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# A CHARACTERIZATION OF BIPOLAR MINIMAL SURFACES IN $S^4$

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1. **Introduction.** There is a generalized Clifford surface on any odd dimensional unit sphere, which is the image of a minimal immersion of $R^2$ [5]. It is an interesting problem to study minimal immersions of a torus into an even dimensional sphere. As such an example, we know the Lawson's bipolar surface of a torus [7]. It is a minimal immersion of a torus into a 4-dimensional unit sphere. Let $S^n \subset R^{n+1}$ be an $n$-dimensional unit sphere and $M$ an oriented Riemannian 2-manifold. Let $x: M \to S^3$ be a minimal immersion. The associated Gauss map $x^*: M \to S^3$ is defined pointwise as the image of the unit normal in $S^3$ translated to the origin of $R^4$. We view each map as $R^4$-valued and define $\tilde{x}: M \to S^5 \subset R^6$ by $\tilde{x} = x \wedge x^*$. (We identify $\Lambda^2 R^4$ with $R^6$.) This mapping induces a non-singular metric on $M$ of the form $d\tilde{s}^2 = (2 - K)ds^2$, where $K$ is the Gaussian curvature for the metric induced by $x$. It is easy to see that $\tilde{x}$ is also a minimal immersion of $M$ into $S^5$ by the calculation of the Laplacian of $\tilde{x}$ [7]. Following Lawson, we shall call this surface the bipolar surface of $x: M \to S^3$. He has shown that the bipolar surface of the minimal torus, $\tau_{m,k}$, is contained in an $S^4 \subset S^5$ where none of the images lies in an $S^3 \subset S^4$. His proof depends heavily on the last theorem of W. Y. Hsiang and H. B. Lawson, Jr [4].

The purpose of this paper is to calculate the local invariants of the bipolar surface by means of the local terminologies of $x$, and we can show the bipolar surface of a torus is not an $R$-surface. I think that this is the first example of surfaces which are not $R$-surfaces (cf. [10]). As a byproduct, we obtain a very elementary proof of the above Lawson's result. At last, we give a characterization of the bipolar minimal surface in $S^4$. Our results are stated in the Theorems 1, 2, 3 and 4.

In this paper, we use freely the concept and the elementary results of higher fundamental forms of $\tilde{x}$ [5].

2. **Frames on the bipolar surface.** Since §§2, 3 and 4 are a local theory, we assume that the minimal immersion $x$ of $M$ is not totally geodesic at every point of $M$. The 2nd fundamental tensor $h_{ij}$ has different

eigenvalues at each point of $M$. Let $e_i$, $i=1,2$, be local tangent unit vector fields on $M$ such that the 2nd fundamental tensor field is diagonalized. We denote the eigenvalue corresponding to the $e_1$ by $h$. The Gauss equation is, then, represented by

$$1+h^2=2-K\,. \tag{2.1}$$

Let $\{w_i\}$ be the dual basis of $\{e_i\}$ and $w_{12}$ the connection form on $M$ for $x$. By the Codazzi equations and the definition of the covariant derivatives, we obtain

$$dh+i(2hw_{12})=(h_{;1}+ih_{;2})(w_1-iw_2)\,, \tag{2.2}$$

where the semi-colon is the covariant derivatives of $h_{ij}$ and $h_{;1}=h_{11;1}$, $h_{;2}=h_{11;2}$. Let $e_3$ be the unit normal vector field of $x$ and $\tilde{x}=x\wedge e_3$. Let

$$\tilde{e}_1=\frac{(-1)}{\sqrt{2-K}}(hx+e_3)\wedge e_1\,,\qquad \tilde{e}_2=\frac{1}{\sqrt{2-K}}(hx-e_3)\wedge e_2\,, \tag{2.3}$$

where we remark that $e_i$ and $e_3$ are considered as maps of $M$ into $S^3$ in $R^4$. The local frame field $\{\tilde{e}_i\}$ is tangent to $\tilde{x}(M)$ and we have $d\tilde{x}=\sum_i \tilde{w}_i\tilde{e}_i$, where $\tilde{w}_i=\sqrt{2-K}\,w_i$.

We define unit normal vector (local) fields $\tilde{e}_\alpha$, $3\leqq\alpha,\beta\leqq 5$, of $\tilde{x}$ in $S^5$ as follows:

$$x\wedge e_1=\frac{1}{\sqrt{2-K}}(\tilde{e}_3-h\tilde{e}_1)\,,\qquad x\wedge e_2=\frac{1}{\sqrt{2-K}}(\tilde{e}_4+h\tilde{e}_2)\,, \tag{2.4}$$
$$e_1\wedge e_2=\tilde{e}_5.$$

Then $\{\tilde{x},\tilde{e}_i,\tilde{e}_\alpha\}$ *is an orthonormal frame field on* $\tilde{x}(M)$ *in* $S^5\subset R^6$ *and we have*

$$e_1\wedge e_3=\frac{1}{\sqrt{2-K}}(h\tilde{e}_3+\tilde{e}_1)\,,\qquad e_2\wedge e_3=\frac{1}{\sqrt{2-K}}(-h\tilde{e}_4+\tilde{e}_2)\,. \tag{2.5}$$

**3. Formulae for $d\tilde{e}_i$.** We wish to calculate the $d\tilde{e}_i$. Let $\langle\cdot,\cdot\rangle$ be the inner product of $R^4$. Since we have $\langle de_i, x\rangle=-w_i$, by virtue of (2.3), (2.4) and (2.5), we get

$$\begin{aligned} d\tilde{e}_1 = &-\sqrt{2-K}\,w_1\tilde{x}+\frac{1-h^2}{1+h^2}w_{12}\tilde{e}_2-\frac{1}{2-K}dh\tilde{e}_3 \\ &-\frac{2h}{2-K}w_{12}\tilde{e}_4+\frac{2h}{\sqrt{2-K}}w_2\tilde{e}_5\,. \end{aligned} \tag{3.1}$$

By the same method, we have

$$(3.2)\qquad d\tilde{e}_2 = -\sqrt{2-K}\,w_2\tilde{x} - \frac{1-h^2}{1+h^2}w_{12}\tilde{e}_1 - \frac{2h}{2-K}w_{12}\tilde{e}_3 + \frac{1}{2-K}dh\tilde{e}_4 + \frac{2h}{\sqrt{2-K}}w_1\tilde{e}_5 .$$

Thus we have

$$(3.3)\qquad \tilde{w}_{14} = \tilde{w}_{23} = \frac{-2h}{2-K}w_{12}\,, \qquad \tilde{w}_{15} = \frac{2h}{2-K}\tilde{w}_2\,,$$

$$(3.4)\qquad \tilde{w}_{24} = -\tilde{w}_{13} = \frac{dh}{2-K}\,, \qquad \tilde{w}_{25} = \frac{2h}{2-K}\tilde{w}_1\,.$$

In particular we get

$$(3.5)\qquad \tilde{h}_{311} = -\tilde{h}_{412},\ \tilde{h}_{411} = \tilde{h}_{312},\ \tilde{h}_{511} = 0\,.$$

(3.3) and (3.4) show that $\tilde{x}$ is a minimal immersion.

We set

$$(3.6)\qquad \sigma^2 = \sum_{\alpha}\tilde{h}^2_{\alpha 11} \quad\text{and}\quad \tau^2 = \sum_{\beta}\tilde{h}^2_{\beta 12}\,.$$

By (3.4) and (3.5), we can verify

$$(3.7)\qquad \sigma^2 = \frac{1}{(2-K)^3}(h^2_{;1} + h^2_{;2}) \quad\text{and}\quad \tau^2 = \sigma^2 + \frac{4h^2}{(1+h^2)^2}\,.$$

We remark that $\sigma^2$ and $\tau^2$ are invariants of $x(M)$. By (2.1) and (2.2), $\sigma = 0$ at $p \in M$ if and only if $dK = 0$ at $p \in M$. We know that by (3.5), we have also

$$(3.8)\qquad \tilde{K}_{(2)} = \sigma^2 + \tau^2,\ \tilde{N}_{(2)} = \sigma^2\tau^2 \quad\text{and}\quad \tilde{f}_{(2)} = (\tau^2 - \sigma^2)^2\,.$$

(See [5] for the definitions of the above invariants of $\tilde{x}$.) We remark that we have $\tilde{f}_{(2)} \leqq 1$ since $h^2$ is real in the second formula of (3.7). Since the immersion $x$ is not totally geodesic at any point of $M$, we have $\tau^2 > 0$ on $M$ and $\tilde{f}_{(2)} > 0$ on $\tilde{x}(M)$.

$\tilde{N}_{(2)} = 0$ on $\tilde{x}(M)$ if and only if $\sigma^2 = 0$ on $x(M)$. Therefore, $\tilde{N}_{(2)} = 0$ on $\tilde{x}(M)$ if and only if $dK = 0$ on $x(M)$. By an Otsuki's lemma [8, p. 96] or [5, Lemma 2], if $\tilde{N}_{(2)} \equiv 0$, then $\tilde{x}(M)$ is contained in a 3-dimensional space of constant curvature 1 in $S^5$.

Next, taking the exterior derivative of the first formula in (2.4), we have

$$(3.9)\qquad \tilde{w}_{34} = \tilde{w}_{12} = \frac{1-h^2}{1+h^2}w_{12}\,, \qquad \tilde{w}_{35} = -\left(\frac{1-h^2}{1+h^2}\right)\tilde{w}_2\,.$$

By the similar way, we have

$$(3.10)\qquad \tilde{w}_{45} = \frac{1-h^2}{1+h^2}\tilde{w}_1 .$$

Summarizing up these result, we have

THEOREM 1. *Let $x: M \to S^3$ be an isometric minimal immersion of an oriented Riemannian 2-manifold into $S^3$. Let $\tilde{x}$ be the bipolar surface of $x$. We assume that $x$ is not totally geodesic at any point of $M$. Then $\tilde{x}$ is a minimal immersion of $M$ into $S^5$. $\tilde{x}$ is not totally geodesic on $M$ and $0 < \tilde{f}_{(2)} \leqq 1$ on $\tilde{x}(M)$.*

*$\tilde{x}(M)$ is contained in a 3-dimensional space of constant curvature 1 in $S^5$ if and only if the Gaussian curvature by the induced metric of $x$ is constant on $M$.*

REMARK. By a Lawson's result [6], such a $K$ takes only the values 0 or 1.

4. **3rd fundamental form of $\tilde{x}$.** From now on we shall assume $\sigma \neq 0$ on $M$. We wish to study the 3rd fundamental form of $\tilde{x}$. Let

$$(4.1)\qquad e_3^* = \frac{1}{\sigma}\sum_\alpha \tilde{h}_{\alpha 11}\tilde{e}_\alpha , \quad e_4^* = \frac{1}{\tau}\sum_\alpha \tilde{h}_{\alpha 12}\tilde{e}_\alpha .$$

(3.5) implies that unit normal vector local fields $e_3^*$ and $e_4^*$ are orthogonal. Since we have

$$\sigma e_3^* + i\tau e_4^* = (\tilde{h}_{311} + i\tilde{h}_{312})(\tilde{e}_3 - i\tilde{e}_4) + i\tilde{h}_{512}\tilde{e}_5 ,$$

we obtain, by (3.5),

$$(4.2)\quad d(\tilde{e}_1 + i\tilde{e}_2) = -(\tilde{w}_1 + i\tilde{w}_2)\tilde{x} - i\tilde{w}_{12}(\tilde{e}_1 + i\tilde{e}_2) + (\tilde{w}_1 - i\tilde{w}_2)(\sigma e_3^* + i\tau e_4^*) .$$

We define a unit normal local vector field $e_5^*$ by

$$(4.3)\qquad e_5^* = \frac{1}{\sigma}\{\tilde{h}_{512}e_4^* - \tau\tilde{e}_5\} .$$

We can then verify that $e_5^*$ is orthogonal to $e_3^*$ and $e_4^*$. The formula (4.2) implies

$$(4.4)\qquad \begin{aligned} &w_{13}^* = \sigma\tilde{w}_1,\ w_{14}^* = \tau\tilde{w}_2,\ w_{i5}^* = 0,\ i = 1, 2 ,\\ &w_{23}^* = -\sigma\tilde{w}_2,\ w_{24}^* = \tau\tilde{w}_1 . \end{aligned}$$

The 2nd osculating space is, then, spanned by $e_3^*$ and $e_4^*$. We have, taking the exterior derivatives of $w_{i5}^* = 0$,

$$(4.5)\qquad \begin{aligned} &\tilde{D}h_{511}^* = \sigma w_{35}^* = h_{511,1}^*\tilde{w}_1 + h_{511,2}^*\tilde{w}_2 ,\\ &\tilde{D}h_{512}^* = \tau w_{45}^* = h_{511,2}^*\tilde{w}_1 - h_{511,1}^*\tilde{w}_2 , \end{aligned}$$

where $h_{5ijk}^* = h_{5ij,k}^*$ are the 3rd fundamental tensors of $\tilde{x}$ (see [5]) and $\tilde{D}$

is the covariant differential operator of the van der Waerden-Bortolotti for the immersion $\tilde{x}: M \to S^5$. Thus $\tilde{D}h^*_{5ij} = 0$ is equivalent to $w^*_{35} = 0$. As the 3rd order invariant $\tilde{K}_{(3)}$ is defined by $\tilde{K}_{(3)} = (h^*_{5111})^2 + (h^*_{5112})^2$, $\tilde{K}_{(3)} = 0$ is equivalent to $w^*_{35} = 0$. Therefore, we shall represent $w^*_{35}$ by means of the terminologies of $x$. At first we find

$$\sum_{\alpha=3}^{4} \tilde{h}_{\alpha 11} d\tilde{e}_\alpha = -\tau \tilde{w}_{12} e^*_4 + \{\text{terms of } \tilde{e}_i\} . \tag{4.6}$$

We prove (4.6) as follows: By (3.5), the left hand side of (4.6) is equal to $-\tilde{h}_{412} d\tilde{e}_3 + \tilde{h}_{312} d\tilde{e}_4$. We have, by (3.3), (3.5) and (3.9),

$$\frac{1-h^2}{1+h^2}(\tilde{h}_{411}\tilde{w}_1 - \tilde{h}_{311}\tilde{w}_2) = -\tilde{h}_{512}\tilde{w}_{12} . \tag{4.7}$$

From (3.5), (3.9), (3.10) and (4.7), (4.6) follows.

By virtue of (3.5) and the latter half of (4.1), we have

$$d\left(\frac{\tilde{h}_{311}}{\sigma}\right)\tilde{e}_3 + d\left(\frac{\tilde{h}_{411}}{\sigma}\right)\tilde{e}_4 = \sigma^{-3}\tau\Phi(\tau e^*_4 - \tilde{h}_{512}\tilde{e}_5) , \tag{4.8}$$

where $\Phi = \tau^{-1}(\tilde{h}_{312} d\tilde{h}_{311} - \tilde{h}_{311} d\tilde{h}_{312})$. Thus we get from (4.6) and (4.8),

$$\sigma w^*_{34} = -\tau \tilde{w}_{12} + \Phi \quad \text{and} \quad \sigma w^*_{35} = \frac{2h}{\sigma(2-K)}\Phi . \tag{4.9}$$

As we have

$$\tilde{h}_{311} = -(1+h^2)^{-3/2} h_{;1} \quad \text{and} \quad \tilde{h}_{312} = -(1+h^2)^{-3/2} h_{;2} ,$$

we get

$$\Phi = \frac{1}{\tau(1+h^2)^3}(h_{;2} dh_{;1} - h_{;1} dh_{;2}) . \tag{4.10}$$

We shall state the properties of $\tilde{K}_{(3)}$ in the following form.

THEOREM 2. *Let $M$ be an oriented Riemannian 2-manifold and let $x: M \to S^3$ be an isometric minimal immersion with $\sigma \neq 0$ on $M$ and not totally geodesic at any point of $M$. Let $\{e_i\}$ be the local principal vector fields of $x$. Then the 3rd fundamental form of $\tilde{x}$ is given by (4.9) and (4.10).*

*$\tilde{K}_{(3)} = 0$ on $\tilde{x}(M)$ if and only if $e_1(h): e_2(h)$ is constant on the each domain of definition.*

**5. The case of $\tau_{m,k}$.** As an application, we shall study the bipolar surface $\tilde{\tau}_{m,k}$ of a minimal torus, $\tau_{m,k}$. The $\tau_{m,k} (m \geqq k \geqq 1)$ is defined by the image of the doubly periodic immersion $\Psi: R^2 \to S^3$ given by

$$\Psi(x_1, x_2) = (\cos mx_1 \cos x_2, \sin mx_1 \cos x_2, \cos kx_1 \sin x_2, \sin kx_1 \sin x_2) . \tag{5.1}$$

Then the first fundamental form of $\tau_{m,k}$ is

$$\sum_{i,j} g_{ij}dx_i dx_j = g^2(x_2)dx_1^2 + dx_2^2 , \tag{5.2}$$

where $g^2(x_2) = m^2 \cos^2 x_2 + k^2 \sin^2 x_2$.

The vector-valued second fundamental form can be expressed as

$$B_{ij} = \Psi_{ij} - \sum_k < \Psi_{ij}, f_k > f_k + g_{ij}\Psi , \tag{5.3}$$

where $f_1 = g(x_2)^{-1}\Psi_1$, $f_2 = \Psi_2$, $\Psi_i = \partial\Psi/\partial x_i$ and $\Psi_{ij} = \partial^2\Psi/\partial x_i \partial x_j$.

It is easily verified that $B_{11} = B_{22} = 0$ and

$$B_{12} = mkg\,(x_2)^{-1}e_3 , \tag{5.4}$$

where

$$e_3 = g(x_2)^{-1}(k \sin mx_1 \sin x_2, -k \cos mx_1 \sin x_2 , \\ -m \sin kx_1 \cos x_2, m \cos kx_1 \cos x_2)$$

is the unit normal vector field of $\tau_{m,k}$. Since $(x_1, x_2)$ is the global coordinates of $\tau_{m,k}$, $e_3$ is also globally defined on $\tau_{m,k}$. We have, then, $B_{12} \neq 0$ on the $\tau_{m,k}$. This shows that $\tau_{m,k}$ is not totally geodesic at any point of the surface. Then, by the Gauss equation, we have

$$K = 1 - m^2k^2g(x_2)^{-2} . \tag{5.5}$$

It follows from the Theorem 1 that the bipolar surface of $\tau_{m,k}$ has $\tilde{N}_{(2)} \equiv 0$ if and only if $m = k$. When the case of $m > k \geqq 1$, we have $\tilde{N}_{(2)} \neq 0$ except the points $(x_1, (s/2)\pi)$ with $s = 0, 1, 2, 3$, but we shall show $\tilde{K}_{(3)} = 0$ on the $\tilde{\tau}^0_{m,k} \equiv \tilde{\tau}_{m,k} - \{(x_1, (s/2)\pi)\}$: We set $e_1 = 2^{-1/2}(f_1 + f_2)$, $e_2 = 2^{-1/2}(f_1 - f_2)$. We denote the 2nd fundamental tensor for the above vectors $e_i$, $h_{ij}$. We then have $h_{11} = -h_{22} = h$ and $h_{12} = 0$, where $h = mkg(x_2)^{-1}$ and $h$ is the globally defined function on the $\tau_{m,k}$. Since we have $w_1 = 2^{-1/2}(g(x_2)dx_1 + dx_2)$ and $w_2 = 2^{-1/2}(g(x_2)dx_1 - dx_2)$, we can verify

$$h_{;1} + h_{;2} = 0 , \tag{5.6}$$

$$h_{;1} = \frac{mk(m^2 - k^2)\sin 2x_2}{2\sqrt{2}\,g(x_2)^3} . \tag{5.7}$$

The formula (5.6) leads to $\tilde{K}_{(3)} = 0$ on $\tilde{\tau}^0_{m,k}$ by Theorem 2. Therefore, by the Otsuki's lemma [8, p. 96] or [5, Lemma 2], $\tilde{\tau}^0_{m,k}$ is contained in a 4-dimensional space of constant curvature in $S^5$.

Moreover, we shall prove the following Lawson's result by an elementary way.

THEOREM 3. [7]. *The bipolar surface $\tilde{\tau}_{m,k}$ of the minimal torus $\tau_{m,k}$ with $m > k$ is contained in an $S^4 \subset S^5$ where none of the images lies* *in an* $S^3 \subset S^4$.

PROOF. From (3.4) and (5.6), we have

$$\tilde{h}_{312} = \tilde{h}_{412} = (2 - K)^{-3/2}h_{;1},\ \tilde{h}_{311} = -\tilde{h}_{411} = -(2 - K)^{-3/2}h_{;1}\ . \tag{5.8}$$

By virtue of the second formula of (5.8), we can define the following orthonormal vector fields $e_\alpha^*$ which are normal vector fields on $\tilde{\tau}_{m,k}$:

$$\begin{aligned} e_3^* &= \frac{1}{\sqrt{2}}(\tilde{e}_3 - \tilde{e}_4)\ , \\ e_4^* &= \frac{1}{\tau}\Big(\sum_\alpha \tilde{h}_{\alpha 12}\tilde{e}_\alpha\Big)\ , \\ e_5^* &= \frac{-1}{\sqrt{2}\cdot\tau}(\tilde{h}_{512}(\tilde{e}_3 + \tilde{e}_4) + 2\tilde{h}_{311}\tilde{e}_5)\ . \end{aligned} \tag{5.9}$$

We remark that vector fields (5.9) are globally defined on $\tilde{\tau}_{m,k}$: By definition $(x_1, x_2)$ is a global coordinates of $\tau_{m,k}$ and $e_1$, $e_2$, $e_3$ are the globally defined field. Since the function $h$ is defined globally on the surface, $\tilde{e}_\alpha$ are also globally defined on $\tilde{\tau}_{m,k}$. By (3.3) and (5.8), $\tilde{h}_{\alpha ij}$ are also defined globally on the $\tilde{\tau}_{m,k}$.

We shall show $de_5^* = 0$ on $\tilde{\tau}_{m,k}$: Since we have $w_{5i}^* = -\langle e_5^*, d\tilde{e}_i\rangle$, making use of (5.8), we have $w_{5i}^* = 0$. From (3.9), (3.10), (4.7) and (5.8), we have $w_{53}^* = 0$. By (5.8), we get $w_{41}^* = -\tau\tilde{w}_2$ and $w_{42}^* = -\tau\tilde{w}_1$. Taking the exterior derivatives of $w_{5i}^* = 0$ and making use of the above results, we get $w_{54}^* = 0$. Thus we have shown $de_5^* = 0$ on $\tilde{\tau}_{m,k}$, and so $e_5^*$ is the constant vector in $R^6$. It follows that the bipolar surface of the $\tau_{m,k}$ is contained in an $S^5 \cap \pi$, where $\pi$ is the hyperplane which is orthogonal to $e_5^*$. Since we have $\langle\tilde{x}, e_5^*\rangle = 0$, we get $S^4 = S^5 \cap \pi$, i.e., the radius of $S^5 \cap \pi$ is 1. This proves Theorem 3.

**6. Characterization of the bipolar surface.** Let $(M, \tilde{g})$ be a 2-dimensional Riemannian manifold with the metric $\tilde{g}$ and let $\tilde{x}\colon (M, \tilde{g}) \to M^4(1)$ be an isometric minimal immersion of $(M, \tilde{g})$ into a 4-dimensional space of constant curvature 1. In this case, we have $\tilde{f}_{(2)} = \tilde{K}_{(2)}^2 - 4\tilde{N}_{(2)}$, where $2\tilde{K}_{(2)}$ is the square of the length of the 2nd fundamental tensor and $4\tilde{N}_{(2)}$ is the square of the normal curvature of $\tilde{x}$. If $\tilde{f}_{(2)} = 0$ on $M$, such a surface is called an $R$-surface and has studied by many mathematicians, for instance, Borůvka [2], Wong [10]. Recently S. S. Chern has proved that the minimal immersion of a 2-sphere in $M^4(1)$ is the $R$-surface and characterized such an immersion completely by a rational curve in $P^4(C)$, ([3] or cf. Barbosa [1]). Following Wong, the surface with $\tilde{f}_{(2)} \neq 0$ is called a general minimal surface. By Theorems 1 and 3, $\tilde{\tau}_{m,k}$ is the general type.

In this section we study a general minimal surface with $0 < \tilde{f}_{(2)} < 1$ on $M$. Since $\tilde{f}_{(2)} > 0$ on $M$, $\tilde{x}$ is not totally geodesic on $M$. Therefore, we can assume that the 2nd fundamental tensors, $\tilde{h}_{\alpha ij}$, $3 \leqq \alpha, \beta \leqq 4$, of $\tilde{x}$ has of the forms,

$$\begin{aligned} \tilde{w}_{13} + i\tilde{w}_{23} &= \tilde{h}_{311}(\tilde{w}_1 - i\tilde{w}_2)\ , \\ \tilde{w}_{14} + i\tilde{w}_{24} &= i\tilde{h}_{412}(\tilde{w}_1 - i\tilde{w}_2)\ , \end{aligned} \tag{6.1}$$

and we have

$$\tilde{f}_{(2)} = (\tilde{h}_{412}^2 - \tilde{h}_{311}^2)^2,\ \tilde{N}_{(2)} = \tilde{h}_{311}^2\tilde{h}_{412}^2 = \frac{1}{4}N^2\ , \tag{6.2}$$

where $N$ is the normal curvature of $\tilde{w}_{34}$, i.e., $d\tilde{w}_{34} = -N\tilde{w}_1 \wedge \tilde{w}_2$. We call the above defined system of (local) vector fields the adapted frame on $(M, \tilde{g})$. We remark that such the adapted frame is uniquely determined up to isometries of the ambiant space, if $N$ is not vanish.

PROPOSITION. *Let $\tilde{x}: (M, \tilde{g}) \to M^4(1)$ be an isometric minimal immersion with $0 < \tilde{f}_{(2)} < 1$. Suppose that the immersion satisfies,*

$$\frac{3}{2}N\tilde{D}\tilde{h}_{312} + (2\tilde{h}_{311}^2 + \tilde{h}_{412}^2)\tilde{D}\tilde{h}_{411} = 0\ . \tag{6.3}$$

*We define a function $h$ on $M$ as follows:*

$$\left(\frac{4h^2}{(1+h^2)^2}\right)^2 = \tilde{f}_{(2)}\ , \tag{6.4}$$

*and we set*

$$g_{ij} = \frac{1}{1+h^2}\tilde{g}_{ij}\ , \tag{6.5}$$

$$h_{11} = -h_{22} = h \quad and \quad h_{12} = 0\ . \tag{6.6}$$

*Then there exists an isometric minimal imbedding $x_U$ of a neighborhood $U$ of any fixed point in $(M, g)$ into a 3-dimensional space of constant curvature 1 such that (6.6) is the second fundamental tensor for $x_U$.*

For a proof of Proposition, we shall need the following Lemmas 6.1 and 6.2. We treat only the case of $\tilde{h}_{412}^2 - \tilde{h}_{311}^2 > 0$. In the other case we can get the same conclusions by the similar way.

LEMMA 6.1. *Under the same hypothesis and notations as Proposition we have*

$$w_{12} = \frac{1+h^2}{1-h^2}\tilde{w}_{12}\ , \tag{6.7}$$

*where $w_{12}$ is the connection form of $(M, g)$.*

PROOF OF LEMMA 6.1. By (6.5), the basic forms on $(M, g)$ are represented by

$$w_i = \frac{1}{\sqrt{1+h^2}}\tilde{w}_i \,. \tag{6.8}$$

Taking the exterior derivatives of (6.8), we have

$$\left\{\frac{h}{1+h^2}dh + i(w_{12} - \tilde{w}_{12})\right\} \wedge (\tilde{w}_1 - i\tilde{w}_2) = 0 \,. \tag{6.9}$$

By $0 < \tilde{f}_{(2)} < 1$, we have $(1 - h^2)^2 > 0$. Taking the exterior derivative of (6.4), we get

$$\frac{h}{1+h^2}dh = \frac{(1+h^2)^2}{4(1-h^2)}\sum_i f_i\tilde{w}_i \,, \tag{6.10}$$

where

$$f_i = \tilde{h}_{412}\tilde{h}_{412,i} - \tilde{h}_{311}\tilde{h}_{311,i} \,,$$

where "," denotes the covariant derivative for $\tilde{g}_{ij}$. By (6.9) and (6.10), we have

$$w_{12} = \tilde{w}_{12} + \frac{(1+h^2)^2}{4(1-h^2)}(-\tilde{h}_{412}\tilde{D}\tilde{h}_{411} - \tilde{h}_{311}\tilde{D}\tilde{h}_{312}) \,, \tag{6.11}$$

where $\tilde{D}$ is the covariant differential operator of the van der Waerden-Bortolotti for the isometric immersion $\tilde{x}\colon (M, \tilde{g}) \to M^4(1)$ and consider $\{\tilde{h}_{\alpha ij}\}$ as the components of the 2nd fundamental form of this immersion. On the other hand, we know

$$\tilde{D}\tilde{h}_{411} = -2\tilde{h}_{412}\tilde{w}_{12} + \tilde{h}_{311}\tilde{w}_{34} \quad \text{and} \quad \tilde{D}\tilde{h}_{312} = 2\tilde{h}_{311}\tilde{w}_{12} - \tilde{h}_{412}\tilde{w}_{34} \,. \tag{6.12}$$

By (6.11) and (6.12), Lemma 6.1 follows. q.e.d.

Making use of the Lemma 6.1, we can show the "Codazzi equation" for $h_{ij}$: By (6.8), (6.9) and Lemma 6.1, we have

$$(dh + i(2hw_{12})) \wedge (w_1 - iw_2) = 0 \,. \tag{6.13}$$

Let $D$ denote the covariant differentiation for $g_{ij}$ and its derivatives ";". Since we can see, by (6.6),

$$Dh_{11} = dh \quad \text{and} \quad Dh_{12} = 2hw_{12} \,, \tag{6.14}$$

(6.13) is equivalent to

$$(Dh_{11} + iDh_{12}) \wedge (w_1 - iw_2) = 0 \,. \tag{6.15}$$

The formula (6.15) implies

$$h_{11;2} = h_{12;1},\ h_{12;2} + h_{11;1} = 0\ . \tag{6.16}$$

By (6.6), (6.16) is equivalent to $h_{ij;k} = h_{ik;j}$. This proves the Codazzi equation for $h_{ij}$. We remark that we do not use (6.3) for the proof of the Codazzi equation for $h_{ij}$. The hypothesis (6.3) is essential in the following Lemma 6.2.

LEMMA 6.2. *Under the same assumptions as Proposition, we have*

$$\tilde{h}_{311}^2 = \frac{h_{11;1}^2 + h_{11;2}^2}{(1+h^2)^3}\ . \tag{6.17}$$

PROOF OF LEMMA 6.2. By the Gauss equation of $\tilde{x}$, we know

$$\tilde{h}_{311}^2 + \tilde{h}_{412}^2 = 1 - \tilde{K}\ , \tag{6.18}$$

where $\tilde{K}$ denotes the Gaussian curvature of $(M, \tilde{g})$. By (6.4) and (6.18), we get

$$2\tilde{h}_{311}^2 = \frac{(1-h^2)^2}{(1+h^2)^2} - \tilde{K}\ . \tag{6.19}$$

By (6.10) we have

$$\frac{h}{(1+h^2)^{3/2}} h_{11;i} = \frac{(1+h^2)^2}{4(1-h^2)} f_i\ . \tag{6.20}$$

And we get

$$\frac{h_{11;1}^2 + h_{11;2}^2}{(1+h^2)^3} = \frac{(1+h^2)^4}{16h^2(1-h^2)}(f_1^2 + f_2^2)\ . \tag{6.21}$$

By (6.12), we have a formula:

$$\begin{aligned} 3\tilde{h}_{311}\tilde{h}_{412}\tilde{D}\tilde{h}_{312} &+ (2\tilde{h}_{311}^2 + \tilde{h}_{412}^2)\tilde{D}\tilde{h}_{411} \\ &= 2(\tilde{h}_{311}^2 - \tilde{h}_{412}^2)(\tilde{h}_{311}\tilde{w}_{34} + \tilde{h}_{412}\tilde{w}_{12})\ . \end{aligned} \tag{6.22}$$

By (6.3), (6.12) and (6.22), we get

$$\tilde{D}\tilde{h}_{411} = -3\tilde{h}_{412}\tilde{w}_{12}\ . \tag{6.23}$$

We remark that under the condition $\tilde{f}_{(2)} > 0$, (6.3) is equivalent to the condition

$$\tilde{h}_{311}\tilde{w}_{34} = -\tilde{h}_{412}\tilde{w}_{12}\ . \tag{6.24}$$

The other formula equivalent to (6.3) is

$$(2\tilde{h}_{311}^2 + \tilde{h}_{412}^2)\tilde{h}_{412,i} = 3\tilde{h}_{311}\tilde{h}_{412}\tilde{h}_{311\ i}\ . \tag{6.3$'$}$$

Taking the exterior derivatives of (6.24), we get

$$(6.25)\qquad (\tilde{h}_{311}d\tilde{h}_{412} - \tilde{h}_{412}d\tilde{h}_{311}) \wedge \tilde{w}_{12} = \tilde{h}_{311}\tilde{h}_{412}(2\tilde{h}^2_{311} + \tilde{K})\tilde{w}_1 \wedge \tilde{w}_2 .$$

By (6.3)$'$ and (6.23), (6.25) is reduced in the following formula,

$$(6.26)\qquad \tilde{h}^2_{412,1} + \tilde{h}^2_{412,2} = \frac{9\tilde{h}^2_{311}\tilde{h}^2_{412}}{\tilde{h}^2_{412} - \tilde{h}^2_{311}}(2\tilde{h}^2_{311} + \tilde{K}) .$$

By (6.3)$'$, we have

$$f^2_1 + f^2_2 = \frac{4}{9}\frac{(\tilde{h}^2_{412} - \tilde{h}^2_{311})^2}{\tilde{h}^2_{412}}(\tilde{h}^2_{412,1} + \tilde{h}^2_{412,2}) .$$

By (6.19), (6.21), (6.26) and the above formula, Lemma 6.2 follows.

q.e.d.

PROOF OF PROPOSITION. We have shown the "Codazzi equation" of $h_{ij}$. We shall show the "Gauss equation", that is,

$$(6.27)\qquad h^2 = 1 - K ,$$

where $K$ is the Gaussian curvature of $(M, g)$. We shall prove (6.27) as follows: Taking the exterior derivative of (6.7), we have

$$(6.28)\qquad -Kw_1 \wedge w_2 = \frac{2}{(1-h^2)(1+h^2)}Dh_{11} \wedge Dh_{12} - \frac{(1+h^2)^2}{1-h^2}\tilde{K}w_1 \wedge w_2 .$$

Therefore we have

$$\begin{aligned} K &= \frac{2(1+h^2)^2}{1-h^2}\left(\frac{h^2_{11;1} + h^2_{11;2}}{(1+h^2)^3}\right) + \frac{(1+h^2)^2}{1-h^2}(1 - \tilde{h}^2_{311} - \tilde{h}^2_{412}) \\ &= \frac{(1+h^2)^2}{1-h^2}(2\tilde{h}^2_{311} + 1 - \tilde{h}^2_{311} - \tilde{h}^2_{412}) \\ &= 1 - h^2 . \end{aligned}$$

From the fundamental theorem for the surface theory (cf. [9]), the Proposition follows. q.e.d.

LEMMA 6.3. *Let $\tilde{x}: M \to S^4$ be an isometric minimal immersion. Under the condition* (6.3), *the following conditions are equivalent*:

(1) $\tilde{h}_{412,1}: \tilde{h}_{412,2} =$ constant,
(2) $\tilde{h}_{311,1}: \tilde{h}_{311,2} =$ constant,
(3) $h_{;1}: h_{;2} =$ constant.

PROOF. From (6.3)$'$ and (6.10), Lemma 6.3 follows. q.e.d.

Thus we have a converse version of the Theorem 2.

THEOREM 4. *Let $(M, \tilde{g})$ be a 2-dimensional Riemannian manifold and let $\tilde{x}: (M, \tilde{g}) \to S^4$ be an isometric minimal immersion with $0 < \tilde{f}_{(2)} <$ 1. Suppose that, for the adapted frame,*

$$\frac{3}{2}N\tilde{D}\tilde{h}_{312} + (2\tilde{h}_{311}^2 + \tilde{h}_{412}^2)\tilde{D}\tilde{h}_{411} = 0, \quad and$$

$\tilde{h}_{412,1}$: $\tilde{h}_{412,2}$ = constant *on the each domain of definition. Then the image of* $M$ *under* $\tilde{x}$ *is locally the bipolar surface of a minimal surface in* $S^3$.

PROOF. Theorem 4 follows from the Proposition and the results in §4. q.e.d.

REMARK. Let $M$ be a torus. Then the Riemann-Roch's theorem implies that $\tilde{f}_{(2)} = 0$ on $M$ or $\tilde{f}_{(2)} > 0$ on $M$. (See [3] or [5].)

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