we get

$$\nabla_j\theta_{sr}\cdot\nabla^j\theta^{sr} = 16n^2 - 4n \,.$$

THEOREM 4.3. *A $(4n+1)$-dimensional special SK-space is an Einstein space with scalar curvature $4n(4n+1)$.*

PROOF. By Proposition 3.2 and 4.2, it is evident.

COROLLARY 4.4. *If a special SK-space is of constant φ-holomorphic sectional curvature with respect to its Sasakian structure, then our space is of constant curvature* 1.

PROOF. Let k be the constant φ-holomorphic sectional curvature. Then, as is well known, the Ricci tensor of the space satisfies

$$2R_{ji} = [2n(k+3) + k - 1]g_{ji} - (2n+1)(k-1)\eta_j\eta_i \,.$$

On the other hand, by Theorem 4.3, our space is an Einstein one. Hence the last equation gives us $k = 1$, which is to be proved.

5. Examples. (i) We take a $(4n+1)$-dimensional number space. We define structure tensors φ, ψ, ξ by (2.1) and η by $(0, \cdots, 0, 1)$ with constant components, then they and $\theta = \varphi\psi$ define an almost contact 3-structure of the second kind and g defined by (2.1) is an associated metric with respect to the almost contact 3-structure of the second kind.

(ii) We take a Cayley space R, that is, a 7-dimensional Euclidean space considered as the space of purely imaginary Cayley numbers. It is well known [1], [11] that a Cayley space is characterized by the existence of a bilinear scalar product $A \cdot B$ and bilinear skew-symmetric vector product $A \times B$ satisfying

$$(A \times B) \cdot C = A \cdot (B \times C) \tag{5.1}$$

$$\begin{aligned} &(A \times B) \times C - (A \cdot C) \cdot B + (B \cdot C) \cdot A \\ &= -A \times (B \times C) + (A \cdot C) \cdot B - (A \cdot B) \cdot C , \end{aligned} \tag{5.2}$$

both members of (5.2) being not identically zero.

We consider a 5-dimensional subspace V in R. If we denote by X the position vector in R, then V will be represented locally by a parametric equation

$$X = X(x^h) ,$$

x^h being coordinates on the subspace where and in this section the indices run over the range $1, 2, \cdots, 5$. The vectors

$$e_i = \partial_i X \qquad (\partial_i = \partial / \partial x^i)$$

tangent to V are linearly independent. The subspace V is a Riemannian space with metric $g_{ji} = e_j \cdot e_i$ naturally induced from R.

Assume that V is oriented and that the orthonormal vectors n and m orthogonal to V may be globally taken along V. We can easily see the existence of such a subspace in R (for example, 5-dimensional sphere in R which will be appear).

Now we put, in (5.1), $A = e_i$, $B = C = n$ (resp. $A = e_i$, $B = C = m$), then we have $(e_i \times n) \cdot n = 0$ (resp. $(e_i \times m) \cdot m = 0$) which shows that $e_i \times n$ (resp. $e_i \times m$) is linear combination of e_j and m (resp. e_j and n):

$$e_i \times n = \varphi_i{}^j e_j + \eta_i m , \qquad e_i \times m = \psi_i{}^j e_j + \rho_i n , \tag{5.3}$$

where the dot of the scalar product will be omitted hereafter.

Furthermore, in (5.1), putting $A = n$, $B = m$, $C = n$ (or $A = m$, $B = n$, $C = m$) we see that $n \times m$ is tangent to V. Therefore we can put

$$n \times m = \xi^j e_j . \tag{5.4}$$

Putting $A = e_i$, $B = C = n$ and $A = e_i$, $B = C = m$ in (5.2) we have

$$(e_i \times n) \times n + e_i = 0\,, \qquad (e_i \times m) \times m + e_i = 0\,. \tag{5.5}$$

In (5.2), putting $A = n$, $B = C = m$ and $A = m$, $B = C = n$, we have by virtue of (5.3) and (5.4),

$$\eta_i \xi^i = 1\,, \quad \varphi_i{}^h \xi^i = 0\,, \quad \psi_i{}^h \xi^i = 0\,, \quad \rho_i \xi^i = -1\,. \tag{5.6}$$

The scalar product of $(5.3)_1$ and m (resp. $(5.3)_2$ and n) gives us

$$\eta_i = \xi_i = -\rho_i\,, \qquad (\xi_i = g_{ir}\xi^r)\,. \tag{5.7}$$

Similarly, taking the vector product $(5.3)_1$ and n (resp. $(5.3)_2$ and m) and using $(5.5)_1$ (resp. $(5.5)_2$) we have

$$\begin{aligned} \varphi_i{}^r \varphi_r{}^h &= -\delta_i^h + \eta_i \xi^h\,, & \varphi_i{}^r \eta_r &= 0\,, \\ \psi_i{}^r \psi_r{}^h &= -\delta_i^h - \rho_i \xi^h\,, & \psi_i{}^r \rho_r &= 0\,. \end{aligned} \tag{5.8}$$

From (5.6), (5.7) and (5.8), we see that V admits two almost contact metric structures (φ, ξ, η, g), (ψ, ξ, η, g) on V (cf. [10]). Next if we put

$$e_j \times e_i = T_{ji}{}^r e_r + T_{ji} n + S_{ji} m\,, \tag{5.9}$$

then using (5.1) we see that

$$T_{ji} = -\varphi_{ji} (= -\varphi_j{}^r g_{ri})\,, \qquad S_{ji} = -\psi_{ji} (= -\psi_j{}^r g_{ri})\,,$$

and

$$T_{jih} = T_{ji}{}^r g_{rh}$$

being skew-symmetric in all of its indices. In (5.2), putting $A = e_j$, $B = n$, $C = e_i$, we have

$$\varphi_j{}^r \psi_{ri} = -\psi_j{}^r \varphi_{ri}\,.$$

And if we put

$$\theta_i{}^h = \varphi_r{}^h \psi_i{}^r\,, \tag{5.10}$$

then it follows that (θ, ξ, η, g) is also an almost contact metric structure. From the above fact we can easily see that V has three almost contact metric structures (φ, ξ, η, g), (ψ, ξ, η, g) and (θ, ξ, η, g) such that

$$\theta = \varphi\psi = -\psi\varphi, \quad \varphi = \psi\theta = -\theta\psi, \quad \psi = \theta\varphi = -\varphi\theta\,.$$

Thus we have

THEOREM 5.1. *In a Cayley space, an orientable 5-dimensional subspace V with unit normals globally defined along V has an almost contact 3-structure of the second kind.*

REMARK. If a space admits r almost contact structures $(\varphi_\lambda, \xi, \eta)(\lambda = 1, 2, \cdots, r)$, then we say that they are linearly independent if

$$a_1\varphi_1 + a_2\varphi_2 + \cdots + a_r\varphi_r = 0 \qquad (a_1, a_2, \cdots, a_r\text{: real constants})$$

always implies $a_1 = a_2 = \cdots = a_r = 0$. We can see that φ, ψ, θ on V are linearly independents in the above sense.

Next, we denotes by ∇_i the so-called van der Waerden-Bortolotti covariant differentiation along V with respect to the Riemannian connection determined by g_{ji}. Then the derived equations of Gauss and Weingarten for V are respectively given by

$$\begin{aligned} &\nabla_j e_i = H_{ji} n + K_{ji} m\,, \\ &\nabla_j n = -H_j{}^r e_r + L_j m\,, \qquad \nabla_j m = -K_j{}^r e_r - L_j n\,, \end{aligned} \tag{5.11}$$

where H_{ji}, K_{ji} and L_j are the second fundamental tensors and the third fundamental tensor of V and we have put $H_j{}^r = H_{js}g^{sr}$, $K_j{}^r = K_{js}g^{sr}$. Operating ∇_j to (5.3) and (5.4) along V and using (5.11), we have after some calculations

$$\begin{aligned} \nabla_j\varphi_{ih} &= H_{jr}T_{ih}{}^r + K_{jh}\eta_i - K_{ji}\eta_h + L_j\psi_{ih}\,, \\ \nabla_j\psi_{ih} &= K_{jr}T_{ih}{}^r - H_{jh}\eta_i + H_{ji}\eta_h - L_j\varphi_{ih}\,, \\ \nabla_j\eta_i &= -H_j{}^r\psi_{ri} + K_j{}^r\varphi_{ri}\,. \end{aligned} \tag{5.12}$$

Now we consider a 5-dimensional sphere S in R which is represented by

$$(x^1)^2 + (x^2)^2 + \cdots + (x^6)^2 = 1,\ x^7 = 0\,. \tag{5.13}$$

An orthonormal normal vectors m, n of S are given respectively by

$$m = -x\,, \qquad n = (0, 0, \cdots, 0, 1)\,.$$

In this case we have

$$H_{ji} = 0\,, \qquad K_{ji} = g_{ji}\,.$$

Substituting this into (5.11), we have

$$\nabla_j e_i = g_{ji} m\,, \qquad \nabla_j n = L_j m\,, \qquad \nabla_j m = -e_j - L_j n\,. \tag{5.11$'$}$$

The equations of Gauss and Codazzi for S are easily seen to be

$$R_{kjih} = g_{ji}g_{kh} - g_{ki}g_{jh}\,, \qquad L_j g_{ih} - L_i g_{jh} = 0\,, \qquad \nabla_j L_i - \nabla_i L_j = 0\,,$$

and so we have $L_j = 0$. Hence (5.12) reduces in this case to

$$\nabla_j\varphi_{ih} = \eta_i g_{jh} - \eta_h g_{ji}\,, \qquad \nabla_j\psi_{ih} = T_{jih}\,, \qquad \nabla_i\eta_h = \varphi_{ih}\,. \tag{5.12$'$}$$

By (5.12)$'$ we see that ψ_{ih} is a Killing tensor. Since in a Riemannian space of constant curvature every Killing tensor is special [8], we have

THEOREM 5.2. *In a Cayley space, let S be a 5-dimensional unit sphere expressed as* (5.13). *Then S has a special SK-structure.*

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DEPARTMENT OF MATHEMATICS
YAMAGATA UNIVERSITY
YAMAGATA, JAPAN