Tôhoku Math. Journ.
26 (1974), 325-332.

ON APPROXIMATELY FINITE ALGEBRAS

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(Received March 12, 1973)

1. Introduction. The notion of the approximately finite (hyperfinite) factors on separable Hilbert space was first introduced by Murray and von Neumann [9; Chapter IV]. These are factors of type II_1 which can be suitably approximated by finite dimensional subalgebras. Four kinds of approximate finiteness were given ([9; Def. 4.1.1, 4.3.1, 4.5.2, and 4.6.1]) and it was shown that all of them are equivalent to each other (Theorem XII). We list here two of them, the approximate finiteness (A) and (B).

DEFINITION 1. A factor $\mathscr{A}$ of type II_1 is approximately finite if, for each $\varepsilon > 0$ and each finite set $A_1, \cdots, A_m$ of operators in $\mathscr{A}$, there exists an $n = n(A_1, \cdots, A_m, \varepsilon)$ such that for every $q \geqq n$ there exists a subfactor of type I_q of $\mathscr{A}$ containing elements $B_1, \cdots, B_m$ with $[[B_i - A_i]] < \varepsilon$ for $i = 1, \cdots, m$, where $[[\cdot]]$ denotes the trace norm in $\mathscr{A}$.

DEFINITION 2. A factor $\mathscr{A}$ of type II_1 is approximately finite if, for each $\varepsilon > 0$ and each finite set $A_1, \cdots, A_m$ of operators in $\mathscr{A}$, there exists a finite dimensional *-subalgebra $\mathscr{B}$ of $\mathscr{A}$ containing elements $B_1, \cdots, B_m$ such that $[[B_i - A_i]] < \varepsilon$ for $i = 1, \cdots, m$.

Now, a finite dimensional *-subalgebra $\mathscr{B}$ is of type I but not of general center. Also in other definitions the sub- algebras $\mathscr{B}$ which approximate $\mathscr{A}$ are required to be factors or of finite dimension. The first purpose of this paper is to show that in Definition 2 the subalgebra $\mathscr{B}$ need not be finite dimensional but only to be of type I (Theorem 1). This result is generalized from the factor case to the case of general center. The notion of approximate finiteness was generalized by Misonou [8] and Widom [12] to the case of von Neumann algebra $\mathscr{A}$ of general center. In this case the type I subalgebra $\mathscr{B}$ which approximates $\mathscr{A}$ is required to have the same center as $\mathscr{A}$. We shall show that it suffices to require that the center of $\mathscr{B}$ contains the center of $\mathscr{A}$ (Theorem 2). This justifies the definition of approximate finiteness by Golodets [5; Def. 3.1.1]. An application of Theorem 2 will be shown in the last section.

2. Preliminaries. Our main tool is the reduction theory of von

Neumann algebras. Let $\mathscr{B}$ be a von Neumann algebra on a separable Hilbert space $\mathscr{H}$. Then there exists a complete separable metric space Λ, a finite positive Borel measure μ on Λ, a measurable family of Hilbert spaces $\mathscr{H}(\lambda)$, $\lambda \in \Lambda$, and a measurable family of factors $\mathscr{B}(\lambda)$ on $\mathscr{H}(\lambda)$ such that

$$\mathscr{H} = \int_\Lambda^\oplus \mathscr{H}(\lambda) d\mu(\lambda) \quad \text{and} \quad \mathscr{B} = \int_\Lambda^\oplus \mathscr{B}(\lambda) d\mu(\lambda)$$

([10; Cor. I. 5.10]). This setup describes the central, direct integral decomposition of $\mathscr{B}$.

If $\mathscr{B}$ is of type I, each $\mathscr{B}(\lambda)$ is a factor of type I for μ-almost all $\lambda \in \Lambda$. Put

$$\Lambda_n = \{\lambda \in \Lambda \mid \mathscr{B}(\lambda) \quad \text{is of type} \quad \mathrm{I}_n\}$$

for $1 \leqq n \leqq \infty$. Then $\mathscr{B}_n = \int_{\Lambda_n}^\oplus \mathscr{B}(\lambda) d\mu(\lambda)$ is a von Neumann algebra of type I_n on $\mathscr{H}_n = \int_{\Lambda_n}^\oplus \mathscr{H}(\lambda) d\mu(\lambda)$ and $\mathscr{B} = \sum_{1 \leqq n \leqq \infty} \oplus \mathscr{B}_n$ ([1; III. 3.1]). Each operator $B = \int_{\Lambda_n}^\oplus B(\lambda) d\mu(\lambda)$ in $\mathscr{B}_n$ may be regarded as an operator in $\mathscr{B}$ defined by $B = \int_\Lambda^\oplus B(\lambda) d\mu(\lambda)$ where $B(\lambda) = 0$ for $\lambda \notin \Lambda_n$. Hence, if a faithful normal trace τ is given in $\mathscr{B}$, the trace norm $[[B]] = \tau(B^*B)^{1/2}$ for B in $\mathscr{B}_n$ will mean always that of B as an element of $\mathscr{B}$.

LEMMA 1. *Let $\mathscr{B}$ be a von Neumann algebra of type $\mathrm{I}_n (n < \infty)$ with finite faithful normal trace τ. Then, for each operator B in $\mathscr{B}$ and each $\varepsilon > 0$, there exists a finite dimensional *-subalgebra $\mathscr{C}$ of $\mathscr{B}$ and an operator C in $\mathscr{C}$ such that $[[C - B]] < \varepsilon$.*

PROOF. *Let* $\mathscr{B} = \int_\Lambda^\oplus \mathscr{B}(\lambda) d\mu(\lambda)$ be the central decomposition of $\mathscr{B}$, each $\mathscr{B}(\lambda)$ hence being a factor of type I_n. So far as algebraic properties concern, we may and do assume that each $\mathscr{B}(\lambda)$ acts on an n-dimensional Hilbert space $\mathscr{H}_n$ (cf. [11; Theorem 3]). Let

$$B = \int_\Lambda^\oplus B(\lambda) d\mu(\lambda)$$

be the decomposition of the operator B. Then $\mathscr{B}(\lambda)$ is a μ-measurable operator-valued function on Λ ([10; p. 19]), i.e., $B(\lambda)\xi(\lambda)$ is a μ-measurable vector-valued function (cf. [2; Def. III. 2.10]) for each $\xi(\lambda)$ in the direct integral Hilbert space $\mathscr{H} = \int_\Lambda^\oplus \mathscr{H}(\lambda) d\mu(\lambda)$, $\mathscr{H}(\lambda) = \mathscr{H}_n$. Hence ([2; III. 2.7]) there exists a sequence of μ-simple vector-valued functions $\eta_k(\lambda)$ $(k = 1, 2, \cdots)$ such that, for each $\varepsilon > 0$,

$$\mu\{\lambda \in \Lambda \mid \|B(\lambda)\xi(\lambda) - \eta_k(\lambda)\| > \varepsilon\} \to 0$$

as $k \to \infty$. Let $\{\xi_p\}_{1 \leq p \leq n}$ be a fixed basis of $\mathscr{H}_n$. Then we may choose sequences of vector-valued functions $\{\eta_{p,k}(\lambda)\}_{k=1,2,\cdots}(p = 1, \cdots, n)$ such that

$$\|B(\lambda)\xi_p - \eta_{p,k}(\lambda)\| \to 0 \qquad (k \to \infty)$$

in μ-measure. Clearly we may assume the functions $\|\eta_{p,k}(\lambda)\|$ to be uniformly bounded. Define the operator-valued functions $B_k(\lambda)$ by

$$B_k(\lambda)\xi_p = \eta_{p,k}(\lambda) .$$

Then we see easily that $B_k(\lambda)$ are well-defined and bounded, that $B_k(\lambda)\xi$ converges to $B(\lambda)\xi$ in μ-measure for each $\xi \in \mathscr{H}_n$ and that each $B_k(\lambda)\xi$ is μ-simple for each $\xi \in \mathscr{H}_n$ since $B_k(\lambda)\xi_p$ are μ-simple. Hence each $B_k(\lambda)$ is a μ-simple function and the direct integral operator

$$B_k = \int_\Lambda^\oplus B_k(\lambda)d\mu(\lambda) \qquad (k = 1, 2, \cdots)$$

is clearly in $\mathscr{B}$. By the uniform boundedness of $\|\eta_{p,k}(\lambda)\|$ we see that

$$\sup_k \|B_k\| < \infty$$

and hence that B_k converges strongly to B ([10; Lemma I. 3.6]). Therefore, if we put $C = B_k$ for sufficiently large k, we see that

$$[[C - B]] < \varepsilon$$

([9; Lemma 1.3.2]).

Now the function $C(\lambda) = B_k(\lambda)$ takes a finite number of values, say, $C^{(1)}, \cdots, C^{(r)}$ and the set

$$\Lambda_j = \{\lambda \in \Lambda \mid C(\lambda) = C^{(j)} \ (j = 1, \cdots, r)\}$$

are all measurable. Since the von Neumann algebra generated by $C^{(j)}$ on $\int_{\Lambda_j}^\oplus \mathscr{H}(\lambda)d\mu(\lambda)$ has a finite linear basis composed of matrix units, the von Neumann algebra $\mathscr{C}$ generated by

$$C = \sum_{j=1}^r \oplus \int_{\Lambda_j}^\oplus C(\lambda)d\mu(\lambda)$$

is also a finite dimensional subalgebra of $\mathscr{B}$.

COROLLARY. *Let $\mathscr{B}$ be a type I_n subalgebra of a factor $\mathscr{A}$ of type* II_1 *and $B_1, \cdots, B_m$ a finite set of operators in $\mathscr{B}$. Then, for each $\varepsilon > 0$, there exists a finite dimensional *-subalgebra $\mathscr{C}$ of $\mathscr{B}$ containing elements $C_1, \cdots, C_m$ such that*

$$[[C_i - B_i]] < \varepsilon \qquad (i = 1, \cdots, m) .$$

PROOF. According to the above lemma, there exists for each operator B_i an operator $C_i = \int_\Lambda^\oplus C_i(\lambda)d\mu(\lambda)$ such that $[[C_i - B_i]] < \varepsilon$, where each $C_i(\lambda)$ takes only a finite number of values $C_i^{(1)}, \cdots, C_i^{(r_i)}$. Then, for any sequence $\{j_i\}$ $(i = 1, \cdots, m;\ 1 \leqq j_i \leqq r_i)$, the set

$$\Lambda_{j_1 \cdots j_m} = \{\lambda \in \Lambda \mid C_i(\lambda) = C_i^{(j_i)} \quad \text{for} \quad 1 \leqq i \leqq m\}$$

is measurable and we see that each von Neumann algebra generated by $C_i^{(j_i)} (i = 1, \cdots, m)$ on $\int_{\Lambda_{j_1 \cdots j_m}}^\oplus \mathscr{H}(\lambda)d\mu(\lambda)$ is of finite dimension by the same argument as in Lemma 1. Therefore, the von Neumann algebra generated by $C_i = \int_\Lambda^\oplus C_i(\lambda)d\mu(\lambda)$ $(i = 1, \cdots, m)$ is a finite dimensional subalgebra of $\mathscr{B}$.

3. Approximate finiteness. First we prove that in Definition 2 the algebra $\mathscr{B}$ need only to be of type I.

THEOREM 1. *Let $\mathscr{A}$ be a factor of type* II_1. *If, for each $\varepsilon > 0$ and each finite set $A_1, \cdots, A_m$ of operators in $\mathscr{A}$, there exists a subalgebra $\mathscr{B}$ of type* I (*not necessarily a factor*) *containing elements $B_1, \cdots, B_m$ such that $[[B_i - A_i]] < \varepsilon$ for $i = 1, \cdots, m$, then $\mathscr{A}$ is approximately finite.*

PROOF. If we show the existence of a finite dimensional *-subalgebra $\mathscr{C}$ of $\mathscr{A}$ and operators $C_1, \cdots, C_m$ in $\mathscr{C}$ such that $[[C_i - B_i]] < \varepsilon$, then, combining with the assumption, we see that $[[C_i - A_i]] < 2\varepsilon$ $(i = 1, \cdots, m)$ and hence that $\mathscr{A}$ is an approximately finite factor by Definition 2.

Let $\mathscr{B} = \int_\Lambda^\oplus \mathscr{B}(\lambda)d\mu(\lambda)$ be the central, direct integral decomposition of $\mathscr{B}$. Put

$$\Lambda_n = \{\lambda \in \Lambda \mid \mathscr{B}(\lambda) \quad \text{is a } \mathrm{I}_n\text{-factor}\}$$

and

$$\Lambda_n = \Lambda - \bigcup_{k=1}^{n} \Lambda_k \ .$$

Since $\mathscr{B}$ is finite and of type I, we have, for each $\varepsilon < 0$,

$$\mu(\Lambda_s) < \varepsilon/2\,(\sup_i \|B_i\|)$$

for sufficiently large s, where $B_i (1 \leqq i \leqq m)$ are given operators. For each operator $B_i = \int_\Lambda^\oplus B_i(\lambda)d\mu(\lambda)$, put

$$B_{i,n} = \int_{\Lambda_n}^\oplus B_i(\lambda)d\mu(\lambda) \ .$$

Then, for each von Neumann algebra

$$\mathscr{B}_n = \int_{\Lambda_n}^{\oplus} \mathscr{B}(\lambda) d\mu(\lambda) \quad \text{on} \quad \mathscr{H}_{\Lambda_n} = \int_{\Lambda_n}^{\oplus} \mathscr{H}(\lambda) d\mu(\lambda)$$

($n = 1, \cdots, s$), there exists by the above corollary a finite dimensional *-subalgebra $\mathscr{C}_n$ containing operators $C_{1,n}, \cdots, C_{m,n}$ such that

$$[[C_{i,n} - B_{i,n}]] < \varepsilon/2s \ .$$

On the residual Hilbert space $\mathscr{H}_{\tilde{\Lambda}_s} = \int_{\tilde{\Lambda}_s}^{\oplus} \mathscr{H}(\lambda) d\mu(\lambda)$ we define $\tilde{\mathscr{C}}_s$ to be the von Neumann algebra composed of only scalar multiples of the identity operator on $\mathscr{H}_{\tilde{\Lambda}_s}$. Then the direct sum

$$\mathscr{C} = \left(\sum_{n=1}^{s} \oplus \mathscr{C}_n\right) \oplus \tilde{\mathscr{C}}_s$$

is clearly of finite dimension. Put $C_i = (\sum_{n=1}^{s} \oplus C_{i,n}) \oplus 0$. Since

$$\begin{aligned}\left[\left[\int_{\tilde{\Lambda}_s}^{\oplus} B_i(\lambda) d\mu(\lambda)\right]\right] &\leqq \operatorname*{ess.\,sup}_{\lambda \in \tilde{\Lambda}_s} \|B_i(\lambda)\| \, \mu(\tilde{\Lambda}_s) \\ &< \|B_i\| \cdot \varepsilon/2 \sup_i \|B_i\| \leqq \varepsilon/2 \ ,\end{aligned}$$

we have

$$\begin{aligned}[[C_i - B_i]] &= \sum_{n=1}^{s} [[C_{i,n} - B_{i,n}]] + \left[\left[\int_{\tilde{\Lambda}_s}^{\oplus} B_i(\lambda) d\mu(\lambda)\right]\right] \\ &< s \cdot \varepsilon/2s + \varepsilon/2 = \varepsilon \ .\end{aligned}$$

This completes the proof of the theorem.

Next we generalize this theorem to the case of general center. Hereafter $\mathscr{A}$ denotes a von Neumann algebra of type II_1 with a faithful normal trace τ. For $A \in \mathscr{A}$, define the trace norm $[[A]] = \tau(A^*A)^{1/2}$.

DEFINITION 3. $\mathscr{A}$ is called approximately finite if, for each $\varepsilon > 0$ and each finite set of operators $A_1, \cdots, A_m$, there exists a type I subalgebra $\mathscr{B}$ of $\mathscr{A}$ with center $\mathscr{Z}_{\mathscr{B}}$ identical with the center $\mathscr{Z}_{\mathscr{A}}$ of $\mathscr{A}$ and a sequence of operators $B_1, \cdots, B_m$ in $\mathscr{B}$ such that $[[A_i - B_i]] < \varepsilon$ ($i = 1, \cdots, m$).

In the terminology of Widom [11], this is approximate finiteness (A2). The condition $\mathscr{Z}_{\mathscr{A}} = \mathscr{Z}_{\mathscr{B}}$ has also been required in [8; Def. 3.1]. Recently, Golodets [5] has defined approximately finite algebra as the von Neumann algebra satisfying the conditions in Definition 3 but replacing $\mathscr{Z}_{\mathscr{A}} = \mathscr{Z}_{\mathscr{B}}$ by $\mathscr{Z}_{\mathscr{A}} \subseteqq \mathscr{Z}_{\mathscr{B}}$. He has given no explicit proof of the equivalence of these two definitions. Our generalization is nothing but to show the validity of this replacement.

THEOREM 2. *Let $\mathscr{A}$ be a von Neumann algebra of type* II_1 *with a faithful normal trace τ. If, for each $\varepsilon > 0$ and each finite set $A_1, \cdots, A_m$ of operators in $\mathscr{A}$, there exists a subalgebra $\mathscr{B}$ of type* I *with center $\mathscr{Z}_{\mathscr{B}}$ containing the center $\mathscr{Z}_{\mathscr{A}}$ of $\mathscr{A}$ and operators $B_1, \cdots, B_m$ in $\mathscr{B}$ such that $[[B_i - A_i]] < \varepsilon (i = 1, \cdots, m)$, then $\mathscr{A}$ is an approximately finite algebra.*

PROOF. Let $\mathscr{A} = \int_{\Lambda}^{\oplus} \mathscr{A}(\lambda) d\mu(\lambda)$ be the central, direct integral decomposition of $\mathscr{A}$, each $\mathscr{A}(\lambda)$ being hence a II_1-factor for almost all $\lambda \in \Lambda$. Since $\mathscr{Z}_{\mathscr{B}} \supseteqq \mathscr{Z}_{\mathscr{A}}$, $\mathscr{B}$ is decomposable as $\int_{\Lambda}^{\oplus} \mathscr{B}(\lambda) d\mu(\lambda)$ where each $\mathscr{B}(\lambda)$ is a subalgebra of type I of $\mathscr{A}(\lambda)$ ([10; Lemma I. 5.6 and Theorem I. 5.9]). The trace τ has the expression

$$\tau(A) = \int_{\Lambda} f(\lambda) \tau_{\lambda}(A(\lambda)) d\mu(\lambda)$$

for $A = \int_{\Lambda}^{\oplus} A(\lambda) d\mu(\lambda) \in \mathscr{A}$, where τ_{λ} is the normalized trace in $\mathscr{A}(\lambda)$ and $f(\lambda)$ is a function belonging to $L_1(\Lambda, \mu)$ and positive almost everywhere ([10; Theorem III. 1.13]). Then, for almost all $\lambda \in \Lambda$ and each $\varepsilon > 0$, there exist by the assumption operators $B_1, \cdots, B_m$ in $\mathscr{B}$ such that

$$[[B_i(\lambda) - A_i(\lambda)]]_{\lambda} < \varepsilon$$

where $[[A(\lambda)]]_{\lambda} = \tau_{\lambda}(A(\lambda)^* A(\lambda))^{1/2}$. Therefore, almost all of the factors $\mathscr{A}(\lambda)$ are approximately finite by Theorem 1. Hence $\mathscr{A}$ is isomorphic to the tensor product $\mathscr{Z} \otimes \mathscr{A}_0$ of the center $\mathscr{Z} = L^{\infty}(\Lambda, \mu)$ and an approximately finite factor $\mathscr{A}_0$ ([11; Theorem 3]). Since this is a von Neumann algebra generated by the simple functions in $L^{\infty}(\Lambda, \mu)$ and $\mathscr{A}_0$, we may assume that each operator A_i given in the assumption is the direct integral of a μ-simple operator-valued function $A_i(\lambda)$ which takes a finite number of values, say, $A_i^{(j)} \in \mathscr{A}_0$ $(j = 1, \cdots, r_i)$. Put

$$\Lambda_{j_1 \cdots j_m} = \{\lambda \in \Lambda \mid A_i(\lambda) = A_i^{(j_i)} \ (i = 1, \cdots, m)\} .$$

According to Definition 1, for each $(j_1, \cdots, j_m)$ and each $\varepsilon > 0$, there exists an integer $n = n(j_1, \cdots, j_m, \varepsilon)$ such that for every integer $q \geqq n$ there exists a subfactor $\mathscr{C}_{j_1 \cdots j_m}$ of type I_q of $\mathscr{A}_0$ containing elements $C_1^{(j_1)}, \cdots, C_m^{(j_m)}$ with

$$[[C_i^{(j_i)} - A_i^{(j_i)}]]_{\lambda} < \varepsilon \quad (i = 1, \cdots, m; \lambda \in \Lambda_{j_1 \cdots j_m}) .$$

Taking the maximum of $n(j_1, \cdots, j_m, \varepsilon)$ for all $(j_1, \cdots, j_m)$, we may assume that n depends only on ε. Therefore, putting $\mathscr{C}(\lambda) = C_{j_1 \cdots j_m}$ for $\lambda \in \Lambda_{j_1 \cdots j_m}$, we get a measurable family $\{\mathscr{C}(\lambda)\}$ of type I factors. The operator-valued functions $\mathscr{C}_i(\lambda)$ defined by

$$C_i(\lambda) = C_i^{(j_i)} \quad \text{for} \quad \lambda \in \Lambda_{j_1 \cdots j_m}$$

$(i = 1, \cdots, m)$ are all measurable and the direct integral operators $C_i = \int_\Lambda^\oplus C_i(\lambda) d\mu(\lambda)$ clearly belongs to the type I algebra $C = \int_\Lambda^\oplus \mathscr{C}(\lambda) d\mu(\lambda)$ which has the same center as $\mathscr{A}$. Further,

$$\begin{aligned} [[C_i - A_i]] &= \{\tau[(C_i - A_i)^*(C_i - A_i)]\}^{1/2} \\ &= \left\{\int_\Lambda f(\lambda)\tau_\lambda[(C_i(\lambda) - A_i(\lambda))^*(C_i(\lambda) - A_i(\lambda))] d\mu(\lambda)\right\}^{1/2} \\ &< \varepsilon\left\{\int_\Lambda f(\lambda) d\mu(\lambda)\right\}^{1/2} \quad (i = 1, \cdots, m) . \end{aligned}$$

Thus $\mathscr{C}$ is a type I subalgebra of $\mathscr{A}$ with the same center as $\mathscr{A}$ and contains operators $C_1, \cdots, C_m$ which approximate $A_1, \cdots, A_m$. Therefore, $\mathscr{A}$ is an approximately finite algebra.

4. **An application.** As an application of Theorem 2, we generalize a theorem of Dye [4; Cor. 6.1] which essentially shows the approximate finiteness of the cross product $G \otimes \mathscr{A}$ of an abelian von Neumann algebra $\mathscr{A}$ by an approximately finite group G of its automorphisms. For the definition of the approximately finite group, see [3; Def. 5.1]. In his proof the approximation of $G \otimes \mathscr{A}$ by a type I subalgebra with center larger than that of $G \otimes \mathscr{A}$ is achieved rather easily and the proof is, for the most part, devoted to the discussion of the center of the type I subalgebra. We can slightly generalize this result as follows.

THEOREM 3. *If $\mathscr{A}$ is a finite von Neumann algebra of type* I *and G is a group of automorphisms of $\mathscr{A}$ acting freely and approximately finite on the center of $\mathscr{A}$, then the cross product $G \otimes \mathscr{A}$ is approximately finite.*

Some generalizations of Dye's results to the cross product have been investigated in [6] and [7]. According to [7; Theorem 4.10], we may follow the first paragraph of the proof of [4; Cor. 6.1] word for word. Then by Theorem 2 we get the conclusion of Theorem 3 without complicated discussion for the coincidence of the center of $\mathscr{A}$ and that of type I subalgebra which approximates $\mathscr{A}$.

REFERENCES

[1] J. DIXMIER, Les algèbres d'opérateurs dans l'espace Hilbertien, Gauthier-Villars, Paris, 1957.

[2] N. DUNFORD AND J. T. SCHWARTZ, Linear Operators, Part I, Interscience, New York, 1958.

[3] H. A. DYE, On groups of measure preserving transformations I, Amer. J. Math., 81 (1959), 119-159.

[4] ———, On groups of measure preserving transformations II, Amer. J. Math., 85 (1963), 551-576.

[5] V. JA. GOLODETS, Crossed products of von Neumann algebras, Uspekhi Mat. Nauk, 26 (1971), 3-50 (Russian).

[6] Y. HAGA AND Z. TAKEDA, Correspondence between subgroups and subalgebras in a cross product von Neumann algebra, Tôhoku Math. J., 24 (1972), 167-190.

[7] Y. HAGA, On subalgebras of a cross product von Neumann algebra, Tôhoku Math. J., 25 (1973), 291-305.

[8] Y. MISONOU, Generalized approximately finite W^*-algebras, Tôhoku Math. J., 7 (1955), 192-205.

[9] F. J. MURRAY AND J. VON NEUMANN, On rings of operators IV, Ann. Math., 44 (1943), 716-808.

[10] J. T. SCHWARTZ, W^*-Algebras, Gordon and Breach, New York, 1967.

[11] M. TAKESAKI, Remarks on the reduction theory of von Neumann algebras, Proc. Amer. Math Soc., 20 (1969), 434-438.

[12] H. WIDOM, Approximately finite algebras, Trans. Amer. Math. Soc., 83 (1956), 275-280.

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