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NOTE ON THE KRICKEBERG DECOMPOSITION

TAKESHI SEKIGUCHI

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The existence and the uniqueness of the Krickeberg decomposition for L^1-bounded martingales are wellknown. Namely, for each L^1-bounded right continuous martingale $(M_t)_{t\geqq 0}$ we have uniquely the following decomposition:

$$M_t = M_t^+ - M_t^- , \qquad t \geqq 0 ,$$

where $(M_t^+)_{t\geqq 0}$ and $(M_t^-)_{t\geqq 0}$ are positive right continuous martingales, and

$$\sup_{t\geqq 0} E[|M_t|] = E[M_0^+] + E[M_0^-] .$$

In particular, if $(M_t)_{t\geqq 0}$ is uniformly integrable, $M_\infty^+ = M_\infty \vee 0$ and $M_\infty^- = (-M_\infty) \vee 0$. We consider whether $(M_t^+)_{t\geqq 0}$ is continuous or not when $(M_t)_{t\geqq 0}$ is continuous. This problem was raised by N. Kazamaki. In this note we show that there exists an L^2-bounded continuous martingale $(M_t)_{t\geqq 0}$ such that $(M_t^+)_{t\geqq 0}$ is not continuous. That is the following theorem.

THEOREM. *There exists a complete probability space $(\Omega, \mathscr{F}, P)$ with a quasi-left continuous and right continuous increasing family $(\mathscr{F}_t)_{t\geqq 0}$ of sub-σ-fields such that, for some square integrable function f, the martingale $(E[f \mid \mathscr{F}_t])_{t\geqq 0}$ has a continuous modification, whereas $(E[f \vee 0 \mid \mathscr{F}_t])_{t\geqq 0}$ does not have any continuous modification.*

First of all we give the following lemmas.

LEMMA 1. *Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $\mathscr{H}$ a closed subspace of $L^2(\mathscr{F})$ containing all constant functions. If the σ-field $\sigma(\mathscr{H})$ generated by $\mathscr{H}$ is equal to $\mathscr{F}$ and if $g \vee 0 \in \mathscr{H}$ for each $g \in \mathscr{H}$ then $\mathscr{H} = L^2(\mathscr{F})$.*

PROOF. For $g \in \mathscr{H}$ define $g_n = [n(g \vee 0)] \wedge 1$. Then as we see easily, $g_n \in \mathscr{H}$ and $\lim g_n = 1_{[g>0]} \in \mathscr{H}$. Therefore we have $\mathscr{H} = L^2(\mathscr{F})$ by the assumption $\sigma(\mathscr{H}) = \mathscr{F}$.

LEMMA 2. *Let $(M_t)_{t>0}$ and $(N_t)_{t\geqq 0}$ be a Brownian motion with $M_0 = 0$ and a Poisson process with $N_0 = 0$ on a complete probability space $(\Omega, \mathscr{F}, P)$ respectively, and suppose that both processes are independent. Let $\mathscr{G}_t$ be the completion of $\sigma(M_s, N_s: s \leqq t)$ in $\mathscr{F}$ and S a stopping* *time relative to* $(\mathcal{G}_t)_{t\geqq 0}$. *Let* $\mathcal{M}^S$, $\mathcal{M}^S_c$ *and* $\mathcal{M}^S_d$ *be the families of all square integrable right continuous martingales, of all square integrable continuous martingales and of all square integrable purely discontinuous martingales relative to* $(\mathcal{G}_{S\wedge t})_{t\geqq 0}$ *with zero at time zero respectively. Then the following statements hold.*

1) $(\mathcal{G}_t)_{t\geqq 0}$ *is a quasi-left continuous, right continuous increasing family.*

2) $(M_t)_{t\geqq 0}$ *and* $(N_t - t)_{t\geqq 0}$ *are square integrable martingales relative to* $(\mathcal{G}_t)_{t\geqq 0}$.

3) $\mathcal{M}^S_c$ *and* $\mathcal{M}^S_d$ *are stable subspaces generated by* $(M_{S\wedge t})_{t\geqq 0}$ *and* $(N_{S\wedge t} - S\wedge t)_{t\geqq 0}$ *in* $\mathcal{M}^S$ *respectively.*

PROOF. Since $(M_t)_{t\geqq 0}$ and $(N_t)_{t\geqq 0}$ are independent, the statement 2) is clear. $(M_t, N_t)_{t\geqq 0}$ is a 2-dimensional stationary additive process and a Hunt process, so that we get 1) by the general theory of Markov processes (see P. A. Meyer [6] p. 116). 3) will be proved by the method employed in C. Dellacherie [4].

PROOF OF THEOREM. Let Ω be the family of all functions $\omega(t) = (\omega_1(t), \omega_2(t))$ on R_+ to R^2 such that $t \longrightarrow \omega_1(t)$ is a continuous function and $t \longrightarrow \omega_2(t)$ a purely discontinuous, increasing function with jumps 1. Put $M_t(\omega) = \omega_1(t)$ and $N_t(\omega) = \omega_2(t)$. Then we know that there exists a probability measure P on $(\Omega, \sigma(M_t, N_t: t \geqq 0))$, for which $(M_t)_{t\geqq 0}$ and $(N_t)_{t\geqq 0}$ are a Brownian motion with $M_0 = 0$ a.s. and a Poisson process with $N_0 = 0$ a.s. respectively, and both processes are independent. Let $\mathcal{F}$ be the completion of $\sigma(M_t, N_t: t \geqq 0)$. We use the same notation $\mathcal{G}_t$, $\mathcal{M}^S$, $\mathcal{M}^S_c$, $\mathcal{M}^S_d$ as in Lemma 2, where we put $S = \inf(t; N_t = 1)$. Since $(N_t)_{t\geqq 0}$ is a Poisson process, $P(S > t) = e^{-t}$, $t \geqq 0$, and $P(N_S \neq N_{S-}) = 1$, from which S is a totally inaccessible stopping time with respect to $(\mathcal{G}_t)_{t\geqq 0}$ and $(\mathcal{G}_{S\wedge t})_{t\geqq 0}$, according to Lemma 2, 1) and 2) (see C. Dellacherie [5] p. 112). Therefore $\mathcal{M}^S_d \neq 0$. Put $\mathcal{F}_t = \mathcal{G}_{S\wedge t}$. Let $\mathcal{H}$ be the family of all square integrable $\mathcal{F}_\infty$-measurable functions g of the following form: $g = \text{constant} + \int_0^\infty H_t dM_{S\wedge t}$, where $(H_t)_{t\geqq 0}$ are previsible processes with respect to $(\mathcal{F}_t)_{t\geqq 0}$. From P. Courrege and P. Priouret ([3] p. 254-255) $\mathcal{F}_t$ is the completion of $\sigma(M_{S\wedge s}, N_{S\wedge s}: s \leqq t)$ in $\mathcal{F}_\infty$, and from C. Doléans [2] $S \wedge t = \langle M^S, M^S\rangle_t$ is measurable with respect to the completion of $\sigma(M_{S\wedge s}: s \leqq t)$ in $\mathcal{F}_\infty$, where $\langle M^S, M^S\rangle$ is the previsible increasing process such that $(M^2_{S\wedge t} - \langle M^S, M^S\rangle_t)_{t\geqq 0}$ is a martingale, and $N_{S\wedge t} = 1_{[S\leqq t]}$ a.s., from which $\mathcal{F}_t$ is equal to the completion of $\sigma(M_{S\wedge s}: s \leqq t)$ in $\mathcal{F}_\infty$. Therefore $\sigma(\mathcal{H}) = \mathcal{F}_\infty$. Now we suppose that $g \vee 0 \in \mathcal{H}$ for each $g \in \mathcal{H}$. Then we have $\mathcal{H} = L^2(\mathcal{F}_\infty)$ by Lemma 1 and $\mathcal{M}^S = \mathcal{M}^S_c$.

This is contrary to $\mathscr{M}_d^s \neq 0$. Consequently, we can choose $f \in \mathscr{H}$ such that $f \vee 0 \notin \mathscr{H}$ and by Lemma 2.3) f satisfies the condition in Theorem.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN