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CONSTRUCTING MANIFOLDS BY HOMOTOPY EQUIVALENCES II

Browder-Novikov-Wall Type Obstruction to Constructing *PL*- and Topological Manifolds from Homology Manifolds

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0. Introduction. Let M be a homology manifold of dimension $n \geqq 5$. If $\partial M \neq \emptyset$, suppose that a neighborhood of ∂M is a *PL*-manifold. In the previous paper [8], we have defined the obstruction $\lambda(M) = \sum_{\sigma:(n-4)\text{-simplexes}} \sigma \otimes \{Lk(\sigma)\}$ in

$$H_{n-4}(M; \mathscr{H}^3),$$

where $\mathscr{H}^3$ is the group of 3-dimensional *PL*-homology spheres modulo those which are the boundary of an acyclic *PL*-manifold. If the obstruction vanishes, then M is pseudo cellular equivalent and simple homotopy equivalent to a *PL*-manifold with the same boundary. In this paper, we search for a *PL*-manifold or a topological manifold which is simple homotopy equivalent or (π_1, H_*)-equivalent to M. We call a map a (π_1, H_*)-equivalence if it induces isomorphisms of the fundamental groups and the homology groups of all dimensions.

We have a surjective homomorphism

$$i: \mathscr{H}^3 \longrightarrow Z_2 .$$

Let $\beta: H_{n-4}(M; Z_2) \longrightarrow H_{n-5}(M; Z)$ be the integral Bockstein homomorphism. Then we have the composition

$$\beta \circ i_*: H_{n-4}(M; \mathscr{H}^3) \longrightarrow H_{n-5}(M; Z) .$$

This composition was firstly considered by Sullivan [20].

Our first theorem is as follows.

THEOREM 1. *If the obstruction*

$$\beta \circ i_*(\lambda(M)) \in H_{n-5}(M; Z)$$

is zero, and if a surgery obstruction in the Wall group

$$L_n(\pi_1(M), \omega)$$

is zero, M is relatively simple homotopy equivalent to a PL-manifold

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N with $\partial N=\partial M$.

If we search for (π_1, H_*)-equivalent *PL*-manifolds, the situation becomes simpler.

THEOREM 2. *If the obstruction*

$$\beta\circ i_*(\lambda(M))\in H_{n-5}(M; Z)$$

is zero, M is relatively (π_1, H_*)-equivalent to a PL-manifold N with $\partial N=\partial M$.

The vanishing of the (π_1, H_*)-surgery obstruction for any normal map is due to Cappell-Shaneson [2] if n is odd. But in the even dimensional cases, it is complicated in general.

If we aim at constructing a topological manifold, the bundle type obstruction vanishes. Suppose that a neighborhood of ∂M is a triangulated topological manifold if $\partial M\neq\emptyset$. We have the following.

THEOREM 3. *Let n be even or odd integer greater than 4. If n is odd, suppose that a surgery obstruction is the Wall group*

$$L_n(\pi_1(M), \omega)$$

is zero, then M is relatively simple homotopy equivalent to a topological manifold N with $\partial N=\partial M$.

The author does not know an example of odd dimensional homology manifold which has non-zero obstruction of Theorem 3 in $L_n^h(\pi_1(M), \omega)$ nor in $L_n(\pi_1(M), \omega)$.

Further we have

THEOREM 4. *M is always relatively (π_1, H_*)-equivalent to a topological manifold N with $\partial N=\partial M$.*

If we regard M as a Poincaré complex, we have already the Browder-Novikov theory. We meet with the two sorts of obstructions, one for the lifting of the Spivak-fibration to the *PL* or topological bundle and the other the surgery obstruction. Our results say that the Spivak fibration of a homological manifold has a lifting to a topological bundle and shows the vanishing of the homology surgery obstruction and the vanishing of even dimensional Wall obstruction to the construction of simple homotopy equivalent topological manifold.

The vanishing of $\beta\circ i_*(\lambda(M))$ is necessary for the construction of (π_1, H_*)-equivalent *PL*-manifold, for by a result of Browder [12], two (π_1, H_*)-equivalent Poincaré complexes have the same Spivak fibrations.

Our result has an application to the uniqueness problem of two pseudo-cellularly equivalent PL-manifolds. The result for the bundle type obstruction is announced by Sullivan in [20].

Further we note that the same methods work for homotopy manifolds. We have the same result if we replace $\mathscr{H}^3$ by Θ^3 the h-cobordisms class group of 3-dimensional homotopy spheres (cf. [19]).

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The results of this paper have been announced in [23].

1. Lemmas on homology surgery. Let (X_1, Y_1) and (X_2, Y_2) be two pairs of finite simplicial complexes. A map

$$f: (X_1, Y_1) \rightarrow (X_2, Y_2)$$

is called a (π_1, H_*)-equivalence if

$$f_*: \pi_1(X_1, Y_1) \rightarrow \pi_1(X_2, Y_2)$$
$$f_*: H_j(X_1, Y_1; Z) \rightarrow H_j(X_2, Y_2; Z)$$

are isomorphisms for all j. By the theorem of Whitehead, if the pairs (X_j, Y_j) are simply connected, (π_1, H_*)-equivalence is a homotopy equivalence.

Let $(M, \partial M)$ be a compact PL-manifolds pair and let (X, Y) be a simple Poincaré complex pair of dimension $n \geqq 5$. Let ν_M be a stable normal bundle of M and let η be a stable PL bundle over X. Let $(f, \tilde{f})$ be a normal map

$$\begin{array}{ccc} \tilde{f}: \quad \nu_M & \longrightarrow & \eta \\ \downarrow & & \downarrow \\ f: (M, \partial M) & \longrightarrow & (X, Y)\,. \end{array}$$

It means that f is a degree one map and $\tilde{f}$ is a bundle map covering f. We assume that $f \mid \partial M$ induces a (π_1, H_*)-equivalence. We can perform a finite sequence of surgeries on M fixing over ∂M to make f k-connected provided $2k < n$ (cf. Wall [11] or § 2 of Cappell-Shaneson [2]). Our situation is a special case of Cappell-Shaneson [2].

PROPOSITION 1.1. *If n is odd $\geqq 5$, $(f, \tilde{f})$ is normally cobordant rel ∂M to a normal map $(f', \tilde{f}')$ such that f' is a (π_1, H_*)-equivalence.*

PROOF. By Theorem 15.4 of Cappell-Shaneson [2], the obstruction lies in $L_n(e)$ which is zero since n is odd.

If $n = 2k \geqq 6$, we can define the surgery obstruction

$$\sigma(f, \tilde{f}) \in Z \quad \text{if} \quad k = \text{even}$$
$$\in Z_2 \quad \text{if} \quad k = \text{odd}$$

which is the index and the Arf-Kervaire obstruction respectively by the same way as in the 1-connected case (Browder [1] or § 1 of [2]).

PROPOSITION 1.2. *Let $n = 2k \geqq 6$. We assume one of the following conditions*

1) $\pi_1(X) = 0$

2) *X is homotopy equivalent to a bouquet of S^1.*

Then if $\sigma(f, \tilde{f}) = 0$, $(f, \tilde{f})$ is normally cobordant rel ∂M *to a normal map $(f', \tilde{f}')$ such that f' is a (π_1, H_*)-equivalence.*

PROOF. Under the condition (1), it is well-known. Assume (2). Since $k \geqq 3$, $H_k(X; Z) = 0$, the kernel group

$$K_k(M) = \operatorname{Ker}\{f_*: H_k(M; Z) \to H_k(X; Z)\}$$

is isomorphic to $H_k(M; Z)$. We want to kill the group $K_k(M)$. By the Poincaré duality, $H_k(M; Z)$ is finitely generated and free. Since the map f is k-connected and $\pi_j(X) = 0$ for $j \geqq 2$, $\pi_j(M) = 0$ for $2 \leqq j \leqq k-1$. Then the classifying space $B(\pi_1(M))$ is constructed from M by attaching cells of dimension greater than $k+1$. Since homology group $H_j(\pi_1(M); Z)$ of $\pi_1(M)$ with trivially acting integer coefficient is isomorphic to $H_j(B(\pi_1(M)); Z)$, we have the exact sequence

$$\pi_k(M) \to H_k(M; Z) \to H_k(\pi_1(M); Z) \to 0 .$$

Since $\pi_1(M)$ is a free product of free groups of one generator,

$$H_k(\pi_1(M); Z) = 0$$

for $k \geqq 2$. So the Herewicz homomorphism

$$H: \pi_k(M) \to H_k(M; Z)$$

is surjective. By Lemmas 7.1, 8.4 of Kervaire-Milnor [6] and Lemma 9 of Milnor [7], we can do the surgery to kill $H_k(M; Z)$ if the obstruction $\sigma(f, \tilde{f})$ vanishes.

Remark that by the plumbing theorem [1, II 1.3], if $Y \neq \emptyset$, we can always sum the plumbed manifold so that the obstruction vanishes.

Later we use the case under the assumption (1) or (2) of Proposition 1.2 even in the case when n is odd. In such cases we can prove Proposition 1.1 by the method of Kervaire-Milnor [6] without using Cappell-Shaneson's theorem.

2. Trivialization lemma. Let $S^k = \partial\Delta^{k+1}$ be the boundary of a typical $(k+1)$-simplex. To any simplex τ^p of $\partial\Delta^{k+1}$, we have the dual cell $D(b_\tau)$ of the barycenter b_τ of τ. Then $D(b_\tau) = D_\tau$ is a k-cell and we have the cellular decomposition

$$S^k = \bigcup_{\tau \in S^k} D_\tau .$$

The number of p-simplexes of S^k is

$$C_{p+1}^{k+2} = (k+2)!/(k+1-p)!\,(p+1)! .$$

Let us denote by $S^k_{(p)}$ and $S^k_{(\bar{p})}$ the union of dual cells D_τ such that dim $\tau \leqq p$ and dim $\tau \geqq p$ respectively. Then we have

$$S^k = S^k_{(1)} \cup \Big(\bigcup_{\dim \tau \geqq 2} D_\tau \Big) .$$

Since the number of cells D_τ for $\dim \tau \geqq 2$ is equal to

$$\varphi(k) = \sum_{k \geqq p \geqq 2} C_{p+1}^{k+2} ,$$

we have the $(\varphi(k)+2)$-ad

$$\Big(S^k, S^k_{(1)}, \bigcup_{\dim \tau \geqq 2} D_\tau \Big) .$$

We call the decomposition

$$S^k = S^k_{(1)} \cup \Big(\bigcup_{\dim \tau \geqq 2} D_\tau \Big)$$

the canonical mod 1-skeleton cellular decomposition.

Obviously we have

LEMMA 2.1. *$\pi_1(S^k_{(1)})$ is a free product of free groups of one generator and $\pi_j(S^k_{(1)}) = 0$ for $j \geqq 2$.*

Now let $(M, M_{(1)}, \bigcup M_\tau)$ be a $(\varphi(k)+2)$-ad of PL-manifolds such that M is $(n+k)$-dimensional. Let H^n be an n-dimensional homological homology sphere(*) and let S^k be the natural sphere. Let CH^n be the cone of H^n. We have the $(\varphi(k)+2)$-ad of homology manifolds

$$(CH^n \times S^k, CH^n \times S^k_{(1)}, \bigcup CH^n \times D_\tau)$$

and its boundary

$$(H^n \times S^k, H^n \times S^k_{(1)}, \bigcup H^n \times D_\tau) .$$

PROPOSITION 2.2. *Suppose given an $(n+k+1)$-dimensional PL-manifold N with $\partial N = M$ and a normal map $(f, \tilde{f})$*

(*) By a homological homology sphere, we mean a homology manifold whose integral homology is isomorphic to that of the natural sphere.

$$\begin{array}{ccc} f\colon \nu_N & \longrightarrow & \eta \\ \downarrow & & \downarrow \\ f\colon (N, M, M_{(1)}, \bigcup M_\tau) & \longrightarrow & (CH^n \times S^k, H^n \times S^k, H^n \times S^k_{(1)}, \bigcup H^n \times D_\tau) \end{array}$$

such that $f \mid M$ induces a (π_1, H_*)-equivalence of $(\varphi(k)+2)$-ads, where ν_N is a stable normal PL-bundle and η is a stable PL-bundle.

If $n+k+1$ is an odd integer greater than 4, $(f, \tilde{f})$ is normally cobordant rel M *to a normal map $(g, \tilde{g})$ where*

$$g\colon \{(N, N_{(1)}, \bigcup N_\tau), (M, M_{(1)}, \bigcup M_\tau) \to$$
$$\{(CH^n \times S^k, CH^n \times S^k_{(1)}, \bigcup CH^n \times D_\tau), (H^n \times S^k, H^n \times S^k_{(1)}, \bigcup H^n \times D_\tau)\}$$

is a (π_1, H_*)-equivalence of pairs of $(\varphi(k)+2)$-ads and $(N, N_{(1)}, \bigcup N_\tau)$ is a $(\varphi(k)+2)$-ad of PL-manifolds.

If $n+k+1$ is an even integer greater than 5, then there exists a pair of PL-manifolds $(U, \partial U = S^{n+k})$ and a normal map $(h, \tilde{h})\colon (U, S^{n+k}) \to (D^{n+k+1}, S^{n+k})$, $h \mid S^{n+k}$ being the identity, such that $(f \amalg h, \tilde{f} \amalg \tilde{h})$ is normally cobordant rel M *to $(g, \tilde{g})$ where g is a (π_1, H_*)-equivalence of pairs of $(\varphi(k)+2)$-ads.*

Especially if the global (π_1, H_*)-surgery obstruction $\sigma(f, \tilde{f}) \in L_{n+k+1}(e)$ is zero, $(f, \tilde{f})$ is normally cobordant rel M *to a (π_1, H_*)-equivalence of pairs of $(\varphi(k)+2)$-ads.*

PROOF. We will prove by the induction of k. Nothing is to be proved for $k=1$. Assume the proposition for $k-1$. We will prove the proposition for k. We will change f on the inverse image of $f^{-1}(CH^n \times D_\tau)$ inductively from the higher dimension of τ. Suppose we have obtained a normal map which is (π_1, H_*)-equivalence of $\psi(p+1)$-ads on $f^{-1}(CH^n \times S^k{}_{(\overline{p+1})})$ for $p+1 \geqq 2$, where $\psi(p+1)$ denotes the sum of dual cells in S^k which is contained in $S^k{}_{(\overline{p+1})}$, but which does not meet with $S^k_{(1)}$. Let τ be a p-simplex. Then $D_\tau \cap S^k{}_{(\overline{p+1})} \cong D^p \times S^{k-p-1}$. By the transversality theorem, we may suppose that $(f, \tilde{f})$ is transversal on $\partial(CH^n \times D_\tau)$. Let $(f', \tilde{f}')$ be the restriction of $(f, \tilde{f})$ on $(f^{-1}(CH^n \times \partial D_\tau), f^{-1}(H^n \times \partial D_\tau))$. We have

$$\begin{aligned} \partial D_\tau &= D^p \times S^{k-p-1} \cup S^{p-1} \times D^{k-p} \\ CH^n \times \partial D_\tau &= CH^n \times D^p \times S^{k-p-1} \cup CH^n \times S^{p-1} \times D^{k-p} \\ H^n \times \partial D_\tau &= H^n \times D^p \times S^{k-p-1} \cup H^n \times S^{p-1} \times D^{k-p} . \end{aligned}$$

Since we assumed that f' is already (π_1, H_*)-equivalence on $f^{-1}(CH^n \times D^p \times S^{k-p-1})$, it is a (π_1, H_*)-equivalence on the inverse image of

$$CH^n \times D^p \times S^{k-p-1} \cup H^n \times D^p \times D^{k-p} .$$

We have

$$\partial(CH^n \times D^p \times S^{k-p-1} \cup H^n \times D^p \times D^{k-p}) \\ = (CH^n \times S^{k-p-1} \cup H^n \times D^{k-p}) \times S^{p-1}\,.$$

Obviously $CH^n \times S^{k-p-1} \cup H^n \times D^{k-p}$ is a homology sphere. The restriction of f' on $(CH^n \times S^{k-p-1} \cup H^n \times D^{k-p}) \times S^{p-1}$ is a (π_1, H_*)-equivalence of $\varphi(p-1)$-ad. Note that

$$\partial(CH^n \times D_\tau) - \mathrm{Int}\,(CH^n \times D^p \times S^{k-p-1} \cup H^n \times D^p \times D^{k-p}) \\ = CH^n \times D^{k-p} \times S^{p-1}\,.$$

Hence $\partial(CH^n \times D_\tau) - \mathrm{Int}\,(CH^n \times D^p \times S^{k-p-1} \cup H^n \times D^p \times D^{k-p})$ is homotopy equivalent to

$$C(CH^n \times S^{k-p-1} \cup H^n \times D^{k-p}) \times S^{p-1}\,.$$

Since $p-1 \leqq k-1$, we can apply the assumption of the induction to the cobordism on

$$(\partial(CH^n \times D_\tau) - \mathrm{Int}\,(CH^n \times D^p \times S^{k-p-1} \cup H^n \times D^p \times D^{k-p}), \\ (CH^n \times S^{k-p-1} \cup H^n \times D^{k-p}) \times S^{p-1})\,.$$

Note that $(f', \tilde{f}')$ is cobordant to zero. Hence by the cobordism property (II 1.5 of [1]), the obstruction $(f', \tilde{f}')$ is zero. Consequently $(f', \tilde{f}')$ is normally cobordant rel $\partial(\overline{M - M_\tau})$ to a (π_1, H_*)-equivalence of $(\varphi(p-1)+2)$-ad. We have obtained a normal map $(f'', \tilde{f}'')$ on $\{CH^n \times (S^k - \mathrm{Int}\, D_\tau), H^n \times (S^k - \mathrm{Int}\, D_\tau)\}$ which is (π_1, H_*)-equivalence on the inverse image of $CH^n \times \partial D_\tau$. Since $\partial(CH^n \times D_\tau)$ is a simply connected homology sphere, $f''^{-1}(\partial(CH^n \times D_\tau))$ is a natural sphere, since the dimension is greater than 4. Attaching a disc, we obtain a parallelizable manifold. We can extend the normal map on $\{CH^n \times S^k, H^n \times S^k\}$. The new normal map $(f^{(3)}, \tilde{f}^{(3)})$ and the original normal map $(f, \tilde{f})$ may differ in the interior of $f^{-1}(CH^n \times D_\tau)$ in the framed cobordism sense. Since the pair $\{CH^n \times D_\tau, \partial(CH^n \times D_\tau)\}$ is simple homotopy equivalent to a PL-manifolds pair, the difference lies in

$$[CH^n \times D_\tau/\partial(CH^n \times D_\tau), F/PL] = \pi_{n+k+1}(F/PL) \\ = L_{n+k+1}(e)$$

(10.6 of [11]). Consequently, by the plumbing theorem [1], we have a normal map $(h_\tau, \tilde{h}_\tau)\colon (U_\tau, S^{n+k}) \to (D^{n+k+1}, S^{n+k})$, $h \mid S^{n+k}$ being the identity, such that $(f \perp\!\!\!\perp h_\tau, \tilde{f} \perp\!\!\!\perp \tilde{h}_\tau)$ and $(f^{(3)}, \tilde{f}^{(3)})$ are normally cobordant. The extension of (π_1, H_*)-equivalence on $f^{-1}(CH^n \times S^k_{(1)})$ is quite the same, for by the Lemma 2.1 and Propositions 1.1, 1.2, the (π_1, H_*)-surgery obstruction is equal to the surgery obstruction of simply connected manifolds. We have a normal map $(h_{(1)}, \tilde{h}_{(1)}): (U_{(1)}, S^{n+k}) \to (D^{n+k+1}, S^{n+k})$ such that

$$(f \amalg (\amalg h_\tau) \amalg h_{(1)}, \tilde{f} \amalg (\amalg \tilde{h}_\tau) \amalg \tilde{h}_{(1)})$$

is normally cobordant to the required map. The global surgery obstruction $\sigma(f, \tilde{f})$ is equal to $-\sigma((\amalg h_\tau) \amalg h_{(1)}, (\amalg \tilde{h}_\tau) \amalg \tilde{h}_{(1)})$. Hence the last part of the proposition follows.

REMARK. Let x be a vertex of $\partial\Delta^{k+1}$ and let $c(x)$ be the union of $D(\tau)$ for all τ such that $\tau \gneqq x$. The union $D(x) \cup (\bigcup_{\sigma \ngtr x} D(\sigma))$ and $c(x)$ gives the cellular decomposition of S^k which is isomorphic to the decomposition

$$S^0 \times D^k \cup D^1 \times S^{k-1} = S^0 \times D^k \cup D^1 \times (\bigcup D^{k-1}),$$

where $S^{k-1} = \bigcup D^{k-1}$ is the canonical decomposition. Similarly we have the $(\varphi(k-1)+3)$-ad

$$(S^k, S^0 \times D^k, D^1 \times S^{k-1}_{(1)}, \bigcup D^1 \times D^{k-1}_\tau).$$

The Proposition 2.2 holds if we are given a (π_1, H_*)-equivalence of $(\varphi(k-1)+3)$-ads on $H^n \times S^k$. We can surgery modulo a plumbed manifold to a (π_1, H_*)-equivalence of pairs of $(\varphi(k-1)+3)$-ads.

3. A lemma concerning 4-dimensional homology spheres. Let M be a $(4+q)$-dimensional PL-manifold for $q \geqq 2$ which satisfies the following conditions.

1) M is homotopy equivalent to $H^4 \times S^q$ where H^4 is a 4-dimensional homological homology sphere.

2) M is the boundary of parallelizable PL-manifold N such that the pair (N, M) is homotopy equivalent to $(H^4 \times D^{q+1}, H^4 \times S^q)$.

3) The generator of $H_q(S^q)$ in $H_q(M)$ is represented by an embedded S^q whose normal block bundle is trivial. Under these assumptions, we prove the following.

PROPOSITION 3.1. *M is the boundary of parallelizable manifold W such that (W, M) is homotopy equivalent to $(CH^4 \times S^q, H^4 \times S^q)$.*

PROOF. Let $f: M \to H^4 \times S^q$ be a homotopy equivalence and let $c: H^4 \to S^4$ be the map obtained by collapsing the outside of an embedded disc in H^4. Let $h = (c \times id) \circ f: M \to S^4 \times S^q$ be the composition. Then h is a homology equivalence. Let $x_1, \cdots, x_\alpha$ be a finite set of generators of $\pi_1(H^4)$. Let $\phi_1, \cdots, \phi_\alpha$ be embeddings of $S^1 \times D^{q+3}$ into M^n with disjoint images representing $x_1, \cdots, x_\alpha$ respectively. We use $\phi_1, \cdots, \phi_\alpha$ to attach α handles to $I \times M$ along $(1) \times M$. Let

$$V_0 = I \times M + (\phi_1) + \cdots + (\phi_\alpha)$$

be the resulting $(q+5)$-manifold. We may assume that the embeddings $\phi_1, \cdots, \phi_\alpha$ have been chosen so that V_0 is parallelizable. Since $S^4 \times S^q$ is 1-connected the obstruction of the extension of h on $(0) \times M$ to whole of V_0 is zero, since it lies

$$H^2(V_0, M; \pi_1(S^4 \times S^q)) \cong 0 .$$

Let $h_0 \colon V_0 \to S^4 \times S^q$ be the extension. Let $N_0 = \partial V_0 - (0) \times M$. It is easy to see that

$$\begin{aligned} &\pi_1(N_0) = \pi_1(V_0) = 1 \\ &H_i(N_0) = H_i(M) \quad \text{for} \quad 3 \leqq i \leqq q+1 . \end{aligned}$$

We have

$$H_2(N_0) = H_2(V_0) = H_{q+2}(N_0) = H_2(S^q) \oplus Z^\alpha .$$

Represent the α-generators of $H_2(N_0)$ which do not come from $H_2(S^q)$ by disjoint embeddings of $S^2 \times D^{q+2}$ in N_0. Attaching the handles $\psi_1, \cdots, \psi_\alpha$, we have

$$V_1 = I \times M + (\phi_1) + \cdots + (\phi_\alpha) + (\psi_1) + \cdots + (\psi_\alpha) .$$

We can do this so that V_1 is parallelizable. We can extend h_0 on V_0 to V_1 to a mapping $h_1 \colon V_1 \to S^4 \times S^q$ since the obstruction element vanishes. Let $N_1 = \partial V_1 - (0) \times M$. We have

$$\begin{aligned} &\pi_1(N_1) = \pi_1(V_1) = 1 \\ &H_*(N_1) \cong H_*(V_1) \cong H_*(M) . \end{aligned}$$

Further the restriction of h_1 on N_1 induces the isomorphism of homology groups. Since N_1 and $S^4 \times S^q$ are 1-connected, $h_1 \mid N_1$ determines a homotopy triangulation of $S^4 \times S^q$. Denote by $hT(S^4 \times S^q)$ the set of concordance classes of homotopy triangulations of $S^4 \times S^q$. By Sullivan [9], we have

$$hT(S^4 \times S^q) = [S^4 \vee S^q, F/PL] = \pi_4(F/PL) \oplus \pi_q(F/PL) .$$

In our case the invariant in $\pi_4(F/PL)$ vanishes by the assumption 2) and one in $\pi_q(F/PL)$ vanishes by the assumption 3). Consequently the manifold N_1 is PL-homeomorphic to $S^4 \times S^q$. It is the boundary of $D^5 \times S^q$. Let W be the attached manifold

$$W = V_1 \cup D^5 \times S^q .$$

Since V_1 is parallelizable, we can attach $D^5 \times S^q$ so that W is parallelizable. By an easy computation, we have

$$\pi_1(W) = 0$$
$$H_*(W) \cong H_*(S^q) .$$

The map $f: M \to H^4 \times S^q$ can be extend to a map $g: W \to CH^4 \times S^q$ which gives a homotopy equivalence of the pairs

$$g: (W, M) \to (CH^4 \times S^q, H^4 \times S^q) .$$

More generally we have the following, which is not necessary for our purpose. Suppose $q \geqq 1$. Let

1)′, 2)′ be conditions as follows

1)′ M is (π_1, H_*)-equivalent to $H^4 \times S^q$ where H^4 is a 4-dimensional homological homology sphere.

2)′ M is the boundary of parallelizable PL-manifold N such that the pair (N, M) is (π_1, H_*)-equivalent to $(H^4 \times D^{q+1}, H^4 \times S^q)$.

PROPOSITION 3.2. *Under the assumptions* 1)′ *and* 2)′, *M is the boundary of a PL-manifold W such that (W, M) is (π_1, H_*)-equivalent to* $(CH^4 \times S^q, H^4 \times S^q)$.

PROOF. Let η be the Spivak fibration of $CH^4 \times S^q$. Since the restriction of η over $H^4 \times S^q$ is also a Spivak fibration of $H^4 \times S^q$, the normal PL-bundle of M gives a lifting of η to a PL-bundle over $H^4 \times S^q$. The obstruction of the extension of this lifting over $CH^4 \times S^q$ lies in $H^5(CH^4 \times S^q, H^4 \times S^q; \pi_4(F/PL))$. But this vanishes by the condition 2)′ and by the result of Browder [13]. Consequently, the fibration η has a lifting to a PL-bundle over $CH^4 \times S^q$. Hence it has a degree one normal map from some PL-manifold. Since $\pi_1(CH^4 \times S^q) = Z$ or 0, by the result of § 1, we can do the surgery modulo a plumbed manifold. Hence we obtain the desired manifold W.

4. General schema of construction of *PL*-manifold. We decompose a homology manifold M of dimension n as in [8]. We call the dual cell $D(\sigma) = D(b_\sigma) = D_\sigma$ i-handle if σ is $(n-i)$-simplex. We denote by M_i the union of i-handles. Let

$$M_{(j)} = \bigcup_{i \leqq j} M_i$$
$$M_{(\bar{j})} = \bigcup_{i \geqq j} M_i .$$

Then we have

$$M = \bigcup_{i \leqq n} M_i = M_{(n)} .$$

Let $\sigma > \tau$ be p-dimensional and q-dimensional simplexes respectively. We define an $(n-1)$-homology submanifold $D(\sigma)_{(\tau)}$ of $\partial D(\sigma)$ to the intersection

$$D(\sigma)_{(\tau)} = D(\sigma) \cap D(\tau) .$$

If $i < n - q$, we define $(n-1)$-dimensional homology submanifold $M_i(\tau)$ and $M_{(i)}(\tau)$ of $\partial M_i(\tau)$ and $\partial M_{(i)}(\tau)$ by

$$M_i(\tau) = M_i \cap D(\tau)$$
$$M_{(i)}(\tau) = M_{(i)} \cap D(\tau) .$$

Note that $\partial D(\sigma) = (M_{n-p-1} \cap D(\sigma)) \cup (\bigcup_{\tau<\sigma} D(\sigma)_{(\tau)})$

$$\begin{aligned} M_{(n-p-1)} \cap D(\sigma) &\cong Lk(\sigma) \times D^p \\ \bigcup_{\tau<\sigma} D(\sigma)_{(\tau)} &\cong (CLk(\sigma)) \times S^{p-1} \\ &\cong (CLk(\sigma)) \times \bigcup_{\tau<\sigma} D_\tau^{p-1} \end{aligned}$$

where $S^{p-1} = \bigcup_{\tau<\sigma} D_\tau^{p-1}$ is the canonical decomposition. We have

$$\begin{aligned} Lk(\sigma) \times D^p &\cong M_{(n-p-1)} \cap D(\sigma) \\ &= \bigcup_{j \leqq n-p-1} M_j(\sigma) = \bigcup_{\mu>\sigma} D(\mu)_{(\sigma)} \\ &= \bigcup_{\mu^r > \sigma^p} CLk(\mu) \times D^{r-p-1} \times D^p . \end{aligned}$$

On the boundary $\partial M_{(i)}$, we have the decomposition

$$\partial M_{(i)} = \bigcup M_{(i)}(\sigma)$$

σ moving all j-simplexes for $j < n - i$.

In this paper, for $i \leqq n - 3$, we will inductively construct *PL*-manifolds

$$N_{(i)} = \bigcup_{k \leqq i} N_k = \bigcup_{k \leqq i} \Big(\bigcup_{\dim \sigma = n-k} E(\sigma) \Big)$$

where $E(\sigma)$ is a contractible manifold with $\partial E(\sigma)$ decomposed by *PL*-submanifolds as

$$\partial E(\sigma) = (N_{(k-1)} \cap E(\sigma)) \cup E(\sigma)_{(1)} \cup \Big(\bigcup_{\substack{\tau<\sigma \\ \dim \tau \geqq 2}} E(\sigma)_{(\tau)} \Big) ,$$

and a map

$$t_{(i)} \colon N_{(i)} \longrightarrow M_{(i)}$$

which satisfies the following conditions, for all $k \leqq i$,

*) $\quad t_{(i)} | \colon (N_{(k-1)} \cap E(\sigma), \partial(N_{(k-1)} \cap E(\sigma)) \longrightarrow (M_{(k-1)} \cap D(\sigma), \partial(M_{(k-1)} \cap D(\sigma))$

is a simple homotopy equivalence,

$$**)\quad t_{(i)}\,|\,\{(\overline{\partial E(\sigma) - N_{(k-1)} \cap E(\sigma)},\ E(\sigma)_{(1)},\ \bigcup E(\sigma)_{(\tau)})\ ,\ (\partial(\overline{\partial E(\sigma) - N_{(k-1)} \cap E(\sigma)}),\ E(\sigma)_{(1)} \cap N_{(k-1)},\ \bigcup E(\sigma)_{(\tau)} \cap N_{(k-1)})\} \longrightarrow \{(CLk\sigma \times S^{n-k-1},\ CLk\sigma \times S^{n-k-1}_{(1)},\ \bigcup CLk\sigma \times D_\tau)\ ,\ (Lk\sigma \times S^{n-k-1},\ Lk\sigma \times S^{n-k-1}_{(1)},\ \bigcup Lk\sigma \times D_\tau)\}$$

is a (π_1, H_*)-equivalence of pairs of $(\varphi(n-k-1)+2)$-ads and is a simple homology equivalence over $Z(\pi_1(Lk\sigma))$ of pairs of $(\varphi(n-k-1)+2)$-ads. (See [2] for the definition of simple homology equivalence.)

We call the pair $(t_{(i)}, N_{(i)})$ a piecewise-linearization of $M_{(i)}$, $\overline{\partial E(\sigma) - N(k-1) \cap E(\sigma)} = (E(\sigma)_{(1)}) \bigcup (\bigcup E(\sigma)_{(\tau)})$ the cellular decomposition mod 1-skeleton and $E(\sigma)_{(\tau)}$ the attaching places.

If $t_{(i)}\colon N_{(i)} \to M_{(i)}$ satisfies $(*)$ and $(**)$, we have;

LEMMA 4.1. *$t_{(i)}\colon (N_{(k)}, \partial N_{(k)}) \to (M_{(k)}, \partial M_{(k)})$ is a simple homotopy equivalence of pairs for $k \leqq i$.*

PROOF. We prove by the induction of k. Since $n-k-1 \geqq 2$, $\{\overline{\partial E(\sigma) - N_{(k-1)} \cap E(\sigma)}, \partial(\overline{\partial E(\sigma) - N_{(k-1)} \cap E(\sigma)})\}$ and $\{CLk\sigma \times S^{n-k-1}, Lk\sigma \times S^{n-k-1}\}$ are simple homotopy equivalent by $**$). Since both $E(\sigma)$ and $D(\sigma)$ are contractible, the proof is straight by using the condition $(*)$.

LEMMA 4.2. *If $t_{(i)}\,|$ is (π_1, H_*)-equivalence of pairs of $(\varphi(n-k-1)+2)$-ads and if $t_{(i)}\,|\,\partial E(\sigma) \cap N_{(k-1)}$ is a simple homology equivalence of $(\varphi(n-k-1)+2)$-ads over $Z(\pi_1(Lk(\sigma)))$, then $t_{(i)}$ satisfies the condition $**$).*

PROOF. $H_j(\overline{\partial E(\sigma) - N_{(k-1)} \cap E(\sigma)}; \quad Z(\pi_1(Lk(\sigma))) \cong H_j(CLk\sigma \times S^{n-k-1};$ $Z(\pi_1(Lk(\sigma))) \cong H_j(S^{n-k-1}; Z)$, the proof is obvious.

LEMMA 4.3. *Let* $\dim \tau = n-i-1$. *Then*

$$t_{(i)}\,|\colon \Big(\bigcup_{\sigma>\tau} E(\sigma)_{(\tau)},\ \partial\Big(\bigcup_{\sigma>\tau} E(\sigma)_{(\tau)},\ \bigcup_{\sigma>\tau} E(\sigma)_{(1)},\ \bigcup_{\gamma}\Big(\bigcup_{\sigma>\tau>\gamma} E(\sigma)_{(\gamma)}\Big)\Big)\Big) \longrightarrow \Big(Lk\tau \times D^{n-i-1},\ Lk\tau \times S^{n-i-2},\ Lk\tau \times S^{n-i-2}_{(1)},\ Lk\tau \times \Big(\bigcup_{\gamma<\tau} D_\gamma^{n-i-2}\Big)\Big)$$

is a (π_1, H_*)-equivalence and simple homology equivalence over $Z(\pi_1(Lk\tau))$ of $(\varphi(n-i-2)+3)$-ads.

We denote $\bigcup_{\sigma>\tau} E(\sigma)_{(\tau)}$ by $N_{(i)}(\tau)$.

REMARK. If $\dim \tau = n-i-1 \geqq 3$,

$$t_{(i)}\colon \Big(\bigcup_{\sigma>\tau} E(\sigma)_{(\tau)},\ \partial\Big(\bigcup_{\sigma>\tau} E(\sigma)_{(\tau)}\Big)\Big) \longrightarrow (Lk\tau \times D^{n-i-1},\ Lk\tau \times S^{n-i-2})$$

is a simple homotopy equivalence of pairs.

PROOF. We have the decomposition

$$\begin{aligned}\bigcup_{\sigma>\tau} D(\sigma)_{(\tau)} &= Lk\tau \times D^{n-i-1} \\ &= \bigcup_{\sigma>\tau} CLk(\sigma) \times D^{k-i-1} \times D^{n-i-1}\,.\end{aligned}$$

It is easy to see that $t_{(i)}$ is a (π_1, H_*)-equivalence by the repeating application of the condition $**$). The simpleness over $Z(\pi_1(Lk(\tau)))$ comes from the fact that $Lk\tau$ is inductively constructed by attaching the cone over $Lk\sigma$ for all $\sigma > \tau$.

LEMMA 4.4. *Let $i \leqq n-4$ and let*

$$t_{(i)}\colon N_{(i)} \longrightarrow M_{(i)}$$

be a piecewise linearization. Suppose we have a pair of $(\varphi(n-i-2)+2)$-ads

$$\{(F, F_{(1)}, \bigcup F_\tau), (\partial F, \partial F_{(1)}, \bigcup \partial F_\tau)\}$$

with

$$(\partial F, dF_{(i)}, \bigcup dF_\tau) = \Big(\partial\Big(\bigcup_{\sigma>\tau} E(\sigma)_{(\tau)}\Big), \bigcup_{\sigma>\tau} E(\sigma)_{(1)}, \bigcup_{\gamma}\Big(\bigcup_{\sigma>\tau>\gamma} E(\sigma)_{(\gamma)}\Big)\Big)$$

and a (π_1, H_*)-equivalence of the pair of $(\varphi(n-i-2)+2)$-ads

$$\begin{aligned} u_\tau\colon &\{(F, F_{(1)}, \bigcup F_\tau), (\partial F, dF_{(1)}, \bigcup dF_\tau)\} \\ &\longrightarrow \Big\{\Big(CLk\tau \times S^{n-i-2}, CLk\tau \times S^{n-i-2}_{(1)}, CLk\tau \times \bigcup_{\tau>\gamma} D^{n-i-2}_\gamma\Big)\,, \\ &\qquad \Big(Lk\tau \times S^{n-i-2}, Lk\tau \times S^{n-i-2}_{(1)}, Lk\tau \times \bigcup_{\tau>\gamma} D^{n-i-2}_\gamma\Big)\Big\}\end{aligned}$$

with

$$u_\tau \mid \partial F = t_{(i)} \mid \partial F$$

for all $(n-i-1)$-simplex τ. Then we have a piecewise-linearization $(N_{(i+1)}, t_{(i+1)})$

$$t_{(i+1)}\colon N_{(i+1)} \longrightarrow M_{(i+1)}\,.$$

PROOF. It is easy to see that the union

$$\Big(\bigcup_{\sigma>\tau} E(\sigma)_{(\tau)}\Big) \cup F$$

is a $(n-1)$-dimensional homotopy sphere. If $n \geqq 6$, by the Poincaré conjecture, it is the natural sphere. It is the boundary of the natural disc D^n_τ. If $n = 5$, it is the boundary of a contractible PL-manifold D^n_τ ([3], [5], [10]). We define $E(\tau)$ by D^n_τ. The map

$$t_{(i)} \cup u_\tau\colon \partial E(\sigma) \longrightarrow (Lk\tau \times D^{n-i-1} \cup CLk\tau \times S^{n-i-1}) = \partial D(\sigma)$$

can be extended to $v_\tau: E(\sigma) \to D(\sigma)$ since both are contractible. We define $N_{(i+1)}$ by

$$N_{(i+1)} = N_{(i)} \cup \left(\bigcup_\tau E(\tau)\right)$$

and

$$t_{(i+1)}: N_{(i)} \cup \left(\bigcup_\tau E(\tau)\right) \to M_{(i+1)} = M_{(i)} \cup \left(\bigcup_\tau D(\tau)\right)$$

by $t_{(i)} \cup (\bigcup v_\tau)$. Then $(N_{(i+1)}, t_{(i+1)})$ is a piecewise-linearization by the Lemma 4.2 and Lemma 4.3.

5. Construction of $N_{(i)}$ for $i \leqq 4$. Since M_1, M_2, M_3 are PL-manifolds, we take them as N_1, N_2 and N_3 respectively. Hence $N_{(3)} = M_{(3)}$ and

$$t_{(3)}: N_{(3)} \to M_{(3)}$$

is the identity.

Let σ be an $(n-4)$-simplex. Then $N_3(\sigma)$ is PL-homeomorphic to $Lk(\sigma) \times D^{n-4}$. Since $H^3_\sigma = Lk(\sigma)$ is a PL-homology 3-sphere, there exists a parallelizable PL-manifold L^4 with boundary H^3_σ. We have a normal cobordism $(f, \tilde{f})$

$$\begin{array}{ccccc} \tilde{f}: & \nu & \longrightarrow & \eta & \\ & \downarrow & & \downarrow & \\ f: & (L^4 \times S^{n-5}, H^3_\sigma \times S^{n-5}) & \longrightarrow & (CH^3_\sigma \times S^{n-5}, H^3_\sigma \times S^{n-5}) & , \end{array}$$

where $f \mid H^3_\sigma \times S^{n-5}$ is the identity. By the Proposition 2.2, there exists a pair of $(\varphi(n-5)+2)$-ads of PL-manifolds

$$(F, F_{(1)}, \bigcup F_\tau)\,, \qquad (\partial F, dF_{(1)}, \bigcup dF_\tau)$$

with

$$(\partial F, dF_{(i)}, \bigcup dF_\tau) = (H^3_\sigma \times S^{n-5}, H^3_\sigma \times S^{n-5}_{(1)}, H^3_\sigma \times \bigcup_\tau D^{n-5}_\tau)$$

and a (π_1, H_*)-equivalence of the pairs of $(\varphi(n-5)+2)$-ads

$$\begin{aligned} h: &\{(F, F_{(1)}, \bigcup F_\tau), (\partial F, dF_{(1)}, \bigcup dF_\tau)\} \\ &\to \{(CH_\sigma \times S^{n-5} \cdot CH_\sigma \times S^{n-5}_{(1)}, CH_\sigma \times \bigcup_\tau D^{n-5}_\tau), (\partial F, dF_{(1)}, \bigcup dF_\tau)\} \end{aligned}$$

with $h \mid \partial F =$ identity. Since it holds for all $(n-4)$-simplex σ, by Lemma 4.4, we have a piecewise-linearization $(N_{(4)}, t_{(4)})$

$$t_{(4)}: N_{(4)} \to M_{(4)}\,.$$

6. Construction of $N_{(5)}$ —first step—. We now construct $(N_{(5)}, t_{(5)})$ for $n \geqq 8$. Let σ be an $(n-5)$-simplex. Then $Lk(\sigma)$ is a 4-dimensional homological homology sphere.

LEMMA 6.1. *The homological homology sphere $Lk(\sigma)$ is pseudo-cellularly equivalent to a 4-dimensional PL-homology sphere $L(\sigma)$.*

PROOF. The obstruction to the existence of pseudo-cellularly equivalent PL-manifold to $Lk(\sigma)$ lies in the 0-chain $C_0(Lk(\sigma), \mathscr{H}^3)$ (see [8]). But since $Lk(\sigma)$ is a homology sphere, the obstruction chain is a boundary. Hence to each 1-simplex α of $Lk(\sigma)$, there exists an element $a(\alpha) \in \mathscr{H}^3$ such that, to any 0-simplex β of $Lk(\sigma)$, we have

$$\sum_{\alpha>\beta} [\alpha, \beta]a(\alpha) = \{Lk(\beta, (Lk\sigma))\} \in \mathscr{H}^3 ,$$

where $[\alpha, \beta]$ denotes the incidence number. Let H_α be a 3-dimensional homology sphere which represents $-a(\alpha)$. In $Lk(\sigma)$, there exists the dual cell

$$((Lk\alpha)*pt.) \times D^1 = D^3 \times D^1 .$$

We make a new manifold

$$(Lk(\sigma)_{(3)} - (((Lk\alpha)*pt.) \times D^1) \bigcup_{S^2\times D^1} (H_\alpha - D^3) \times D^1 .$$

We do this process for all 1-simplexes of $Lk(\sigma)$ and we denote by $K(\sigma)$ the resulting manifold. The boundary of $K(\sigma)$ is a disconnected PL-manifold, the order of the component being equal to the number of 0-simplexes of $Lk(\sigma)$. A component of $\partial K(\sigma)$ has the form

$$Lk(\sigma)_{(2)} \bigcup_{S^2} \Big(\bigcup_{\alpha>\beta} (H_\alpha - D^3)\Big) .$$

It is a 3-dimensional PL-homology sphere. We denote it by H_β. By our construction it is the boundary of an acyclic PL-manifold W_β. We define $L(\sigma)$ by

$$L(\sigma) = K(\sigma) \bigcup_{H_\beta} (\bigcup W_\beta) .$$

Then $L(\sigma)$ is a 4-dimensional PL-homology sphere pseudo cellularly equivalent to $Lk(\sigma)$. The proof of the lemma finishes.

By taking the join $\alpha*\sigma$ and $\beta*\sigma$, we have a 1-1 correspondence between the set of 1-simplexes of $Lk(\sigma)$ and the set of $(n-3)$-simplexes λ^{n-3} so that $\lambda^{n-3} > \sigma^{n-5}$ and a 1-1 correspondence between the set of 0-simplexes of $Lk(\sigma)$ and the set of $(n-4)$-simplexes μ^{n-4} so that $\mu^{n-4} > \sigma^{n-5}$. By this correspondence, we write H_α, H_β, W_β by $H_{\lambda\sigma}$, $H_{\mu\sigma}$ and $W_{\mu\sigma}$. Then we have

$$L(\sigma) = K(\sigma) \bigcup_{H_{\mu\sigma}} \Big(\bigcup_{\mu} W_{\mu\sigma}\Big) .$$

We think $(H_{\lambda\sigma} - D^3) \times D^1$ to be a 3-handle of $L(\sigma)$ and we denote the union for all $\lambda > \sigma$ by $L(\sigma)_3$. Further we think $W_{\mu\sigma}$ to be a 4-handle of $L(\sigma)$ and we denote its union for all $\mu > \sigma$ to be $L(\sigma)_4$. As usual we define

$$L(\sigma)_{(3)} = L(\sigma)_{(2)} \cup L(\sigma)_3$$
$$L(\sigma)_{(4)} = L(\sigma)_{(3)} \cup L(\sigma)_4 .$$

Since any 4-dimensional PL-homology sphere is the boundary of a contractible PL-manifold ([3], [5], [10]), we have a contractible manifold $P(\sigma)$ such that $\partial P(\sigma) = L(\sigma)$. We want to attach $P(\sigma) \times D^{n-5}$ on $L(\sigma) \times D^{n-5}$ to the attaching place $N_{(4)}(\sigma)$.

We have

$$\begin{aligned} N_2(\sigma) &= L(\sigma)_2 \times D^{n-5} \\ N_3(\sigma) &= \bigcup ((Lk\alpha)*pt.) \times D^1 \times D^{n-5} \\ &= \bigcup D^3 \times D^1 \times D^{n-5} \end{aligned}$$

where α moves all 1-simplex of $Lk(\sigma)$.

On the other hand

$$\begin{aligned} L(\sigma)_3 &= \bigcup (H_{\lambda\sigma} - D^3_{\lambda\sigma}) \times D^1 \\ L(\sigma)_3 \times D^{n-5} &= \bigcup (H_{\lambda\sigma} - D^3_{\lambda\sigma}) \times D^1 \times D^{n-5} . \end{aligned}$$

If we attach $N_3(\sigma)$ and $L(\sigma)_3 \times D^{n-5}$ on $S^2 \times D^1 \times D^{n-5}$ by the identity, we have manifolds

$$\{(H_{\lambda\sigma} - D^3_{\lambda\sigma}) \cup D^3\} \times D^1 \times D^{n-5} = H_{\lambda\sigma} \times D^1 \times D^{n-5} .$$

Its boundary is $H_{\lambda\sigma} \times S^{n-5} = H_{\lambda\sigma} \times (S^0 \times D^{n-5} \cup D^1 \times S^{n-6})$. We have an $(\varphi(n-6)+3)$-ad

$$(H_{\lambda\sigma} \times S^{n-5}, H_{\lambda\sigma} \times S^0 \times D^{n-5}, H_{\lambda\sigma} \times D^1 \times S^{n-6}_{(1)}, \bigcup H_{\lambda\sigma} \times D^1 \times D^{n-6}_{\tau}) .$$

Since $H_{\lambda\sigma}$ is the boundary of the 4-dimensional parallelizable manifold $K_{\lambda\sigma}$, we have a normal map $(f, \tilde{f})$ where

$$f: (K_{\lambda\sigma} \times S^{n-5}, H_{\lambda\sigma} \times S^{n-5}) \to (CH_{\lambda\sigma} \times S^{n-5}, H_{\lambda\sigma} \times S^{n-5}) .$$

By the remark after the proposition of § 2, we can change f rel $H_{\lambda\sigma} \times S^{n-5}$ to a (π_1, H_*)-equivalence of pair of $(\varphi(n-6)+3)$-ad.

Let $\{(L_{\lambda\sigma}, L_{\lambda\sigma_0} \cup L_{\lambda\sigma_1}, L_{\lambda\sigma_{(1)}}, \bigcup L_{\lambda\sigma\tau}),$

$$(\partial L_{\lambda\sigma}, d(L_{\lambda\sigma_0} \cup L_{\lambda\sigma_1}), dL_{\lambda\sigma_{(1)}}, \bigcup dL_{\lambda\sigma\tau})\}$$

be the resulting pair of $(\varphi(n-6)+3)$-ad, where

$$\begin{aligned} &(\partial L_{\lambda\sigma}, d(L_{\lambda\sigma_0} \cup L_{\lambda\sigma_1}), dL_{\lambda\sigma_{(1)}}, \bigcup dL_{\lambda\sigma\tau}) \\ &\quad = (H_{\lambda\sigma} \times S^{n-5}, H_{\lambda\sigma} \times S^0 \times D^{n-5}, H_{\lambda\sigma} \times D^1 \times S^{n-6}_{(1)}, \bigcup H_{\lambda\sigma} \times D^1 \times D^{n-6}) . \end{aligned}$$

Since the dimension is greater than 4, by the Poincaré conjecture, we know that

$$H_{\lambda\sigma} \times D^{n-4} \bigcup_{H_{\lambda\sigma}\times S^{n-5}} L_{\lambda\sigma}$$

is PL-homeomorphic to S^{n-1}. It is the boundary of the natural disc $\bar{D}^n_{\lambda\sigma}$

$$\partial \bar{D}^n_{\lambda\sigma} = H_{\lambda\sigma} \times D^{n-4} \cup L_{\lambda\sigma} .$$

We attach $\bar{D}^n_{\lambda\sigma}$ to $P(\sigma) \times D^{n-5}$ by the identity on $(H_{\lambda\sigma} - D^3_{\lambda\sigma}) \times D^{n-4}$ for all λ such that $\lambda > \sigma$.

Let us denote by $F(\sigma)$ the resulting manifold

$$F(\sigma) = P(\sigma) \times D^{n-5} \bigcup_{(H_{\lambda\sigma}-D^3_{\lambda\sigma})\times D^{n-4}} \Big(\bigcup_{\lambda>\sigma} \bar{D}^n_{\lambda\sigma}\Big)$$

$$\begin{aligned}\partial F(\sigma) &= (L(\sigma) - \operatorname{Int}(H_{\lambda\sigma} - D_{\lambda\sigma}) \times D^1) \times D^{n-5} \cup P(\sigma) \times S^{n-6} \\ &\quad \bigcup \Big(\bigcup_{\lambda>\sigma} (L_{\lambda\sigma} \cup D^3_{\lambda\sigma} \times D^{n-4})\Big) \\ &= (L(\sigma)_{(2)} \cup L(\sigma)_4) \times D^{n-5} \cup P(\sigma) \times S^{n-6} \\ &\quad \bigcup \Big(\bigcup_{\lambda>\sigma} (L_{\lambda\sigma_0} \cup L_{\lambda\sigma_1} \cup L_{\lambda\sigma_{(1)}} \bigcup (\bigcup L_{\lambda\sigma\tau}))\Big) \bigcup D^3_{\lambda\sigma} \times D^{n-4} .\end{aligned}$$

We define $F(\sigma)_{(2)}$, $F(\sigma)_3$ and $F(\sigma)_4$ by

$$\begin{aligned} F(\sigma)_{(2)} &= L(\sigma)_{(2)} \times D^{n-5} \\ F(\sigma)_3 &= \bigcup_{\lambda>\sigma} D^3_{\lambda\sigma} \times D^{n-4} \\ F(\sigma)_4 &= L(\sigma)_4 \times D^{n-5} \bigcup \Big(\bigcup_{\lambda>\sigma} L_{\lambda\sigma_0} \cup L_{\lambda\sigma_1}\Big) .\end{aligned}$$

There exists two $(n-4)$-simplexes μ_0 and μ_1 such that $\lambda > \mu_i > 0$, $i = 0, 1$. We can naturally write $L_{\lambda\sigma_0} = L_{\lambda\mu_0\sigma}$, $L_{\lambda\sigma_1} = L_{\lambda\mu_1\sigma}$. We have the cellular decomposition on $P(\sigma) \times S^{n-6}$. The cellular decomposition mod 1-skeleton $L_{\lambda\sigma_{(1)}} \bigcup (\bigcup L_{\lambda\sigma\tau})$ coincides on $(H_{\lambda\sigma} - D^3_{\lambda\sigma}) \times D^1 \times S^{n-4}$ with that of $P(\sigma) \times S^{n-6}$.

We have

$$F(\sigma)_4 \cap F(\sigma)_{(3)} = \bigcup_{\mu>\sigma} H_\mu \times D^{n-5} \,(\text{disjoint})$$

where

$$F(\sigma)_{(3)} = F(\sigma)_{(2)} \cup F(\sigma)_3 \quad \text{and} \quad H_\mu = Lk(\alpha, Lk(\sigma)) ,$$

$\mu = \alpha * \sigma$.

7. Construction of $N_{(5)}$ —second step—. By the construction of the previous section, we have $F(\sigma)_{(3)}$ and $F(\sigma)_4$ so that

$$F(\sigma)_{(3)} = N_3(\sigma) .$$

We should attach $F(\sigma)_4$ to $N_4(\sigma)$. We have constructed $N_4(\sigma)$ in § 5, which is a union of $E(\mu)_{(\sigma)}$ for all $(n-4)$-simplexes $\mu > \sigma$. We have

$$\partial E(\mu)_{(\sigma)} = H_\mu \times D^{n-5} \bigcup_{H_\mu \times S^{n-6}} L_\mu^{(n-6)}$$

where $L_\mu^{(n-6)}$ is a PL-manifold (π_1, H_*)-equivalent to $CH_\mu \times S^{n-6}$ with boundary $H_\mu \times S^{n-6}$. We have

$$\begin{aligned} N_4(\sigma) \cap N_{(3)}(\sigma) &= F(\sigma)_4 \cap F(\sigma)_{(3)} \\ &= \bigcup_{\mu>\sigma} H_\mu \times D^{n-5} \quad \text{(disjoint)} . \end{aligned}$$

Now let

$$Y_{\sigma\mu} = \left(W_{\mu\sigma} \times D^{n-5} \bigcup_{(H_{\lambda\sigma} - D_{\lambda\sigma}) \times D^{n-5}} \left(\bigcup_{\lambda>\sigma} L_{\lambda\mu\sigma} \right) \right) \bigcup_{H_\mu \times D^{n-5}} E(\mu)_{(\sigma)} .$$

Then

$$N_4(\sigma) \bigcup_{(\cup H_\mu \times D^{n-5})} F(\sigma)_4 = \bigcup_\sigma Y_{\sigma\mu} .$$

We have

$$\partial Y_{\sigma\mu} = \left(W_{\mu\sigma} \times S^{n-6} \bigcup_{(H_{\lambda\sigma} - D_{\lambda\sigma}) \times S^{n-6}} \left(\bigcup_{\mu>\sigma} \tilde{L}_{\lambda\mu\sigma} \right) \right) \bigcup_{H_\mu \times D^{n-5}} L_\mu^{(n-6)}$$

where

$$\tilde{L}_{\lambda\mu\sigma} = \overline{\partial L_{\lambda\mu\sigma} - H_{\lambda\sigma} \times D^{n-5}} .$$

By our construction, it is easy to see that $(Y_{\sigma\mu}, \partial Y_{\sigma\mu})$ is (π_1, H_*)-equivalent to

$$\left(CH_\mu \bigcup_{\substack{\mu<\lambda \\ \mu>\sigma}} \left(\left(\bigcup CH_{\lambda\sigma} \right) \bigcup_{H_{\lambda\sigma} - D^3_{\lambda\sigma}} W_{\mu\sigma} \right) \right) \times (D^{n-5}, S^{n-6}) .$$

Further it is simple homology equivalent over

$$Z(\pi_1(CH_\mu \cup ((\cup CH_{\lambda\sigma}) \cup W_{\mu\sigma}))) .$$

Since $n \geqq 8$, they are simple homotopy equivalent. Note that

$$CH_\mu \cup ((\cup CH_{\lambda\mu}) \cup W_{\mu\sigma})$$

is a homological homology sphere of dimension 4. We denote it by $K_{\sigma\mu}$.

On $\partial Y_{\sigma\mu}$, we have a cellular decomposition mod 1-skeleton which is defined by the decompositions of the boundary of $W_{\mu\sigma} \times S^{n-6}$, $L_{\lambda\mu\sigma}$ and $L_\mu^{(n-6)}$.

If $Y_{\sigma\mu}$ is parallelizable, we can apply the result of § 3. Since $Y_{\sigma\mu}$ is parallelizable as a Poincaré complex, the obstruction to PL-trivialization is

$$c(\sigma\mu) \in [K_{\sigma\mu}, F/PL] \cong \pi_4(F/PL) \cong Z .$$

We define $c(\sigma)$ by

$$c(\sigma) = \sum_{\mu>\sigma} [\mu, \sigma] c(\sigma\mu) \in Z ,$$

where $[\mu, \sigma]$ denotes the incidence number. If $c(\sigma) = 0$, we can change the attaching map of $P(\sigma) \times D^{n-5}$ on $L(\sigma) \times D^{n-5}$ in its isotopy class so that, for all $\mu > \sigma$, $c(\sigma\mu) = 0$. We define a chain $c \in C_{n-5}(M; Z)$ by

$$c = \sum_{\sigma} c(\sigma) .$$

LEMMA 7.1. *c is a cycle.*

PROOF. To any $(n-6)$-simplex γ, the sum

$$\sum_{\sigma>\gamma} [\sigma, \gamma] c(\sigma) = \sum_{\sigma>\gamma} [\sigma, \gamma] \sum_{\mu>\sigma} [\mu, \sigma] c_{\sigma\mu} = 0 .$$

Now we suppose that it is a boundary. Then there exists a chain

$$b = \sum b(\mu) \in C_{n-4}(M; Z)$$

such that $\partial b = c$. In the construction of N_4, we have started with $L^4 \times S^{n-5}$ and trivial bundle ν and η (§ 5). But we may take η' which is trivial as a spherical fibration but may not be trivial as a PL-bundle. (10.2 of [11]). The isomorphism class of such bundle is equal to $[CH^3_\sigma \times S^{n-5}/H^3_\sigma \times S^{n-5}, F/PL] = \pi_4(F/PL) = Z$. Corresponding to $-b(\mu) \in Z$, we take η'. Then we have new N'_4 such that the class

$$c'(\sigma) = c(\sigma) + \partial(-b)(\sigma) = 0 .$$

Hence we have

LEMMA 7.2. *If c is a boundary, we can take $N_{(4)}$ so that $c(\sigma\mu) = 0$ for all pairs.*

Now by the result of § 3, we have a manifold $Z_{\sigma\mu}$ with $\partial Z_{\sigma\mu} = \partial Y_{\sigma\mu}$ such that

$$(Z_{\sigma\mu}, \partial Y_{\sigma\mu}) \simeq (CK_{\sigma\mu} \times S^{n-6}, K_{\sigma\mu} \times S^{n-6}) .$$

We can apply the Proposition 2.2 so that we have the cellular decomposition mod 1-skeleton of $Z_{\sigma\mu}$ which extends the one on $\partial Y_{\sigma\mu}$. Since $Z_{\sigma\mu} \cup Y_{\sigma\mu}$ is a natural sphere, it is the boundary of n-disc $D_{\sigma\mu}$. We define $E(\sigma)$ by

$$E(\sigma) = F(\sigma) \bigcup_{Y_{\sigma\mu} - E(\mu)_{(\sigma)}} \Big(\bigcup_{\mu>\sigma} D_{\sigma\mu}\Big) .$$

We define $E(\sigma)_{(4)}$ by

$$E(\sigma)_{(4)} = F(\sigma)_{(3)} \cup \Big(\bigcup_{\mu>\sigma} E(\mu)_{(\sigma)}\Big)\,.$$

Then $N_{(4)}(\sigma) = E(\sigma)_{(4)}$ and we have the decomposition on $\overline{\partial E(\sigma) - E(\sigma)_4}$. As is shown in § 4, we have a piecewise-linearization $(N_{(5)}, t_{(5)})$ by defining

$$N_{(5)} = N_{(4)} \cup (\cup E(\sigma))\,.$$

Summarizing we have

PROPOSITION 7.3. *If the cyclic $c \in C_{n-5}(M; Z)$ is a boundary, we have a piecewise-linearization*

$$t_{(5)}\colon N_{(5)} \longrightarrow M_{(5)}\,.$$

8. **Construction of $N_{(i)}$ for $i \leqq n-3$.** Suppose we have obtained a piecewise-linearization $t_{(p-1)}\colon N_{(p-1)} \to M_{(p-1)}$ for $6 \leqq p \leqq n-3$. We want to extend it to $t_{(p)}\colon N_{(p)} \to M_{(p)}$.

Let σ be an $(n-p)$-simplex of M, and let $t_{(p-1)}\,|\colon N_{(p-1)}(\sigma) \to M_{(p-1)}(\sigma)$ be the restriction of $t_{(p-1)}$. Since $n-p \geqq 3$, by Lemma 4.3, it is a simple homotopy equivalence of pairs $(N_{(p-1)}(\sigma), \partial N_{(p-1)}(\sigma))$ and $(H_\sigma \times D^{n-p}, H_\sigma \times S^{n-p-1})$ where H_σ is $Lk(\sigma, M)$ which is a $(p-1)$-dimensional homological homology sphere. Since $H_{p-1-4}(H_\sigma; Z) = H_{p-1-5}(H_\sigma; Z) = 0$, by the result of [8], H_σ is simple homotopy equivalent to a PL-homology sphere K_σ. By the embedding theorem up to homotopy (see 11.3 of Wall [11]), we can embed K_σ in $N_{(p-1)}(\sigma)$. The normal block bundle $T(\sigma)$ is homotopically trivial. Let

$$\begin{aligned} c(\sigma) \in [H_\sigma, G_{n-p}/\widetilde{PL}_{n-p}] &\cong \pi_{p-1}(G_{n-p}/\widetilde{PL}_{n-p}) \\ &\cong \pi_{p-1}(F/PL) \end{aligned}$$

be the obstruction to a PL-trivialization. If $c(\sigma)$ is zero, the normal block bundle is trivial and the boundary $\partial N_{(p-1)}(\sigma)$ is PL-homeomorphic to $K_\sigma \times S^{n-p-1}$ by the s-cobordism theorem. We have a chain

$$c = \sum c(\sigma)$$

in

$$C_{n-p}(M; \pi_{p-1}(F/PL))\,.$$

We will show that this is a boundary. We need the following. Let $\{\lambda_j\}$ be the set of simplexes in M such that $\lambda_j > \sigma$. The union $\bigcup D(\lambda_j)$ in $M_{(p-1)}$ is an n-dimensional homology manifold which is a total space of a homology cobordism bundle (Martin-Maunder [12]) over $Lk(\sigma)$. It is a trivial bundle and especially it is a stable trivial spherical fibration. We denote $\bigcup D(\lambda_j)$ by $M_{R(\sigma)}$. We have also the union $\bigcup E(\lambda_j)$ in $N_{(p-1)}$, which we denote by $N_{R(\sigma)}$. The restriction of $t_{(p-1)}$ on $N_{R(\sigma)}$ maps $N_{R(\sigma)}$

onto $M_{R(\sigma)}$. By the same way as the proof of Lemma 4.3 and the remark after that, we have the following.

LEMMA 8.1. *The restriction*

$$t_{(p-1)}|: (N_{R(\sigma)}, \partial N_{R(\sigma)}) \to (M_{R(\sigma)}, \partial M_{R(\sigma)})$$

is a simple homotopy equivalence of pairs.

Consequently $(N_{R(\sigma)}, \partial N_{R(\sigma)})$ defines a stable parallelizable spherical fibration over H_σ. We can embed K_σ in $N_{R(\sigma)}$ so that the embedding $K_\sigma \to N_{R(\sigma)}$ is a simple homotopy equivalence (11.3 of [11]). It has a normal block bundle $S(\sigma)$. By the s-cobordism theorem, $N_{R(\sigma)}$ is PL-homeomorphic to $S(\sigma)$. Since it is stably homotopically trivial, the obstruction to the stable PL-trivialization lies in

$$[H_\sigma, F/PL] \cong \pi_{p-1}(F/PL) .$$

Let $e(\sigma) \in \pi_{p-1}(F/PL)$ be the obstruction. Consequently we have the chain

$$e = \sum_\sigma e(\sigma) \in C_{n-p}(M; \pi_{p-1}(F/PL)) .$$

LEMMA 8.2. *The chain e is a boundary.*

PROOF. Let $N_{R(\sigma)_{(p-2)}}$ denote the intersection

$$N_{R(\sigma)} \cap N_{(p-2)} .$$

Then the inclusion $N_{R(\sigma)_{(p-2)}} \to N_{R(\sigma)}$ induces the homomorphism

$$h: [N_{R(\sigma)}, F/PL] \to [N_{R(\sigma)_{(p-2)}}, F/PL] .$$

But $[N_{R(\sigma)}, F/PL] = [H_\sigma, F/PL] = \pi_{p-1}(F/PL)$, and $N_{R(\sigma)_{(p-2)}}$ is homotopically $(p-2)$-dimensional. Hence the map h is trivial. Consequently we have a PL-trivialization on $N_{R(\sigma)_{(p-2)}}$. We can extend this trivialization on $N_{R(\sigma')_{(p-2)}}$ for all $(n-p)$-simplexes σ' in M. Hence each obstruction $e(\sigma)$ comes from $(n-p-1)$-simplexes which shows that e is a boundary.

LEMMA 8.3. $e(\sigma) = c(\sigma) \in \pi_{p-1}(F/PL)$.

PROOF. $T(\sigma)$ may be regarded as an embedded PL-manifold in $\partial S(\sigma)$. Hence $S(\sigma)$ is equal to the Whitney sum of $T(\sigma)$ with a trivial 1-dimensional bundle. Then the stabilization isomorphism

$$\pi_{p-1}(G_{n-p}/\widetilde{PL}_{n-p}) = \pi_{p-1}(F/PL)$$

maps $c(\sigma)$ to $e(\sigma)$.

Consequently, the chain c is a boundary. By the same argument as in the construction of $N_{(5)}$, we can change the attaching of $N_{(p-1)}$ to $N_{(p-2)}$ so that $c(\sigma)$ is zero for all $(n-p)$-simplex σ.

Now suppose $c(\sigma)$ is zero. Then $N_{(p-1)}(\sigma)$ is PL-homeomorphic to $K_\sigma \times S^{n-p-1}$. By [3], [5], [10], we have a contractible PL-manifold W_σ so that $\partial W_\sigma = K_\sigma$. We have a normal map $(f, \tilde{f})$ where

$$f: (W_\sigma \times S^{n-p-1}, K_\sigma \times S^{n-p-1}) \longrightarrow (CH_\sigma \times S^{n-p-1}, H_\sigma \times S^{n-p-1}) .$$

We can apply the Proposition 2.2. We have a PL-manifold Z_σ with $\partial Z_\sigma = K_\sigma \times S^{n-p-1}$ and the cellular decomposition mod 1-skeleton. We can apply Lemma 4.4 to the construction of $t_{(p)}$ and $N_{(p)}$. We have obtained the following.

PROPOSITION 8.4. *If $\sigma \leqq p \leqq n-3$, then the piecewise-linearization $t_{(p-1)}: N_{(p-1)} \longrightarrow M_{(p-1)}$ can be extended to a piecewise linearization*

$$t_{(p)}: N_{(p)} \longrightarrow M_{(p)} .$$

9. Construction of simple homotopy equivalent manifold. Suppose we have constructed a piecewise-linearization $t_{(n-3)}: N_{(n-3)} \longrightarrow M_{(n-3)}$ for $n \geqq 5$. Then $t_{(n-3)} |: \partial N_{(n-3)} \longrightarrow \partial M_{(n-3)}$ is a simple homotopy equivalence by Lemma 4.1. We have defined $M_{(\overline{n-2})}$ by

$$M_{(\overline{n-2})} = \bigcup_{j \geqq n-2} M_j .$$

Then $\partial M_{(\overline{n-2})} = \partial M_{(n-3)}$.

Since $M_{(\overline{n-2})}$ is a simple Poincaré pair, it has a Spivak normal fibration [18]. The normal bundle of PL-manifold $\partial N_{(n-3)}$ gives a section of the associated bundle with fibre G/PL over $\partial M_{(\overline{n-2})}$. If the section can be extended over $M_{(\overline{n-2})}$, we have a normal map over the pair $(M_{(\overline{n-2})}, \partial M_{(\overline{n-2})})$ (See for example 10.2 of [11]).

LEMMA 9.1. *If $n \geqq 8$, the section over $\partial M_{(\overline{n-2})}$ is extendable to a section over $M_{(\overline{n-2})}$.*

PROOF. The obstruction to an extension lies in

$$H^j(M_{(\overline{n-2})}, \partial M_{(\overline{n-2})}; \pi_{j-1}(F/PL)) .$$

But we have

$$\begin{aligned} H^j(M_{(\overline{n-2})}, \partial M_{(\overline{n-2})}; Z) &= 0 \qquad j \leqq n-3 \\ &= \text{free} \quad j = n-2 . \end{aligned}$$

Hence the first obstruction lies in the chain $C^{n-2}(M_{(\overline{n-2})}, \partial M_{(\overline{n-2})}; \pi_{n-3}(F/PL))$. This obstruction is represented by the obstruction to the triviality of stable PL-bundles over the homological homology $(n-3)$-spheres in $N_{(n-3)}$

which represent the links of 3-simplexes of M. As in the proof of Lemma 8.2, it is a coboundary. As in § 7, we may change the construction of N_{n-3} so that the chain is zero for all links of 3-simplexes. Similarly the next obstruction in $H^j(M_{(\overline{n-2})}, \partial M_{(\overline{n-2})}; \pi_{j-1}(F/PL))$ is zero, for $j = n - 1$ and $j = n$.

LEMMA 9.2. *Let $5 \leqq n \leqq 7$. Then the section over $M_{(\overline{n-2})}$ is extendable to a section over $M_{(\overline{n-2})}$ if the obstruction in $H_{n-5}(M; Z)$ vanishes.*

PROOF. As before, the obstructions lie in

$$H^j(M_{(\overline{n-2})}, \partial M_{(\overline{n-2})}; \pi_{j-1}(F/PL))$$

for $n \geqq j \geqq n - 2$, $j \geqq 5$. If $5 \leqq n \leqq 6$, the non-zero element is only in

$$\begin{aligned} H^5(M_{(\overline{n-2})}, \partial M_{(\overline{n-2})}; \pi_4(F/PL)) &= H_{n-5}(M_{(\overline{n-2})}; Z) \\ &= H_{n-5}(M; Z) . \end{aligned}$$

If $n = 7$, $H_{n-5}(M_{(\overline{n-2})}; Z)$ is free. But the obstruction element in $H_{n-5}(M_{(\overline{n-2})}; Z)$ reduces to an element in $H_{n-5}(M; Z)$. We have another obstruction in

$$H^7(M_{(\overline{n-2})}, \partial M_{(\overline{n-2})}; \pi_6(F/PL)) .$$

But this element vanishes by the same reason as in Lemma 9.2.

The obstruction in $H_{n-5}(M; Z)$ has the same nature as c of § 7. We also denote it by the same simbol c. A homotopy theoretic interpretation is given in § 11. The result is that

$$c = \beta \circ i_*(\lambda(M)) \in H_{n-5}(M; Z) ,$$

where $\lambda(M) \in H_{n-4}(M; \mathscr{H}^3)$ is the obstruction of [8], $i: \mathscr{H}^3 \to Z_2$ is the homomorphism defined by taking the mod 2 class of the 1/8 index of a bounding PL-manifold, and $\beta: H_{n-4}(M; Z_2) \to H_{n-5}(M; Z)$ is the Bockstein homomorphism.

As a corollary, it follows easily,

COROLLARY 9.3. *If $n = 5$, the section over $\partial M_{(\overline{n-2})}$ is always extendable over $M_{(\overline{n-2})}$.*

Now suppose that the obstruction $c \in H_{n-5}(M; Z)$ is zero for $n \geqq 6$. Since it is a lifting of the Spivak fibration, there is a normal map ϕ on $M_{(\overline{n-2})}$.

Then the obstruction to make ϕ simple homotopy equivalence relative to the boundary lies in

$$L_n(\pi_1(M_{(\overline{n-2})}), \omega) \cong L_n(\pi_1(M), \omega) .$$

If this obstruction is zero, we have a PL-manifold pair $(W, \partial N_{(n-3)})$ and a simple homotopy equivalence

$$\psi\colon (W, \partial N_{(n-3)}) \to (M_{(\overline{n-2})}, \partial M_{(\overline{n-2})})$$

such that $\psi \mid N_{(n-3)} = t_{(n-3)}$. Then we can add W to $N_{(n-3)}$ obtaining a PL-manifold N which is simple homotopy equivalence to M.

Combining the results of previous sections, we have the following theorem.

THEOREM 1. *Suppose given a homology manifold M of dimension greater than 4. If the obstruction*

$$c \in H_{n-5}(M; Z)$$

vanishes and if the obstruction in

$$L_n(\pi_1(M), \omega)$$

vanishes, then M is simple homotopy equivalent to a PL-manifold.

10. Construction of (π_1, H_*)-equivalent manifold. As in the previous section, suppose that we have a piecewise-linearization

$$t_{(n-3)}\colon N_{(n-3)} \to M_{(n-3)} .$$

In this section, we will search for a PL-manifold N which is (π_1, H_*)-equivalent to M.

At first suppose that n is odd. Then the method of previous section goes. We have a homology surgery obstruction. But by a result of Cappell-Shaneson (Th. 15.4 of [2]), it vanishes. Hence we have always a (π_1, H_*)-equivalent homology manifold.

Now suppose n is even. Then the calculation of homology surgery obstruction is difficult. We proceed as follows.

Let σ be a 2-simplex. In $\partial N_{(n-3)}$, we have a distinguished submanifold $N_{(n-3)}(\sigma)$. By Lemma 4.3, $(N_{(n-3)}(\sigma), \partial N_{(n-3)}(\sigma))$ is (π_1, H_*)-equivalent and simple homology equivalent over $Z(\pi_1(Lk(\sigma)))$ to $(Lk\sigma \times D^2, Lk\sigma \times S^1)$. Then $N_{(n-3)}(\sigma)$ is simple homotopy equivalent to $Lk(\sigma)$. Denote by H_σ the $Lk(\sigma)$ which is $(n-3)$-dimensional homological homology sphere. If $n \geqq 8$, $H_{n-4}(H_\sigma; \mathscr{H}^3) = 0$. Then we have a PL-homology sphere K_σ which is simple homotopy equivalent to H_σ.

LEMMA 10.1. *If n is even and $N_{(n-3)}(\sigma)$ is parallelizable we can embed K in $N_{(n-3)}(\sigma)$ so that the inclusion*

$$K_\sigma \to N_{(n-3)}(\sigma)$$

is a simple homotopy equivalence.

PROOF. By the result of Kato-Matsumoto [4] or Cappell-Shaneson (Th. 8.2 of [2]), the obstruction to the existence of such embedding is equal to the abstract surgery obstruction. If $N_{(n-3)}(\sigma)$ is parallelizable, it is normally cobordant to the PL-homology sphere. So the obstruction vanishes.

Since $N_{(n-3)}(\sigma)$ is homotopically parallelizable, the obstruction $c(\sigma)$ to the stable trivialization as a PL-bundle lies in

$$[H_\sigma, F/PL] = \pi_{n-3}(F/PL)$$

which is zero since n is even. We can embed K_σ in the interior of $N_{(n-3)}(\sigma)$. Since the codimension is two, it has a trivial block bundle. Let $\bigcup_\sigma K_\sigma$ be the disjoint union of embedded K_σ in $\partial N_{(n-3)}$. We have the disjoint union $\bigcup_\sigma H_\sigma$ in $\partial M_{(n-3)}$ as the core of $(n-2)$-handles. By the homotopy extension property, we may change $t_{(n-3)} \mid \partial N_{(n-3)}$ by the homotopy so that the restriction of $t_{(n-3)}$ on K_σ maps K_σ onto H_σ by a simple homotopy equivalence. Since H_σ has trivial block bundle, we may suppose that f is transverse regular on each H_σ. Consequently we may suppose that f maps each regular neighborhood of H_σ by a bundle map to the regular neighborhood of K_σ. Since this bundle is two dimensional, we may think that it is a trivial bundle map.

Since K_σ is a homology sphere, it is a boundary of contractible manifold L_σ. Let $F(\sigma) = L_\sigma \times D^2$ and attach it on the regular neighborhood of K_σ by the identity map on $K_\sigma \times D^2$. Doing this attachment for all 2-simplex, we obtain a PL-manifold

$$N_{(n-2)} = N_{(n-3)} \bigcup_{K_\sigma \times D^2} \Big(\bigcup_\sigma F(\sigma)\Big) .$$

LEMMA 10.2. *We can extend the simple homotopy equivalence*

$$t_{(n-3)}: (N_{(n-3)}, \partial N_{(n-3)}) \longrightarrow (M_{(n-3)}, \partial M_{(n-3)})$$

to a (π_1, H_*)*-equivalence*

$$t_{(n-2)}: (N_{(n-2)}, \partial N_{(n-2)}) \longrightarrow (M_{(n-2)}, \partial M_{(n-2)}) .$$

PROOF. If $n = 6$, we have already proved in § 5. We may assume $n \geqq 8$. As remarked before, we can homotope $t_{(n-3)} \mid \partial N_{(n-3)}$ to a mapping which is a bundle map of $D^2 \times K_\sigma$ onto $D^2 \times H_\sigma$. Note that

$$t_{(n-3)} \mid : \overline{\partial N_{(n-3)} - \bigcup (D^2 \times K_\sigma)} \longrightarrow \overline{\partial M_{(n-3)} - \bigcup D^2 \times H_\sigma}$$

is a (π_1, H_*)-equivalence. Since both L_σ and CH_σ are contractible, we can extend $t_{(n-3)}$ on $\overline{\bigcup \partial(D^2 \times L_\sigma) - D^2 \times K_\sigma}$ to $\overline{\bigcup \partial(D^2 \times CH_\sigma - D^2 \times H_\sigma)}$. Next we can extend it to the interior of handles. The resulting map

$t_{(n-2)}$ is a (π_1, H_*)-equivalence on the boundary and a simple homotopy equivalence absolutely.

Let ν be a normal bundle of $\partial N_{(n-2)}$. Then $t_{(n-2)} \mid \partial N_{(n-2)}$ and $t_{(n-2)}\mid_*$ give a degree one normal map.

As in the previous section,

LEMMA 10.3. *Let $n \geqq 5$. If $n \leqq 7$ suppose the obstruction c in $H_{n-5}(M; Z)$ vanishes. Then the Spivak fibration of $M_{\overline{(n-1)}}$ has a lifting to a PL-bundle which is equal to ν on $\partial M_{\overline{(n-1)}}$.*

Suppose the degree one normal map can be extended to the interior $M_{\overline{(n-1)}}$. Then $\pi_1(M_{\overline{(n-1)}})$ is a free product of free groups of one generator and $\pi_j(M_{\overline{(n-1)}}) = 0$ for $j \geqq 2$. By Proposition 1.2, the (π_1, H_*)-surgery obstruction is equal to the simply connected surgery obstruction. By the plumbing theorem, we can always sum the plumbed manifold so that the obstruction vanishes.

Hence we have

PROPOSITION 10.4. *Let $n \geqq 5$ and suppose given a piecewise linearization*

$$t_{(n-3)}: N_{(n-3)} \longrightarrow M_{(n-3)} .$$

If $n \leqq 7$ suppose the obstruction c in $H_{n-5}(M; Z)$ vanishes. Then we have a PL-manifold N and a (π_1, H_*)-equivalence

$$t: N \longrightarrow M$$

which is an extension of $t_{(n-3)}$.

Combining the result of previous section, we have

THEOREM 2. *Suppose given a homology manifold M of dimension greater than 4. If the obstruction*

$$c \in H_{n-5}(M; Z)$$

vanishes, then M is (π_1, H_*)-equivalent to a PL-manifold N.

11. Calculus of the obstruction $c(M) \in H_{n-5}(M; Z)$. Let us denote by *HOM* the s. s. complex of stable homology cobordism bundle ([12], [14]). We have naturally the s. s. complex *F/HOM*, *HOM/PL* which are the group of homotopically trivial homology automorphism or homologically trivial *PL*-automorphism of R^n, n: large, respectively. We have a fibration

$$HOM/PL \longrightarrow F/PL \longrightarrow F/HOM$$

and an exact sequence

$$\to \pi_j(HOM/PL) \to \pi_j(F/PL) \to \pi_j(F/HOM) \to \pi_{j-1}(HOM/PL)\,.$$

By the result of [8], we can directly deduce that*),

PROPOSITION 11.1.

$$\begin{aligned} \pi_j(HOM/PL) &= \mathscr{H}^3 \qquad j = 3 \\ &= 0 \qquad \textit{otherwise}\,. \end{aligned}$$

Consequently we have a short exact sequence

$$(1) \qquad 0 \longrightarrow \pi_4(F/PL) \xrightarrow{p} \pi_4(F/HOM) \xrightarrow{q} \mathscr{H}^3 \longrightarrow 0\,.$$

As is shown in [8], by taking the link of every $(n-4)$-simplex, we have the obstruction

$$\lambda(M) = \sum_{\sigma:\,(n-4)\text{-simplexes of } M} \sigma \otimes \{Lk(\sigma)\} \in H_{n-4}(M; \mathscr{H}^3)$$

to the existence of a pseudo-cellularly equivalent PL-manifold. Associated to the exact sequence (1), we have an element,

$$\partial\lambda(M) \in H_{n-5}(M; \pi_4(F/PL)) = H_{n-5}(M; Z)\,.$$

On the other hand, in §§ 7, 9 we have reached the obstruction

$$c(M) \in H_{n-5}(M; Z)$$

to the construction of $N_{(5)}$.

PROPOSITION 11.2.

$$\partial\lambda(M) = c(M)\,.$$

PROOF. We must recall the construction of $N_{(5)}$ for $n \geqq 8$. First we have constructed $N_{(4)}$, which is just equal to a choice of a chain $d \in C_{n-4}(M; \pi_4(F/HOM))$ such that

$$q \circ d = \lambda \in C_{n-4}(M; \mathscr{H}^3)\,.$$

The boundary

$$\partial(q \circ d) \in C_{n-5}(M; \pi_4(F/HOM))$$

is equal to taking the characteristic class of the union $N_{(4)}(\sigma)$ as a homology bundle for each $(n-5)$-simplex σ. Hence to take its characteristic class as a PL-bundle is just equal to the choice of $e \in C_{n-5}(M; \pi_4(F/PL))$ so that

$$p \circ e = \partial(q \circ d)\,.$$

By the definition of the boundary

*) This is suggested to the author by Takao Matumoto. See [24].

$$\partial\lambda(M) = \{e\} \in H_{n-5}(M; \pi_4(F/PL)) .$$

Since we have defined the class $c(M) \in H_{n-5}(M; \pi_4(F/PL))$ by $\{e\}$, we obtain the proposition. For $5 \leqq n \leqq 7$, the proof is similar.

Let TOP be the structure group of stable topological micro-bundle. By the work of Kirby-Siebenmann [13], we have

$$\begin{aligned} \pi_j(TOP/PL) &= Z_2 \qquad j = 3 \\ &= 0 \qquad \text{otherwise} . \end{aligned}$$

Consequently the exact sequence

$$(2) \qquad 0 \to \pi_4(F/PL) \to \pi_4(F/TOP) \to \pi_3(TOP/PL) \to 0$$

is equal to the exact sequence

$$(2') \qquad 0 \longrightarrow Z \xrightarrow{\times 2} Z \longrightarrow Z_2 \longrightarrow 0 .$$

By taking the Pontrjagin class ([15]), we have a homomorphism

$$s\colon \pi_4(F/HOM) \to Z .$$

Let H^3 be a 3-dimensional PL-homology sphere. It is the boundary of a parallelizable manifold W. Let $t(W)$ be the index of the homology manifold $W \cup CH^3$. Let us denote by $i(H^3)$ the class of $t(W)$ in $8Z/16Z \cong Z_2$. Then we have a homomorphism

$$i\colon \mathscr{H}^3 \to Z_2$$

by $i(\{H^3\}) = \{t(W)\}$. The well definedness comes from the Rohlin's Theorem [16] (cf. Sullivan [19]).

The isomorphism $r\colon \pi_4(F/PL) = Z$ and the homomorphisms s and t define a homomorphism of the short exact sequence (1) to (2)′,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_4(F/PL) & \longrightarrow & \pi_4(F/HOM) & \longrightarrow & \mathscr{H}^3 & \longrightarrow & 0 \\ & & \wr\Vert \big\downarrow r & & \cap \big\downarrow s & & \cap \big\downarrow t & & \\ 0 & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & Z_2 & \longrightarrow & 0 . \end{array}$$

Let us denote by β the integral Bockstein homomorphism

$$\beta\colon H_{n-4}(M; Z_2) \to H_{n-5}(M; Z) .$$

It is easy to check the following,

PROPOSITION 11.3. $\beta \circ i_* = \partial\colon H_{n-4}(M; \mathscr{H}^3) \to H_{n-5}(M; Z)$.

COROLLARY 11.4. *The obstruction* $c(M) \in H_{n-5}(M; Z)$ *is two torsion, that is,*

$$2c(M) = 0 \in H_{n-5}(M; Z) .$$

12. Construction of topological manifolds. Now we want to construct a topological manifold. We have analogue of § 2 and § 3 by changing PL-manifold and PL-block bundle by topological manifold and (stable) topological micro-bundle word by word. This is possible by the work of Kirby-Siebenmann [13]. Further we can improve a (π_1, H_*)-equivalence of Proposition 2.2 to a simple homotopy equivalence. This is possible by using the following proposition which T. Matumoto suggested to the author and is essentially due to Siebenmann.

PROPOSITION 12.1. *Let H^3 be a 3-dimensional homology sphere. Then there exists a parallelizable topological manifold N with boundary homeomorphic to $H^3 \times S^1$ and a relative homotopy equivalence*

$$g: (N, \partial N) \to (CH^3 \times S^1, H^3 \times S^1)$$

such that $g|\partial N$ is the identity.

PROOF. As in the proof of Proposition 3 of [8], we have the normal map and the surgery obstruction lies in $L_5(Z) = L_4(0) = Z$. But Siebenmann proved (e.g. [22, Proposition 5.2]) the existence of closed 5-manifold W with $\pi_1(W)$ isomorphic to Z such that, if we add W along S^1, then the surgery obstruction in $L_5(Z) = L_4(0)$ is changed by the element corresponding to the generator. Consequently by adding W, if necessary, we can always finish the surgery so that the normal map is a simple homotopy equivalence.

Further, for the global surgery we need the following which is a corollary of more general theorem of Cappell [21].

PROPOSITION 12.2. $L_n(Z * \cdots * Z) = L_n(0) + \sum L_{n-1}(0)$.

In the construction of $N_{(5)}$, as in § 7, we define an element

$$c'(\sigma\mu) \in [K_{\sigma\mu}, F/TOP] = \pi_4(F/TOP) = Z$$

to the obstruction to the topological trivialization of $Y_{\sigma\mu}$. We define $c'(\sigma)$ by

$$c'(\sigma) = \sum_{\mu>\sigma} [\mu, \sigma] c'(\sigma\mu) \in Z$$

and we have a chain $c' \in C_{n-5}(M; Z)$ by

$$c' = \sum_{\sigma} c'(\sigma) .$$

As before, it is a cycle.

LEMMA 12.1. $c'(\sigma\mu) = 2c(\sigma\mu)$.

PROOF. This follows from the fact that the inclusion homomorphism

$$\pi_4(F/PL) \rightarrow \pi_4(F/TOP)$$

is equal to the maltiplication by two

$$Z \xrightarrow{\times 2} Z .$$

Hence by the Corollary 11.4, we have

LEMMA 12.2. *The cycle c' is a boundary.*

Hence, in the construction of a topological $N_{(5)}$, there is no obstruction. The construction of higher dimensional handles for topological manifold is the same as previous.

If n is even, we can construct $(n-2)$-handle by the Kato-Matsumoto, Cappell-Shaneson Theorem. Since $L_n(Z^* \cdots {}^*Z) = L_n(0) + \sum L_{n-1}(0)$, we can always do the plumbing so that the surgery obstruction vanishes.

Suppose given a homology manifold M of dimension n. If $\partial M \neq \phi$, suppose that there exists a neighborhood U of ∂M such that U is a triangulation of topological manifold. We have the following theorems consequently.

THEOREM 3. *Suppose $n > 4$. If n is odd, suppose that an obstruction in*

$$L_n(\pi_1(M), \omega) ,$$

vanishes, then M is relatively simple homotopy equivalent to a topological manifold N with $\partial N = \partial M$.

THEOREM 4. *If $n > 4$, then M is always (π_1, H_*)-equivalent to a topological manifold N with $\partial N = \partial M$.*

The Theorem 4 says that a homological manifold is equivalent to a topological manifold in S-category of Spanier-Whitehead [17]. We may obtain some applications from the theorem.

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