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ON BOUNDED FUNCTIONS IN THE ABSTRACT HARDY SPACE THEORY III

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1. **Introduction.** We continue from our studies [13], [14] on bounded functions in the abstract Hardy space theory. We mean an abstract H^∞ space by the following space $H = H(X, \Sigma, m)$: Let (X, Σ, m) be a probability measure space and H be a weak* closed subalgebra of the sup-norm algebra L^∞ of the bounded m-measurable functions, satisfying $1 \in H$ and $\int uvdm = \int udm \int vdm$ for all $u, v \in H$. We assume always H is non-trivial, i.e., $H \neq \boldsymbol{C}$. We have shown in [13, 14] that to every non-constant $u \in H$ there corresponds a unique Carathéodory domain $\Gamma(u) = \Gamma(u, m)$ such that $m\{x: u(x) \in \overline{\Gamma(u)}\} = 1$, $\int udm \in \Gamma(u)$ and $m\{x: |u(x) - b| < \varepsilon\} > 0$ for all $\varepsilon > 0$ and all $b \in \partial\Gamma(u)$, and further that if f is in $H^\infty(\Gamma(u))$, i.e., bounded and holomorphic in $\Gamma(u)$, then the composition function $f(u)$ is well-defined and lies in $H(u)$: the weak* closure of the linear spans of $\{1, u, u^2, \cdots\}$. We shall show in this note a converse of this fact, roughly speaking, if D is a bounded domain in the complex plane $\boldsymbol{C}$ and for an abstract H^∞ space H it holds $f(u) \in H(u)$ for all $f \in H^\infty(D)$ and all $u \in H$ with $m\{x: u(x) \in D\} = 1$ and $\int udm \in D$, then D is a Carathéodory domain (Theorem 4.4). This is a characterization of Carathéodory domains. We state other related results in the same Section 4. In Section 5 we give some applications to the classical case. The first one is an extension of Brown-Shields-Zeller and Hoffman-Rossi's result and a characterization of Carathéodory domains in terms of their theorem. Another one is a remark to a result on absolutely convergent exponential sums. Fundamental tools are Lemmas 3.1 and 3.2 in Section 3. In the final section a construction of non-trivial abstract H^∞ spaces on almost every measure space is given.

2. **Preliminaries and Notation.** We mention first some function theoretical facts. Let D be a domain on the Riemann sphere S, which supports non-constant bounded holomorphic functions. We denote by $H^\infty(D)$ the set of all bounded holomorphic functions in D with supremum norm, which we write as $\| \;\|_\infty$ or $\| \;\|_D$. W. Rudin [10] defines a boundary

point b of D as an essential boundary point if there exists an $h \in H^\infty(D)$ such that h does not extend to be holomorphic in any neighborhood of b. If each point in ∂D is essential we say that D is maximal for $H^\infty(D)$. A closed set E is a Painlevé null-set if the algebra of bounded holomorphic functions on $S\backslash E$ consists of the constants alone. A complex number $b \in \partial D$ is an essential boundary point if and only if for every closed disc $B(b, r) = \{z \in C; |z-b| \leqq r\}$ the set $\partial D \cap B(b, r)$ is not a Painlevé null-set [10, p. 334]. Two conformally equivalent domains are simultaneously maximal or not [10, p. 335], and hence each simply connected domain is maximal, since the open unit disc U is obviously maximal. We notice further if b is an essential boundary point of D, there exists for every $B = B(b, r)$ with $B(b, r) \cap D \neq \emptyset$ an $a \in B(b, r) \cap D$ and an $f \in H^\infty(D)$ such that $f(a) = 1$ and $|f(z)| < 1/2$ $(z \in D \cap B^c)$. In fact, if $B \cap (D^c)^0 \neq \emptyset$, this is trivially shown by a rational function. If $B \cap (D^c)^0 = \emptyset$, i.e., $B \cap D^c = B \cap \partial D$, the set $K = B(b, r/2) \cap \partial D$ is not a Painlevé null-set. Hence there exists a non-constant bounded holomorphic function $f(z)$ on $S\backslash K$. We can assume $|f(z)| < 1/2$ $(|z - b| > r)$ and $f(a) = 1$ for some $a \in \partial B(b, 3r/4)$ by the maximum modulus principle. Since $B \cap D^c$ has no interior and K^c is connected, we can assume $a \in D \cap B(b, r)$. This $f(z)$ satisfies all the desired conditions.

We recall next the definition of H^p class on arbitrary domains. For any domain D and any $0 < p < \infty$, one defines $H^p(D)$ as the set of all functions f which are holomorphic in D and for which there exists a harmonic function u in D, such that $|f(z)|^p \leqq u(z)$ $(z \in D)$. There exists then the least harmonic majorant u_f. If we fix a point a in D and define $\|f\|_p = (u_f(a))^{1/p}$, $H^p(D)$ becomes a Banach space with norm $\|\ \|_p$ when $1 \leqq p < \infty$. $H^2(D)$ is a Hilbert space.

We recall then the definition of Carathéodory domain. Let D be a bounded simply connected domain in the complex plane, and let D_∞ be the unbounded component of $(\bar{D})^c$. Then D is said to be a Carathéodory domain if D and D_∞ have the same boundary. Every Jordan domain is a Carathéodory domain. Now let D be a bounded domain and D_∞ the unbounded component of $(\bar{D})^c$. Then D_∞^c coincides with the polynomial convex hull $\hat{\bar{D}}$ of $\bar{D}$. Each component of the interior $(\hat{\bar{D}})^0$ of $\hat{\bar{D}}$ is bounded and simply connected. Exactly one of them contains D, since D is connected and contained in $(\hat{\bar{D}})^0$. Let it be denoted by D_*. Then D_* is a Carathéodory domain and we have $\partial D_* = \partial\hat{\bar{D}} \subset \partial D$. In fact, we have $\bar{D} \subset \bar{D}_* \subset \hat{\bar{D}}$ and $\partial\hat{\bar{D}} = \partial D_\infty \subset \partial\bar{D} \subset \partial D$, and hence $\hat{\bar{D}}_* = \hat{\bar{D}}$, i.e., $(D_*)_\infty = D_\infty$ and $\partial D_* \supset \partial\bar{D}_* \supset \partial\hat{\bar{D}} = \partial D_\infty$. On the other hand, by definition we have $\partial D_\infty \supset \partial D_*$, and so $\partial D_* = \partial(D_*)_\infty = \partial D_\infty \subset \partial D$. Thus D_* is a Carathéodory domain.

Finally we notice that if H is an abstract H^∞ space and D is a Carathéodory domain, then there exist functions $u \in H$ with $\Gamma(u) = D$. Indeed, let $v \in H$ be non-constant. Then $\Gamma(v)$ is non-empty. Let ϕ be a conformal mapping of $\Gamma(v)$ onto D. Then for $u = \phi(v)$ it holds $\Gamma(u) = D$ by definition of $\Gamma(\cdot)$.

3. Fundamental lemmas. We shall give in this section two fundamental lemmas (Lemma 3.1 and Lemma 3.2) and their corollary to prove our main results. The first one is a generalization of a theorem of Brown-Shields-Zeller and Hoffman-Rossi. For a $t\colon |t| = 1$ we define $\Delta(t; \alpha, h)$ as the intersection of the disc $\{|z - t| < h < 1\}$ and the angle of opening 2α, placed symmetrically about the radius to t.

LEMMA 3.1. *Let H be an abstract H^∞ space. Let u be a non-constant function in H and $\Gamma(u)$ be the Carathéodory domain corresponding to u. Then it holds*

$$\|f(u)\|_\infty = \|f(z)\|_{\Gamma(u)} \qquad \textit{for all} \quad f \in H^\infty(\Gamma(u)) .$$

PROOF. By [14, Theorem 3.3] it is enough to show the above in the case where $\Gamma(u)$ is the open unit disc U via conformal mapping. (i) Now assume there exists an $f \in H^\infty(U)$ such that

$$1 = \|f(u)\|_\infty < \|f\|_U .$$

Then there is a Lebesgue measurable subset E on T of positive Lebesgue measure such that to every $t \in E$ there corresponds a truncated angle $\Delta_t = \Delta(t; \alpha, h(t))$ which satisfies $m\{x; u(x) \in \Delta_t\} = 0$ and $|f(z)| > 1$ for all $z \in \Delta_t$, where α is a fixed constant. Therefore there exists an $h > 0$ such that the set $\{t \in E; h(t) \geqq h\}$ is of positive measure and hence has uncountably many elements, since $E = \bigcup_j \{t \in E; h(t) \geqq 1/j\}$ and $L(E) > 0$. Hence there are two points t_1, t_2 with $\Delta_{t_1} \cap \Delta_{t_2} \neq \emptyset$. (ii) Let a be a point in U with $|f(a)| > 1 = \|f(u)\|_\infty$. Then $|f^n(a)| \to \infty$ as $n \to \infty$ but $\|f^n(u)\|_\infty = 1$. Let $g_n = 1 - f^n/f^n(a)$. Then $g_n(u)$ tends to 1 in $L^\infty(m)$-norm. But $g_n(a) = 0$, so $(z - a)^{-1}g_n \in H^\infty(U)$, and hence we have $(u - a)^{-1}g_n(u) \in H$ by [14, Theorem 3.3]. Therefore $(u - a)^{-1}g_n(u)$ tends to $(u - a)^{-1}$ in $L^\infty(m)$-norm, and hence $(u - a)^{-1}$ is in H. (iii) From (i) and (ii) it follows that $u - z$ is invertible in H for $z \in \Delta_{t_1} \cup \Delta_{t_2} = W$. Of course $u - z$ is invertible in H for all z with $|z| > 1$. (iv) We next notice that $U \backslash \bar{W}$ has exactly two connected components. Let W_1, W_2 be these ones. Then one has $m\{x; u(x) \in \bar{W}_1 \cup \bar{W}_2\} = 1$. Set

$$g(z) = \begin{cases} (1 - \bar{t}_1 z)(1 - \bar{t}_2 z) & \text{for} \quad z \in \bar{W}_1 \\ 0 & \text{for} \quad z \in \bar{W}_2 . \end{cases}$$

Then $g(z)$ is continuous on $K = \bar{W}_1 \cup \bar{W}_2$ and holomorphic on $K^0 = W_1 \cup W_2$. Since every boundary point of K also is a boundary point of $K^c = W \cup \bar{U}^c$, $g(z)$ can be approximated uniformly on K by rational functions with poles off K in virtue of a criterion for $A(K) = R(K)$ (Gamelin [3, p. 219]). Since for every rational function h with poles off K it holds $h(u) \in H$ by (iii), we see that $g(u)$ is an L^∞-norm limit of a sequence of elements in H, and hence is in H. Now set $v = (1 - \bar{t}_1 u)(1 - \bar{t}_2 u)$, $v_1 = g(u)$, and $v_2 = v - v_1$. Then we have $v, v_1, v_2 \in H$ and

$$\left(\int v_1 dm\right)^2 + \left(\int v_2 dm\right)^2 = \int v_1^2 dm + \int v_2^2 dm = \int (v_1 + v_2)^2 dm$$
$$= \left(\int (v_1 + v_2) dm\right)^2 = \left(\int v_1 dm + \int v_2 dm\right)^2 .$$

This implies $\int v_1 dm = 0$ or $\int v_2 dm = 0$. Let $\int v_1 dm = 0$, say. But the essential range $\omega(v_1)$ of v_1 is contained in the closure of the set $G_1 = \{(1 - z_1)(1 - z_2); |z_1| < 1, |z_2| < 1\}$ and it holds $G_1 = G_2 = \{(1 - z)^2; |z| < 1\}$ as we shall see soon later. Since $(\bar{G}_2)^c$ is connected, i.e., $\bar{G}_2$ is polynomially convex and $0 \in \partial G_2$, $\int v_1 dm = 0$ implies $v_1 = 0$ by [13, Lemma 1], that is, $m\{x; u(x) \in \bar{W}_2\} = 1$, which contradicts $\Gamma(u) = U$. It is impossible by the same reason that $\int v_2 dm = 0$. Therefore there exist no functions $f \in H^\infty(U)$ with $\|f(u)\|_\infty < \|f\|_U$. Trivially we have $\|f(u)\|_\infty \leqq \|f\|_U$. Hence we obtain

$$\|f(u)\|_\infty = \|f\|_U \qquad \text{for all} \quad f \in H^\infty(U) .$$

(v) Finally we have to show $G_1 = G_2$. Let $1 - z_j = r_j e^{i\theta_j}$ $(j = 1, 2)$. Then we have $-\pi/2 < \theta_j < \pi/2$ and $r_j < 2\cos\theta_j$ $(j = 1, 2)$. Let $\theta = (\theta_1 + \theta_2)/2$, $\theta_1 = \theta + \alpha$ and $\theta_2 = \theta - \alpha$. Let $r = 2\cos\theta$ and $z = 1 - re^{i\theta}$. Then we obtain $|z| = 1$ and

$$r^2 - r_1 r_2 > 4(\cos^2\theta - \cos(\theta + \alpha)\cos(\theta - \alpha))$$
$$= 4\sin^2\alpha \geqq 0 .$$

This means $|(1 - z_1)(1 - z_2)| < |1 - z|^2$ and $\operatorname{Arg}(1 - z_1)(1 - z_2) = \operatorname{Arg}(1 - z)^2$. Hence we have $(1 - z_1)(1 - z_2) \in G_2$. This completes the proof of the lemma. q.e.d.

The second lemma is the following.

LEMMA 3.2. *Let D be a bounded domain in the complex plane. Let (X, Σ, m) be a finite positive measure space. Let T be a multiplicative linear mapping from $H^\infty(D)$ into $L^\infty(m)$ which is isometric and satisfies $T1 = 1$. Further suppose that if a uniformly bounded sequence $f_n \in H^\infty(D)$*

converges to an $f \in H^\infty(D)$ *in* $H^2(D)$*-norm, there exists a subsequence* f_{n_j} *such that* Tf_{n_j} *converges to* Tf *m-almost everywhere. Then we have:* $T(H^\infty(D))$ *is weak* closed and for every* $a \in D$ *the mapping* Φ: $Tf \longrightarrow f(a)$ *is a unique multiplicative linear functional* Φ *on* $T(H^\infty(D))$ *with* $\Phi(Tz) = a$. *This* Φ *is weak* continuous and there exists a non-negative function g in* $L^1(m)$ *such that*

$$f(a) = \Phi(Tf) = \int (Tf) g dm \quad \text{for all} \quad f \in H^\infty(D) .$$

PROOF. (i) We shall show first that $T(H^\infty(D))$ is weak* closed. Let $f_n \in H^\infty(D)$ be a sequence such that $\|Tf_n\| \leqq 1$ and $Tf_n \longrightarrow g$ m-almost everywhere. Since T is isometric, there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges to an $f \in H^\infty(D)$ weakly in $H^2(D)$. Since $H^2(D)$ is a Hilbert space, a sequence $\{g_k \in H^\infty(D)\}$ of finite convex combinations of f_{n_j} converges to f in $H^2(D)$, where g_k is of the form $\sum_{j \geqq k} \alpha_j(k) f_{n_j}$, $\alpha_j(k) > 0$, $\sum \alpha_j(k) = 1$. By the assumption Tg_k converges to Tf m-almost everywhere. On the other hand, since g_k is of the form $\sum_{j \geqq k} \alpha_j(k) f_{n_j}$ and Tf_{n_j} converges to g m-almost everywhere, Tg_k also converges to g m-almost everywhere. Hence one gets $g = Tf$. This shows via L^∞-version of Krein-Shmulian theorem [6, p. 454] that $T(H^\infty(D))$ is weak* closed. (ii) Let Ψ be a multiplicative linear functional on $T(H^\infty(D))$ such that $\Psi(Tz) = a$. Let $f \in H^\infty(D)$. Since $g(z) = (f(z) - f(a))/(z - a) \in H^\infty(D)$, we have $\Psi(Tf - f(a)) = \Psi(T(f - f(a))) = \Psi(TgT(z - a)) = \Psi(Tg)\Psi(Tz - a) = 0$, i.e., $\Psi(Tf) = f(a)$. (iii) We show next the weak* continuity of Φ. Let Tf_α be a directed sequence in $T(H^\infty(D))$ which converges to a Tf weakly*. We can assume $\|Tf_\alpha\| \leqq 1$ to show the weak* continuity. Let Tf_n be a countable subsequence of Tf_α such that $f_n(a)$ tends to a complex number b. Since T is isometric, there exists a subsequence f_{n_j} which converges to a $g \in H^\infty(D)$ weakly in $H^2(D)$ and $f_{n_j}(a)$ tends to $g(a) = \Phi(Tg)$. Further there exists a sequence $g_k = \sum_{j=k} \alpha_j(k) f_{n_j}$ of finite convex combinations of f_{n_j} which converges to g in $H^2(D)$. We can hence assume, by assumption, Tg_n converges to Tg a.e. by choosing subsequence if necessary. Now, since Tf_{n_j} converges to Tf weakly*, Tg_n also converges to Tf weakly*. Hence we obtain $Tg = Tf$, and hence $b = \lim f_n(a) = \lim f_{n_j}(a) = g(a) = \Phi(Tg) = \Phi(Tf) = f(a)$. Therefore one can conclude that $\lim \Phi(Tf_\alpha) = \Phi(Tf)$. Hence Φ is weak* continuous. (iv) Since Φ is a weak* continuous multiplicative linear functional on the subalgebra $T(H^\infty(D))$ of $L^\infty(m)$, there exists by a theorem of König-Hoffman-Rossi a non-negative function $g \in L^1(m)$ such that

$$f(a) = \Phi(Tf) = \int (Tf) g dm \quad \text{for all} \quad f \in H^\infty(D) .$$

The proof is thus complete. q.e.d.

Combining Lemmas 3.1 and 3.2 we have a result on the subalgebra $H(u)$ of an abstract H^∞ space H, generated by a non-constant function u in H.

THEOREM 3.3. *Let H be an abstract H^∞ space. Let $u \in H$ be non-constant and $\Gamma(u)$ be the Carathéodory domain corresponding to u. Let $H(u)$ be the weak* closure of linear spans of $\{u^n: n = 0, 1, 2, \cdots\}$. Then we have*:

(1) $$H(u) = \{f(u); f \in H^\infty(\Gamma(u))\}$$

and $H(u)$ is isometrically isomorphic to $H^\infty(U)$.

(2) *For every $a \in \Gamma(u)$ there exists a unique multiplicative linear functional Φ on $H(u)$ with $\Phi(u) = a$. This Φ is weak* continuous and there exists a non-negative $g \in L^1(m)$ such that*

$$\Phi(v) = \int vgdm \quad \text{for all} \quad v \in H(u)\ ,$$

and in addition

$$f(a) = \Phi(f(u)) = \int f(u)gdm \quad \text{for all} \quad f \in H^\infty(\Gamma(u))\ ,$$

and

$$\|f(u)\|_{L^\infty(gdm)} = \|f\|_{\Gamma(u)} \quad \text{for all} \quad f \in H^\infty(\Gamma(u))\ .$$

PROOF. Let T be the composition operator from $H^\infty(\Gamma(u))$ to $H \subset L^\infty(m)$, defined by $Tf = f(u)$ for $f \in H^\infty(\Gamma(u))$. Then T satisfies all the conditions for Lemma 3.2 in virtue of [14, Theorem 3.3] and Lemma 3.1. Let $H_1 = T(H^\infty(\Gamma(u))) = \{f(u): f \in H^\infty(\Gamma(u))\}$. Then by Lemma 3.2 H_1 is weak* closed. Since $H_1 \subset H(u)$ by [14, Theorem 3.3] and each finite linear combination of $\{1, u, u^2, \cdots\}$ is contained in H_1, we have thus $H(u) = H_1$. Since H_1 is isometrically isomorphic to $H^\infty(\Gamma(u))$ by Lemma 3.1, and since $\Gamma(u)$ is simply connected, $H(u) = H_1$ is isometrically isomorphic to $H^\infty(U)$. Let $a \in \Gamma(u)$. Then by Lemma 3.2 Φ: $f(u) \longrightarrow f(a)$ is a unique multiplicative linear functional on $H_1 = H(u)$ such that $\Phi(Tz) = \Phi(u) = a$. Now let C_n be Jordan curves in $\Gamma(u)$ and D_n be domains bounded by C_n and $\partial\Gamma(u)$ such that $a \notin D_n$, $D_{n+1} \subset D_n$ $(n = 1, 2, \cdots)$ and $\lim D_n = \emptyset$. Then by Lemma 3.1 and the maximum modulus principle we have

$$\operatorname*{ess\,sup}_{u(x) \in \bar{D}_n} |Tf(x)| = \|f\|_{\Gamma(u)} \quad \text{for all} \quad f \in H^\infty(\Gamma(u))\ .$$

Hence by Lemma 3.2 there exist non-negative $g_n \in L^1(m)$ such that $g_n(x) = 0$ for $x: u(x) \notin \bar{D}_n$ and

$$f(a) = \int (Tf)g_n dm \quad \text{for all} \quad f \in H^\infty(\Gamma(u)) .$$

Set $g = \sum_{n=1}^{\infty} 2^{-n} g_n$. Then g is non-negative and we have

$$f(a) = \int (Tf) g dm \quad \text{for all} \quad f \in H^\infty(\Gamma(u)) ,$$

that is

$$f(a) = \int f(u) g dm \quad \text{for all} \quad f \in H^\infty(\Gamma(u)) .$$

Let $dm_n = g_n dm$ and $dm_0 = g dm$. Now $H(u, m_n)$ and $H(u, m_0)$ are abstract H^∞ spaces and we have by a characterization of $\Gamma(\cdot)$ mentioned in the introduction

$$\Gamma(u, m_n) \subset \bar{D}_n \qquad n = 1, 2, \cdots .$$

Let $b \in \partial D$ and $\varepsilon > 0$. Since $\lim D_n = \varnothing$, we have $\{|z - b| < \varepsilon\} \cap \Gamma(u, m_n) \neq \varnothing$ for sufficiently large n and hence $m_n\{x: |u(x) - b| < \varepsilon\} > 0$ by a property of $\Gamma(\cdot)$, and so $m_0\{x: |u(x) - b| < \varepsilon\} > 0$. Since $m_0\{x: u(x) \in \overline{\Gamma(u)}\} = 1$ and $\int u dm_0 = a \in \Gamma(u)$, we obtain $\Gamma(u, m_0) = \Gamma(u)$ by a characterization of $\Gamma(\cdot)$. Hence we have by Lemma 3.1

$$\|f(u)\|_{L^\infty(g dm)} = \|f\|_{\Gamma(u)} \quad \text{for all} \quad f \in H^\infty(\Gamma(u)) .$$

The proof is thus complete.

4. **Main results.** At first we shall give two sufficient conditions for a maximal domain to be simply connected.

LEMMA 4.1. *Let D be a maximal domain in the complex plane. Let H be an abstract H^∞ space. Suppose for every $f \in H^\infty(D)$ it holds $f(u) \in H(u)$ for all $u \in H$ with $m\{x: u(x) \in D\} = 1$ and $\int u dm \in D$. Then D is simply connected. Here $H(u)$ denotes the weak* closure of linear spans of $\{1, u, u^2, \cdots\}$.*

PROOF. Let J be an arbitrarily fixed Jordan curve in D and E be the Jordan domain bounded by J. Now assume $E \cap \partial D \neq \varnothing$. Fix a $b \in E \cap \partial D$. Then since D is maximal, b is an essential boundary point of D. Let $B = B(b, r)$ be a closed disc with center at b and radius r, which lies completely in E. As is mentioned in §2, there exists an $f \in H^\infty(D)$ such that $f(a) = 1$ for some $a \in D \cap B$ and $|f(z)| < 1/2$ for $z \in D \cap B^c$. Naturally $a \in E \cap B$. Let J_1 be a Jordan curve situated inside E and surrounding a, B and $E \cap D^c$. Let F be the domain bounded by J and J_1. Then there exists a $u \in H$ such that $\int u dm \in F$, $m\{x: u(x) \in \bar{F}\} = 1$ and

$\Gamma(u) = E$. In fact, let $v \in H$ satisfy $\Gamma(v) = U$ and $\int v dm = 0$ (there exists always such one as is mentioned in §2), and ϕ be a uniformization function of U onto F, so that ϕ is holomorphic in U and the range of ϕ is precisely F. Then $u = \phi(u)$ is in H and $\int u dm = \phi(0) \in F$ by [14, Theorem 3.3] and it can be easily shown that J and J_1 are contained in the essential range of u. Hence $\Gamma(u) = E$ by a characterization of $\Gamma(u)$. Now for this u we obtain by assumption $g(u) \in H(u)$ for all $g \in H^\infty(D)$. Let Φ be the multiplicative linear functional on $H(u)$ such that $\Phi(u) = a$. Then by Theorem 3.3 there exists an m-absolutely continuous positive measure m_0 such that $\Phi(v) = \int v dm_0$ for all $v \in H(u)$ and hence

$$\Phi(g(u)) = \int g(u) dm_0 \quad \text{for all} \quad g \in H^\infty(D) .$$

Since for every $g \in H^\infty(D)$ it holds $g(z) - g(a) = (z - a)h(z)$ for some $h \in H^\infty(D)$, and since Φ is multiplicative on $H(u)$, we have $\Phi(g(u) - g(a)) = \Phi((u - a)h(u)) = \Phi(u - a)\Phi(h(u)) = 0$. That is, we have

$$\int g(u) dm_0 = g(a) \quad \text{for all} \quad g \in H^\infty(D) ,$$

especially

$$(*) \qquad \int f(u) dm_0 = f(a) = 1 .$$

On the other hand, since $m\{x: u(x) \in \bar{F}\} = 1$ and $|f(z)| < 1/2$ for $z \in D \cap B^c$, we have $m_0\{x: u(x) \in \bar{F}\} = 1$ and $|f(z)| < 1/2$ for $z \in \bar{F}$. Hence we have

$$\left| \int f(u) dm_0 \right| \leqq \int |f(u)| dm_0 \leqq 1/2 .$$

This contradicts $(*)$. Hence we obtain $E \cap \partial D = \emptyset$, and hence $E \subset D$. This means D is simply connected. The proof is complete.

LEMMA 4.2. *Suppose H be an abstract H^∞ space such that there exists a non-constant $u \in H$ with $|u| = 1$. Let D be a maximal domain in the complex plane. Then, if for every $f \in H^\infty(D)$ it holds $f(v) \in H$ for all $v \in H$ with $m\{x: v(x) \in D\} = 1$ and $\int v dm \in D$, it follows that D is a simply connected domain.*

PROOF. We notice that $f(v) \in H$ for all $f \in H^\infty(D)$ implies $\int f(v) dm = f\left(\int v dm\right)$ for all $f \in H^\infty(D)$ as in the proof of Lemma 4.1. Now let J be an analytic Jordan curve in D and G the Jordan domain bounded by J.

Let a be an arbitrarily fixed point of $G \cap D$. Let g be a conformal mapping of U onto G satisfying $g\left(\int udm\right) = a$. Then we see by [14, Theorem 3.1] that $v = g(u)$ satisfies $m\{x: v(x) \in J\} = 1$, $\int vdm = a$ and $(*)$ $m\{x: v(x) \in E\} = m_a(E)$ for every harmonically measurable set $E \subset J$, where m_a is the harmonic measure with respect to a. Now let $f(z) \in H^\infty(D)$. Then we have by $(*)$ and the assumption

$$\int f(z)dm_a(z) = \int f(v)dm = f\left(\int vdm\right) = f(a) .$$

This equation implies that $f(z)$ can be continued harmonically from $G \cap D$ onto G and hence holomorphically. Since D is maximal, we obtain that $G \cap \partial D$ is empty, and so $G \subset D$. This means that D is simply connected, which completes the proof.

Next we shall state a condition for a bounded simply connected domain to be a Carathéodory domain.

LEMMA 4.3. *Let D be a bounded simply connected domain in the complex plane and ϕ be a conformal mapping U onto D. Let H be an abstract H^∞ space and u be a non-constant function in H with $\Gamma(u) = U$. Then, if we have $u \in H(\phi(u))$, it follows that D is a Carathéodory domain.*

PROOF. Let D_* be the component of the interior of $\hat{\hat{D}}$, which contains D. Then we have $\partial D_* = \partial \hat{\hat{D}} \subset \partial D$ and D_* is a Carathéodory domain, as is shown in §2. We have further $\phi(u) \in H$ and $\Gamma(\phi(u)) = D_*$, since $\bar{\Gamma}(\phi(u)) \supset \bar{D}$ and $\Gamma(\phi(u)) \subset D_*$ by [13, Theorem A]. We have in addition by Lemma 3.1

$$\|f(\phi(u))\|_\infty = \|f\|_{D_*} \quad \text{for all} \quad f \in H^\infty(D_*) .$$

Assume now D is not a Carathéodory domain. Then there exists a point $b \in \partial D \cap D_*$ such that dist $(b, \partial D_*) = 2\delta > 0$. Set $D_1 = D \cap \{|z - b| > \delta\}$ and $E = \{x: \phi(u(x)) \in \bar{D}_1\}$ and fix a point $a \in D$. Then by the maximum modulus principle we have

$$\operatorname*{ess\,sup}_{x \in E} |f(\phi(u(x)))| = \|f\|_{D_*} \quad \text{for all} \quad f \in H^\infty(D_*) .$$

Hence by Lemma 3.2 there exists an m-absolutely continuous positive measure m_0 on X such that

$$f(a) = \int f(\phi(u))dm_0 \quad \text{for all} \quad f \in H^\infty(D_*)$$

and m_0 is concentrated on E. By Theorem 3.3 we have $H(\phi(u), m) = \{f(\phi(u)): f \in H^\infty(D_*)\}$, because $\Gamma(\phi(u), m) = D_*$. Hence one can construct an abstract H^∞ space $H(\phi(u), m_0) = H(X, \Sigma, m_0)$ starting from $\phi(u)$ and m_0. Since $u \in H(\phi(u))$ by assumption, u is also in $H(\phi(u), m_0)$. Since m_0 is m-absolutely continuous, we have $\Gamma(u, m_0) \subset U$ by [13. Lemma 2] and hence either $\Gamma(u, m_0) = U$ or there exists a point c in $U \backslash \Gamma(u, m_0)$. Assume the first case takes place. Then we have by Lemma 3.1

$$(1) \qquad \|f(u)\|_{m_0} = \|f\|_U \quad \text{for all} \quad f \in H^\infty(U) .$$

However, as we have mentioned in the preliminaries, there exists a $g \in H^\infty(D)$ such that $g(d) = 1$ for some $d \in D \cap \{|z - b| < \delta\}$ and $|g(z)| < 1/2$ for $z \in D_1$. This implies that $f = g(\phi) \in H^\infty(U)$ does not satisfy the above equality (1), a contradiction. Assume next the second case takes place. As in the proof of Theorem 3.3 we can choose m_0 so that

$$\|f(\phi(u))\|_{L^\infty(m_0)} = \|f\|_{D_*} \quad \text{for all } f \in H^\infty(D_*) .$$

Then, if we regard the composition operator as an operator from $H^\infty(D_*)$ into $L^\infty(m_0)$, there exists again by Lemma 3.2 an m_0-absolutely continuous positive measure m_1 such that

$$f(\phi(c)) = \int f(\phi(u))dm_1 \quad \text{for all} \quad f \in H^\infty(D_*) ,$$

and especially (2) $\phi(c) = \int \phi(u)dm_1$. We can construct also an abstract H^∞ space $H(u, m_1)$ like as $H(\phi(u), m_0)$. Since m_1 is m_0-absolutely continuous, we have $\int udm_1 \in \Gamma(u, m_0)$ and $\Gamma(u, m_1) \subset \Gamma(u, m_0)$. Then by [14, Theorem 3.3] we have $f(u) \in H(u, m_1)$ for all $f \in H^\infty(\Gamma(u, m_1))$. Since the integration with respect to m_1 is multiplicative and linear on $H(u, m_1)$, we have in the same way as in the proof of Lemma 4.1 $\int f(u)dm_1 = f\left(\int udm_1\right)$ for all $f \in H^\infty(\Gamma(u, m_1))$. Since ϕ is in $H^\infty(\Gamma(u, m_0))$, we have hence $\int \phi(u)dm_1 = \phi\left(\int udm_1\right)$. However, since $\int udm_1 \in \Gamma(u, m_0)$, $c \in U \backslash \Gamma(u, m_0)$ and ϕ is one-to-one, we get $\phi(c) \neq \phi\left(\int udm_1\right) = \int \phi(u)dm_1$. This contradicts (2). Hence D is a Carathéodory domain and we are done.

Now we are in the position to state our main theorem.

THEOREM 4.4. *Let D be a bounded maximal domain in the complex plane. Let H be an abstract H^∞ space. Suppose it holds $f(u) \in H(u)$ for all $f \in H^\infty(D)$ and all $u \in H$ with $m\{x: u(x) \in D\} = 1$ and $\int udm \in D$. Then D is a Carathéodory domain.*

PROOF. D is simply connected by Lemma 4.1. We shall next show that there exists a $u \in H$ with (*) $m\{x: u(x) \in U\} = 1$ and $\Gamma(u) = U$. Let

G be the simply connected domain decided by two curves: $\varphi_1 = \tan(\pi r/2)$ and $\varphi_2 = \tan(\pi r/2) + \pi$, $0 \leqq r < 1$, (r, φ; polar coordinates), which contains the point (1/2, 0). Let $v \in H$ satisfy $\Gamma(v) = U$ as in §2 and g a conformal mapping of U onto G. Then $u = g(v)$ satisfies (*) in virtue of properties of $\Gamma(\cdot)$ in §1. Now let ϕ be a conformal mapping of U onto D. Since u satisfies $m\{x: u(x) \in U\} = 1$ we have $u = \phi^{-1} \circ \phi(u)$ and $\phi(u) \in H$, $\int \phi(u) dm = \phi\left(\int u dm\right) \in D$ as before. And since $\phi^{-1} \in H^\infty(D)$, we have $u \in H(\phi(u))$ by the assumption. Hence by Lemma 4.3 the domain D is a Carathéodory domain. q.e.d.

Using Lemma 4.2 and the proof method of Lemma 4.3 we have another characterization of Carathéodory domains.

THEOREM 4.5. *Let D be a bounded maximal domain in the complex plane. Suppose for any abstract H^∞ space H it holds $f(u) \in H$ for all $f \in H^\infty(D)$ and all $u \in H$ with $m\{x: u(x) \in D\} = 1$ and $\int u dm \in D$. Then D is a Carathéodory domain.*

PROOF. Since there exist abstract H^∞ spaces which satisfy the conditions in Lemma 4.2, D is simply connected. Assume D is not a Carathéodory domain. Let D_*, D_1, a, b be the same as in the proof of Lemma 4.3. Let $\{a_j \in U\}_{j=1}^\infty$ satisfy $\overline{\{a_j\}} \supset \partial U$ and

$$f(0) = \sum_{j=1}^\infty f(a_j)\alpha_j \quad \text{for all} \quad f \in H^\infty(U) ,$$

where $\alpha_j > 0$ and $\sum_{j=1}^\infty \alpha_j = 1$. There exist such pairs $\{a_j; \alpha_j\}$, for example, the example of J. Wolff: Let $U_n = \{|z - a_n| < r_n\}$ be a sequence of mutually disjoint subdiscs of U with center a_n and radius r_n such that the planar measure of $U \backslash \bigcup U_n$ is zero. Set $u(j) = a_j$ for $j = 1, 2, \cdots$ and let $m = \sum_{j=1}^\infty \alpha_j \delta_j$, where δ_j is the Dirac measure at j. Then $H(u)$: the m-weak* closure of linear spans of $\{1, u, u^2, \cdots\}$ is an abstract H^∞ space. Hence if we let ϕ be a conformal mapping of U onto D, one gets $\phi(u) \in H(u)$ and $\Gamma(\phi(u)) = D_*$ by virtue of [13, Theorem A]. Thus we have by Lemma 3.1

$$\sup_j |f(\phi(a_j))| = \sup_{z \in D_*} |f(z)| \quad \text{for all} \quad f \in H^\infty(D_*) .$$

Let $\{c_j\}_{j=1}^\infty = \{\phi(a_n)\} \cap \{z \in D: |z - b| > \delta\}$. Then one has by the maximum modulus principle

$$\sup_j |f(c_j)| = \sup_{z \in D_*} |f(z)| \quad \text{for all} \quad f \in H^\infty(D_*) .$$

Applying Lemma 3.2 there exists a probability measure $m_0 = \sum_{j=1}^\infty \beta_j \delta_j$ on N such that

$$\sum_{j=1}^{\infty} f(c_j)\beta_j = f(a) \quad \text{for all} \quad f \in H^\infty(D_*)\ .$$

Let $v(j) = c_j$ for $j \in N$. Then the weak* closure $H(v, m_0)$ of linear spans of $\{v^n : n = 0, 1, 2, \cdots\}$ is an abstract H^∞ space and we have $m_0\{x: v(x) \in D\} = 1$ and $\int v dm_0 = a \in D$. Hence by assumption we have $\phi^{-1}(v) \in H(v, m_0)$. Put $w = \phi^{-1}(v)$. Then we get $\phi(w) = v$, and hence $w \in H(\phi(w), m_0)$. The rest of the proof follows along the same lines as the last step of the proof of Lemma 4.3. The proof is thus complete.

REMARKS TO THIS SECTION. Let $dm = \pi^{-1}dxdy$ be the 2-dimensional normalized Lebesgue measure on $U = \{|x + iy| < 1\}$. Then we have

$$f(0) = \int_U f(x + iy)dm \quad \text{for all} \quad f \in H^\infty(U)\ .$$

Hence $H = H(U, m)$ is an abstract H^∞ space. Let D be the annulus $\{1/2 < |z| < 1\}$. Then it is clear that for every $f \in H^\infty(D)$ it holds $f(u) \in H$ for all $u \in H$ with $m\{u(z) \in D\} = 1$ and $\int udm \in D$, because the condition for u implies that the range of u is contained in D. Hence the assumptions $f(u) \in H(u)$ in Lemma 4.1 and Theorem 4.4 are not superfluous. That the maximality conditions for D in this section are not superfluous is trivially shown by an elemental example: Let D be the punctured disc $U\backslash\{0\}$. Then every $f \in H^\infty(D)$ can be defined as a holomorphic function in U. Hence for every abstract H^∞ space we have $f(u) \in H$ for all $f \in H^\infty(D)$ and $u \in H$ with $m\{x: u(x) \in D\} = 1$ and $\int umd \in D$, as is mentioned in the introduction. It is plausible that the assumptions in Lemma 4.2 alone implies D is a Carathéodory domain, if D is bounded.

5. Applications.

One of our applications is as follows.

THEOREM 5.1. *Let D be a Carathéodory domain in the complex plane and $\{a_n\}$ a sequence in D. Let $a \in D$ be fixed. Then the following are equivalent.* (i) *There exist non-negative numbers $\{c_n\}$ such that the closure of the subset $\{a_{n_j}\}$ with $c_{n_j} > 0$ contains ∂D and*

$$(1) \qquad p(a) = \sum c_n p(a_n) \quad \textit{for all polynomials}\ .$$

(ii) *For all $f \in H^\infty(D)$ we have*

$$(2) \qquad \sup_n |f(a_n)| = \|f\|_D\ .$$

(iii) *For every $b \in D$, there exist non-negative numbers $\{d_n\}$ such that the closure of the subset $\{a_{n_j}\}$ with $d_{n_j} > 0$ contains ∂D and*

$$(3) \qquad h(b) = \sum d_n h(a_n) \quad \textit{for all} \quad h \in H^\infty(D)\ .$$

(iv) *For every $b \in D$, there exist non-negative numbers $\{e_n\}$ such that*

(4) $$p(b) = \sum e_n p(a_n) \quad \textit{for all polynomials} .$$

PROOF. Suppose $\{a_n\}$ satisfies (i). Let $g(j) = a_j$ and $m = \sum c_j \delta_j$, where δ_j is the Dirac measure at $j \in N$. Then $\int dm = 1$ by (1). If we set $H = H(g, m)$: the $\sigma(L^\infty(m), L^1(m))$ closure of linear spans of $\{1, g, g^2, \cdots\}$, H is an abstract H^∞ space and $\Gamma(g) = D$. Hence we have (2) by Lemma 3.1. Next suppose $\{a_n\}$ satisfies (ii). Let $b \in D$ and $m = \sum_{j=1}^\infty 2^{-j}\delta_j$. Let T be the mapping from $H^\infty(D)$ into $L^\infty(m)$ defined by

$$(Tf)(j) = f(a_j) \quad \text{for all} \quad f \in H^\infty(D) .$$

Then T satisfies all the conditions in Lemma 3.2. Hence there exist non-negative numbers $\{d_n\}$ such that

$$f(b) = \sum d_n f(a_n) \quad \text{for all} \quad f \in H^\infty(D) .$$

And one can easily show that we can take the subset $\{a_{n_j}\}$ with $d_{n_j} > 0$ so that the closure of $\{a_{n_j}\}$ contains the boundary of D. The proof follows along the same lines as that of the last equation in Theorem 3.3. In this case one has further from the step (i) to (ii)

$$\sup_j |f(a_{n_j})| = \|f\|_D \quad \text{for all} \quad f \in H^\infty(D) .$$

(iii) trivially implies (i) and (iv). Suppose now (iv). For every $b \in D$, $b \notin \{a_n\}$, choose a sequence $\{d_n(b)\}$ satisfying (4). Then infinitely many $d_n(b)$ are positive. We construct an abstract H^∞ space $H(g, m(b))$ as in (i), where $m(b) = \sum d_n(b)\delta_n$. Then $\Gamma(g, m(b))$ is a non-empty open set containing b and contained in D. It holds by the step (i) to (ii)

$$\sup_{d_n(b)>0} |h(a_n)| = \sup_{\Gamma(g,m(b))} |h(z)| \quad \text{for all} \quad h \in H^\infty(D) .$$

Since clearly we have $D = \bigcup_{b \in D \setminus \{a_n\}} \Gamma(g, m(b))$, we obtain for all $f \in H^\infty(D)$

$$\begin{aligned} \sup_n |f(a_n)| &\geqq \sup_{b \in D\setminus\{a_n\}} \sup_{d_n(b)>0} |f(a_n)| \\ &= \sup_{b \in D\setminus\{a_n\}} \sup_{\Gamma(g,m(b))} |f(z)| = \sup_D |f(z)| . \end{aligned}$$

Trivially $\sup_D |f(z)| \geqq \sup_n |f(a_n)|$. Hence we obtain (2). The proof is thus complete.

As a corollary we have the following result which contains a theorem of Hoffman-Rossi.

COROLLARY 5.2. *Let D be a Carathéodory domain, and $\{a_n\}$ a sequence in D with no limit points in D. Let an $a \in D$, $a \notin \{a_n\}$ be fixed. These are equivalent:*

(i) *There exist non-negative numbers $\{c_n\}$ such that*

$$p(a) = \sum c_n p(a_n) \quad \textit{for all polynomials} .$$

(ii) *For all* $f \in H^\infty(D)$ *we have*

$$\sup_n |f(a_n)| = \sup_D |f(z)| .$$

(iii) *For every* $b \in D$, *there exist non-negative numbers* $\{d_n\}$ *such that*

$$h(b) = \sum d_n h(a_n) \quad \textit{for all} \quad h \in H^\infty(D) .$$

PROOF. It is enough to show that (i) implies (ii). The other implications follow immediately from Theorem 5.1. Suppose (i). Let $m = \sum c_n \delta_n$ and $u(n) = a_n$ for $n = 1, 2, \cdots$. Then $H(u, m)$ is an abstract Hardy space and $\Gamma(u, m)$ is non-empty, since $a \notin \{a_n\}$ and hence u is non-constant in $L^\infty(m)$. Since each boundary point of $\Gamma(u, m)$ is a limit point of $\{a_{n_j}\}$ with $c_{n_j} > 0$ by a property of $\Gamma(\cdot)$, and since $\{a_n\}$ has no limit points in D, we have $\Gamma(u, m) = D$. Hence by Lemma 3.1 we obtain (ii). q.e.d.

REMARK. If $a \in \{a_n\}$, the corollary is still valid when we replace (i) by the following proposition (i'): There exist non-negative numbers $\{c_n\}$ such that infinitely many c_n are positive and

$$p(a) = \sum c_n p(a_n) \quad \text{for all polynomials} .$$

One can prove in this case with a slight modification of the proof.

Hoffman and Rossi showed the above in the case where D is the unit disc. We can further characterize Carathéodory domain in terms of Theorem 5.1.

THEOREM 5.3. *Let D be a bounded maximal domain in the complex plane. The following three propositions are equivalent*:

(i) *D is a Carathéodory domain.*

(ii) *Let $\{a_n\}$ be a sequence of distinct points in D with no interior limit points, for which there exist an $a \in D$ and non-negative numbers $\{c_n\}$ with infinitely many positive ones such that*

$$p(a) = \sum c_n p(a_n) \quad \textit{for all polynomials} .$$

Then one has

$$\sup_n |f(a_n)| = \sup_D |f(z)| \quad \textit{for all} \quad f \in H^\infty(D) .$$

(iii) *Let $\{a_n\}$ be a sequence of distinct points in D such that for every $c \in D$ there exist non-negative numbers $\{d_n\}$ satisfying*

$$p(c) = \sum d_n p(a_n) \quad \textit{for all polynomials} .$$

Then one has

$$\sup_n |f(a_n)| = \sup_D |f(z)| \quad \textit{for all} \quad f \in H^\infty(D) .$$

PROOF. At first we shall show that for every bounded domain D there exists a sequence $\{a_n\}$ in D which satisfies the assumptions in (ii) and (iii). Let D_* be the component of the interior of $\hat{\hat{D}}$, which contains D. Then we have $\partial D_* = \partial\hat{\hat{D}} \subset \partial D_*$ and D_* is a Carathéodory domain. It holds further

$$(5.1) \qquad \sup_D |f(z)| = \sup_{D_*} |f(z)| \quad \text{for all} \quad f \in H^\infty(D_*) .$$

In fact, let H be the abstract H^∞ space constructed in the remark at the end of the last section and ϕ be a uniformization function from U onto D, and so ϕ is holomorphic in U and its range is precisely D. Then we have clearly $\phi(u) \in H$ and $\Gamma(\phi(u)) = D_*$, since $\partial D_* \subset \partial D$. Hence by Lemma 3.1 we obtain

$$\sup_{z \in U} |f(\phi(z))| = \|f(\phi(z))\|_{L^\infty(m)} = \sup_{w \in D_*} |f(w)| \quad \text{for all} \quad f \in H^\infty(D_*) ,$$

and so we have (5.1). Now let $\{b_n\}$ be a sequence of distinct points in the unit disc U with no interior limit points, $0 \notin \{b_n\}$, such that

$$\sup_n |f(b_n)| = \sup_U |f(z)| \quad \text{for all} \quad f \in H^\infty(U) .$$

An example in [1, p. 172] is such a sequence. Let ψ be a conformal mapping of U onto D_* and $c_n = \psi(b_n)$. Then $\{c_n\}$ is a sequence of distinct points in D_* with no interior limit points and we have through the conformal mapping ψ

$$(5.2) \qquad \sup_n |f(c_n)| = \sup_{D_*} |f(z)| \quad \text{for all} \quad f \in H^\infty(D_*) .$$

Set $\{a_n\} = \{c_n \in D\}$. Then by (5.1) and (5.2) we have

$$(5.3) \qquad \sup_n |f(a_n)| = \sup_{D_*} |f(z)| \quad \text{for all} \quad f \in H^\infty(D_*) .$$

This $\{a_n\}$ satisfies the assumptions in (ii) and (iii) by Corollary (5.2). Now we shall show the theorem. (i) implies (ii) and (iii) by Theorem 5.1 and Corollary 5.2. Suppose next (ii) or (iii) and let $\{a_n\}$, D_* be as above. Assume D is not a Carathéodory domain. Then there exists a $b \in \partial D$ such that dist $(b, \partial D_*) = 2\delta > 0$. Set $D_1 = D \cap \{|z - b| > \delta\}$. By the maximum modulus principle and (5.3) we have

$$\sup_{a_n \in D_1} |f(a_n)| = \sup_{D_*} |f(z)| \quad \text{for all} \quad f \in H^\infty(D_*) .$$

Hence by Theorem 5.1 the set $\{a_n: a_n \in D_1\}$ satisfies the assumptions in (ii) and (iii), and hence by assumption we get

$$(5.4) \qquad \sup_{a_n \in D_1} |f(a_n)| = \sup_D |f(z)| \quad \text{for all} \quad f \in H^\infty(D) .$$

On the other hand, since D is maximal, there exists a $g \in H^\infty(D)$ such that $\sup_{z \in D} |g(z)| > 1$ and $|g(z)| < 1/2$ for $z \in D_1$. This contradicts (5.4). q.e.d.

Our second application is some remarks to the work of Brown-Shields-Zeller [1]. We remark that almost all of their results for Jordan domains are still valid for Carathéodory domains, but we do not describe them here. We shall however answer to some of their questions. The first one is the following: Their Lemma 3 [1, p. 165] is valid for Carathéodory domains and it is in a sence the largest class of domains for which their Lemma 3 holds. We state it as follows.

PROPOSITION 5.4. *Let D be a bounded maximal domain in the complex plane. The following proposition* $(*)$ *holds if and only if D is a Carathéodory domain.*

$(*)$ *If a measure μ in D satisfies $\int e^{zw} d\mu(w) \equiv 0$, then it follows*

$$\int f(w) d\mu(w) = 0 \quad \textit{for all} \quad f \in H^\infty(D) .$$

This follows immediately from our Theorem 5.3 and Corollary 5.2 and the following two facts: (1) $\int e^{zw} d\mu(w) \equiv 0$ is equivalent to

$$\int p(w) d\mu(w) = 0$$

for all polynomials. (2) If D is a Carathéodory domain, every $f \in H^\infty(D)$ can be approximated pointwise boundedly by polynomials.

Next let D be a bounded domain in the complex plane. By $E(D)$ we denote the set of all those entire functions $h(z)$ that admit a representation of the form

$$h(z) = \sum d_n \exp(a_n z),\ \sum |d_n| < \infty,\ a_n \in D .$$

Then we have the following (cf. [1, p. 167]).

PROPOSITION 5.5. *Let D be a bounded domain in the complex plane and D_* the component of the interior of the polynomial convex hull $\hat{\bar{D}}$ of the closure of D, which contains D. Then we have $E(D) = E(D_*)$.*

PROOF. As is shown in the proof of Theorem 5.3, there exists a sequence of distinct points in D such that

$$\sup_n |f(a_n)| = \sup_{D_*} |f(z)| \quad \text{for all} \quad f \in H^\infty(D_*) .$$

This implies by Theorem 5.1 that for every $a \in D_*$ there exist non-negative numbers $\{d_n\}$ such that

$$p(a) = \sum d_n p(a_n) \quad \text{for all polynomials .}$$

Since for every $w \in \boldsymbol{C}$ e^{wz} can be approximated uniformly on $\bar{D}_*$ by polynomials, we have

$$e^{aw} = \sum d_n e^{a_n w} \quad \text{for all} \quad w \in \boldsymbol{C} .$$

Hence we obtain $E(D_*) \subset E(D)$ and hence $E(D_*) = E(D)$. q.e.d.

6. Finally we remark that one can construct an abstract H^∞ space on any positive measure space except exceptional cases. In fact, we can show the following.

PROPOSITION 6.1. *Let (X, Σ, m) be a positive measure space such that there exist infinitely many disjoint m-measurable set A_j with $0 < m(A_j) < \infty$. Then there exist an m-absolutely continuous positive measure m_0 and a m_0-measurable function $f \in L^\infty(m_0)$ such that*

$$\int f^n dm_0 = \left(\int f dm_0\right)^n \quad \textit{for} \quad n = 1, 2, \cdots .$$

That is to say, the $\sigma(L^\infty(m_0), L^1(m_0))$ closure of linear spans of $\{1, f, f^2, \cdots\}$ is an abstract Hardy space.

PROOF. Let c_j, α_j satisfy the following equations, as in the proof of Theorem 4.4. $|c_j| < 1$, $\alpha_j > 0$, $\sum \alpha_j = 1$ and

$$\sum_j c_j^n \alpha_j = \left(\sum c_j \alpha_j\right)^n \quad \text{for} \quad n = 1, 2, \cdots .$$

Set $f(x) = c_j$ for $x \in A_j$ $(j = 1, 2, \cdots)$, $= 0$ otherwise and $g(x) = \alpha_j / m(A_j)$ for $x \in A_j$ $(j = 1, 2, \cdots)$, $= 0$ otherwise. Let $dm_0 = g dm$. Then we have $dm_0 \geqq 0$ and $\int dm_0 = \sum \alpha_j = 1$, and

$$\int f^n dm_0 = \sum_j c_j^n \alpha_j = \left(\sum c_j \alpha_j\right)^n = \left(\int f dm_0\right)^n$$

for $n = 1, 2, \cdots$. q.e.d.

COROLLARY 6.2. *Let (X, Σ, m) be a non-atomic probability measure space. Then there exist bounded m-measurable functions f such that*

$$\int f^n dm = \left(\int f dm\right)^n \quad \textit{for} \quad n = 1, 2, \cdots .$$

In fact, let c_j, α_j be the same as in the proof of Proposition 6.1. Since the space is non-atomic, it holds that if $A \in \Sigma$ with $m(A) > 0$ and $0 \leqq \alpha \leqq m(A)$, there exists a $B \in \Sigma$ with $B \subset A$ such that $\alpha = m(B)$. (See, for example, Halmos [4, p. 174].) Hence there exist countably many disjoint $A_j \in \Sigma$ such that $m\{A_j\} = \alpha_j$ and hence $\sum m(A_j) = 1$. Set $f(x) = c_j$ for $x \in A_j$ $(j = 1, 2, \cdots)$. Then $f \in L^\infty(m)$ and satisfies the desired conditions.

REMARK. The above spaces H contain no inner functions, i.e., there are no non-constant functions $u \in H$ with $|u| = 1$. Indeed, let $u \in H(X, \Sigma, m_0)$ in Proposition 6.1 and $|u| = 1$. Then there are at most countably many a_j on the unit circle $|z| = 1$ such that $m_0\{x: u(x) \in \{a_j\}\} = 1$ and so $m_0\{x: u(x) = a_k\} > 0$ for some k. This implies however $u = a_k$ by [12, Theorem 1].

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