Tôhoku Math. Journ.
29 (1977), 427-437.

ON COMMON BOUNDARY POINTS OF MORE THAN TWO COMPONENTS OF A FINITELY GENERATED KLEINIAN GROUP

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(Received March 19, 1976)

1. Introduction. Let G be a Kleinian group and denote by $\Omega(G)$ and $\Lambda(G)$ the region of discontinuity and the limit set of G, respectively. A component of $\Omega(G)$ will be called a component of G. The component subgroup G_Δ for a component Δ of G is the maximal subgroup of G which keeps Δ invariant. The quotient $\Delta/G_\Delta = S$ is a Riemann surface and the cannonical mapping $\Delta \mapsto S$ is holomorphic.

The modern theory of Kleinian groups was initiated by Ahlfors, who proved the finiteness of a finitely generated Kleinian group, known as the finiteness theorem. That is to say, if G is finitely generated, then there is a finite complete list $\{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ of non-conjugate components of G and $\Omega(G)/G$ is the disjoint union of finite Riemann surfaces $S_1 + S_2 + \cdots + S_n$, where $S_i = \Delta_i/G_{\Delta_i}$. As a corollary of this theorem, we can easily see that the component subgroup G_Δ for any component Δ of G is a finitely generated Kleinian group with the invariant component Δ and that the boundary of each component Δ of G is identical with the limit set of the component subgroup G_Δ.

Recently, in [3] Maskit found the remarkable facts about boundaries of components of a Kleinian group G and about elements of G which have their fixed points on the boundary of a component of G. For the frequent use of those in our later discussion, we shall restate them here.

THEOREM A. *Let G_{Δ_i} $(i = 1, 2)$ be the component subgroup of the component Δ_i of a Kleinian group G. Assume that Δ_i/G_{Δ_i} is a finite Riemann surface, $i = 1, 2$. Then $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) = \Lambda(G_{\Delta_1}) \cap \Lambda(G_{\Delta_2}) = \partial\Delta_1 \cap \partial\Delta_2$.*

THEOREM B. *Let G_Δ be the component subgroup of the component Δ of a Kleinian group G. Assume that Δ/G_Δ is a finite Riemann surface. Let g be a loxodromic element of G with at least one fixed point in $\partial\Delta$. Then $g^n \in G_\Delta$ for some positive integer n.*

THEOREM C. *Let G_Δ, Δ, G be as in Theorem* B. *Let g be a para-*

bolic element of G whose fixed point z lies on the boundary of Δ. Then there is a parabolic element $h \in G_\Delta$ which has z as the fixed point.

Giving two examples, he showed that n in Theorem B is not equal to 1 in general and that g in Theorem C is not an element of G_Δ in general. His examples also imply the existence of two kinds of Kleinian groups. The one is a finitely generated Kleinian group G_1 such that there are finite and more than two components of G_1 having at least two common boundary points. The other is a finitely generated Kleinian group G_2 for which there are an infinite number of components of G_2 having at least one common boundary point.

Those kinds of finitely generated Kleinian groups are ruled out from the space of the finitely generated function groups (see [4]). So, in this paper, we shall treat the intersection of boundaries of more than two components of a finitely generated Kleinian group being not necessarily a function group.

First we shall generalize Theorem A for arbitrarily many (possibly infinite) components of a finitely generated Kleinian group G and next we shall show that the intersection of the boundaries of more than two components of G consists of at most two points and that the common boundary points of infinitely many components of G consists of at most one point z. In the later case, as the Maskit's example is so, there is a parabolic element of G which has the point z as a fixed point and does not keep invariant any component of G. We also give some criteria for the number of common boundary points of components to be one or two.

2. Let Δ_i and Δ_j, $i \neq j$, be two disjoint components of a Kleinian group G. An auxiliary domain D_{ij} of Δ_i relative to Δ_j is defined as follows: Let Δ_{ij}^* be a component of the complement of $\bar{\Delta}_i$ such that $\Delta_{ij}^* \supset \Delta_j$. Then D_{ij} is the component of the complement of $\overline{\Delta_{ij}^*}$ such that $D_{ij} \supset \Delta_i$. It was shown in [4] that $D_{ij} \cap D_{ji} = \phi$ and $\partial D_{ij} \cap \partial D_{ji} = \partial \Delta_i \cap \partial \Delta_j$.

LEMMA 1. $D_{ij} \subset \Delta_{ji}^*$.

PROOF. Since $\Delta_j \subset \Delta_{ij}^*$, for each component D of the complement of $\overline{\Delta_{ij}^*}$ there is a component Δ^* of the complement of $\bar{\Delta}_j$ such that $D \subset \Delta^*$. If D is the component containing Δ_i, then $D = D_{ij}$ and $\Delta^* = \Delta_{ji}^*$. Thus we have $D_{ij} \subset \Delta_{ji}^*$.

Now, let G be (non-elementary and) finitely generated. Then, as mentioned in introduction, the component subgroup G_Δ for any component Δ of G is a finitely generated Kleinian group with an invariant component Δ and we can see from Maskit's result [2] that, for each component Δ^* $(\neq \Delta)$ of G_Δ, the component subgroup G_{Δ^*} for Δ^* of G_Δ is a finitely generated quasi-Fuchsian group of the first kind with the fixed closed Jordan curve $\partial\Delta^*$. Hence we have the following.

LEMMA 2. *If G is finitely generated, then $D_{ij} = \overline{\Delta_{ij}^{*}}^{c}$ and each ∂D_{ij} is a closed Jordan curve.*

The next lemma is basic in our later discussion.

LEMMA 3. *Let $\Delta_1, \Delta_2, \Delta_3$ be three distinct components of a finitely generated Kleinian group G. Then $D_{ij} \neq D_{ik}$ holds for at most one triple (i, j, k), $i, j, k = 1, 2, 3$. Moreover, $D_{ij} \neq D_{ik}$ if and only if $\Delta_{ij}^* \neq \Delta_{ik}^*$.*

PROOF. By Lemma 2, D_{ij} is the complement of $\overline{\Delta_{ij}^*}$. Hence the second statement of our lemma follows. We assume $D_{12} \neq D_{13}$. Since Δ_{12}^* and Δ_{13}^* are components of the complement of $\overline{\Delta}_1$, we have $\Delta_{12}^* \cap \Delta_{13}^* = \phi$ by our assumption. Since $\Delta_2 \subset \Delta_{12}^*$ and $\Delta_3 \subset \overline{\Delta_{12}^{*}}^{c}$, we see that Δ_{23}^* contains the complement of $\overline{\Delta_{12}^*}$ which is D_{12}. Hence $\Delta_{23}^* \supset \Delta_1$. Thus $\Delta_{23}^* = \Delta_{21}^*$ and $D_{23} = D_{21}$. In the same way we have $D_{32} = D_{31}$. Thus the lemma is proved.

We shall write $D_{ij} = D_i$ if $D_{ij} = D_{ik}$. Now we can prove the following.

PROPOSITION. *Let $\Delta_1, \Delta_2, \Delta_3$ be three distinct components of a finitely generated Kleinian group G. Then $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$ consists of at most two points. Moreover, if $D_{ij}=D_i$ for any i, then $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3 = \partial D_1 \cap \partial D_2 \cap \partial D_3$. Otherwise, there is a triple (i, j, k) such that $D_{ij} \neq D_{ik}$ and $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3 = \partial D_j \cap \partial D_k$. In the later case $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$ consists of at most one point.*

PROOF. First note that each ∂D_{ij} is a closed Jordan curve.

The case where $D_{ij} = D_i$ for any i. Since $D_{ij} \cap D_{ji} = \phi$, we see that D_1, D_2 and D_3 are mutually disjoint. Since $\partial\Delta_1 \cap \partial\Delta_2 = \partial D_{12} \cap \partial D_{21}$ and $\partial\Delta_2 \cap \partial\Delta_3 = \partial D_{23} \cap \partial D_{32}$, we also see that $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3 = \partial D_1 \cap \partial D_2 \cap \partial D_3$. We shall show that this set consists of at most two points.

Assume that there are three points z_1, z_2, z_3 in $\partial D_1 \cap \partial D_2 \cap \partial D_3$. Join z_1 and z_2 by Jordan arcs C_{12} in D_1 and C_{12}' in D_2, respectively. Then C_{12}, C_{12}', z_1 and z_2 make a closed Jordan curve K_{12} lying in $D_1 \cup D_2 \cup \{z_1, z_2\}$. Let I_{12} be a component of the complement of K_{12} containing z_3. In the same manner, we can drow a closed Jordan curve K_{13} (or K_{23}) lying in $D_1 \cup D_2 \cup \{z_1, z_3\}$ (or $D_1 \cup D_2 \cup \{z_2, z_3\}$) and passing through z_1, z_3 (or z_2, z_3).

Let I_{13} (or I_{23}) be a component of the complement of K_{13} (or K_{23}) containing z_2 (or z_1). Since z_i $(i = 1, 2, 3)$ is a boundary point of D_3, $D_3 \subset I_{12} \cap I_{13} \cap I_{23}$. On the other hand $I_{12} \cap I_{13} \cap I_{23} \subset D_1 \cup D_2$. Hence $D_3 \cap (D_1 \cup D_2) \neq \phi$. This contradicts the fact that D_1, D_2, D_3 are mutually disjoint. Hence $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$ consists of at most two points.

The case where there is a triple (i, j, k) such that $D_{ij} \neq D_{ik}$. We may assume $i = 1$, $j = 2$ and $k = 3$. By Lemma 3, $D_{21} = D_{23} = D_2$ and $D_{31} = D_{32} = D_3$. Hence $D_2 \cap D_3 = \phi$. If $\partial\Delta_2 \cap \partial\Delta_3$ $(=\partial D_2 \cap \partial D_3)$ contains two points, then there is a closed Jordan curve K passing through these two points such that $K \subset D_2 \cup D_3 \cup \Lambda(G)$. Since $\Delta_{12}^* \cap \Delta_{13}^* = \phi$ by Lemma 3 and since $D_2 \subset \Delta_{12}^*$, $D_3 \subset \Delta_{13}^*$ by Lemma 1, both the interior and the exterior of K contain points of $\partial\Delta_{12}^* \subset \partial\Delta_1$ and hence also contain points of Δ_1. This contradicts connectedness of Δ_1. Hence $\partial\Delta_2 \cap \partial\Delta_3$ consists of at most one point. Therefore, $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$ $(\subset \partial\Delta_2 \cap \partial\Delta_3)$ consists of at most one point.

Next we show that $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3 = \partial D_2 \cap \partial D_3$. As was just stated above, it holds that $D_2 \subset \Delta_{12}^*$, $D_3 \subset \Delta_{13}^*$ and $\Delta_{12}^* \cap \Delta_{13}^* = \phi$. Hence, if $\partial D_2 \cap \partial D_3 \neq \phi$, then $\partial D_2 \cap \partial D_3$ contains a point of $\partial\Delta_{12}^* \subset \partial\Delta_1$. Since $\partial D_2 \cap \partial D_3$ consists of at most one point, $\partial D_2 \cap \partial D_3 \subset \partial\Delta_1$. Combining this with the equality $\partial\Delta_2 \cap \partial\Delta_3 = \partial D_2 \cap \partial D_3$, we have the inclusion relation $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3 = \partial\Delta_1 \cap (\partial D_2 \cap \partial D_3) = \partial D_2 \cap \partial D_3$. Thus we have shown $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3 = \partial D_2 \cap \partial D_3$ and completed the proof of our proposition.

For common subgroups we have the following.

THEOREM 1. *Let G be a finitely generated Kleinian group and let $\{\Delta_i\}$ be any collection of more than two components of G. Then $\bigcap G_{\Delta_i}$ is an elementary group, where the intersection is taken over all elements of $\{\Delta_i\}$.*

PROOF. Since $\Lambda(G_{\Delta_i}) = \partial\Delta_i$, we have $\Lambda(\bigcap G_{\Delta_i}) \subset \bigcap \partial\Delta_i$. By the above Proposition, the limit set of $\bigcap G_{\Delta_i}$ consists of at most two points. From this, the theorem is immediately obtained.

We shall see later that if $D_{ij} = D_i$ $(i = 1, 2, 3)$ and if $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3 \neq \phi$, then $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$ consists of exactly two points.

3. Ahlfors' finiteness theorem and Theorem A imply the fact that if Δ_1 and Δ_2 are components of a finitely generated Kleinian group G, then $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) = \partial\Delta_1 \cap \partial\Delta_2$. We can extend this as follows.

THEOREM 2. *Let G be a finitely generated Kleinian group and let $\{\Delta_i\}$ be any collection of the components of G. Then $\Lambda(\bigcap G_{\Delta_i}) = \bigcap \partial\Delta_i$, where the intersections in both sides are taken over all elements of $\{\Delta_i\}$.*

PROOF. From the fact stated in the beginning of this section, it suffices to prove Theorem 2 for any collection $\{\varDelta_i\}$ consisting of more than two components. The inclusion relation $\Lambda(\bigcap G_{\varDelta_i}) \subset \bigcap \partial\varDelta_i$ was already proved in the proof of Theorem 1. To prove the opposite inclusion relation we note that $\bigcap \partial\varDelta_i$ consists of at most two points and may suppose that $\bigcap \partial\varDelta_i$ is not empty. We divide the proof into three cases corresponding to the number of elements of $\{\varDelta_i\}$.

The case I where $\{\varDelta_i\} = \{\varDelta_1, \varDelta_2, \varDelta_3\}$. First we assume that $D_{ij} = D_i$ $(i = 1, 2, 3)$ and that $\partial\varDelta_1 \cap \partial\varDelta_2 \cap \partial\varDelta_3$ consists of two points z_1, z_2. If either $G_{\varDelta_1} \cap G_{\varDelta_2}$ or $G_{\varDelta_1} \cap G_{\varDelta_3}$, say $G_{\varDelta_1} \cap G_{\varDelta_2}$, is an elementary group, then, by Theorem A, $G_{\varDelta_1} \cap G_{\varDelta_2}$ contains a loxodromic element g of G with z_1 and z_2 as the fixed points. By Theorem B, there is an integer n such that $g^n \in G_{\varDelta_3}$. Then g^n is an element of $G_{\varDelta_1} \cap G_{\varDelta_2} \cap G_{\varDelta_3}$ and has the fixed points z_1, z_2. This is the required. If both $G_{\varDelta_1} \cap G_{\varDelta_2}$ and $G_{\varDelta_1} \cap G_{\varDelta_3}$ are non-elementary, then, since D_1, D_2, D_3 are mutually disjoint and each of their boundaries is a closed Jordan curve, D_3 lies in a component of $(\overline{D_1 \cup D_2})^c$ which is bounded by two Jordan subarcs C_1 of ∂D_1 and C_2 of ∂D_2 with the same end points z_1, z_2. We show that there is a loxodromic element $g \in G_{\varDelta_1} \cap G_{\varDelta_2}$ with both endpoints of C_1 as the fixed points. Let G_{D_i} be the maximal subgroup of $G_{\varDelta_i}$ which keeps D_i invariant, $i = 1, 2$. Then it is shown in [4] that G_{D_i} is a quasi-Fuchsian group of the first kind and $\Lambda(G_{D_1} \cap G_{D_2}) = \partial D_1 \cap \partial D_2$. We can obtain the required g in $G_{D_1} \cap G_{D_2}$ as follows. If the quasi-Fuchsian group $G_{D_1} \cap G_{D_2}$ is of the first kind with two invariant curves ∂D_1 and ∂D_2, then $\Lambda(G_{D_1} \cap G_{D_2}) = \partial D_1 = \partial D_2$ and $\overline{D_1 \cup D_2} = C \cup \{\infty\}$ and $D_3 = \phi$, which is absured. Hence $G_{D_1} \cap G_{D_2}$ must be of the second kind. Let w be a conformal mapping of the upper half plane onto D_1 with $w([0, 1]) = C_1$ and let Γ be a Fuchsian model of $G_{D_1} \cap G_{D_2}$ such that $G_{D_1} \cap G_{D_2} = w\Gamma w^{-1}$. Since D_3 lies in a component bounded by C_1 and C_2 and since $\partial D_1 \cap \partial D_2 = \Lambda(G_{D_1} \cap G_{D_2})$, any point of C_1 except for its both end points lies in $\Omega(G_{D_1} \cap G_{D_2})$. Hence we see that the open interval $(0, 1)$ on the real axis lies in $\Omega(\Gamma)$. On the other hand, since both end points of C_1 lie in $\Lambda(G_{D_1} \cap G_{D_2})$, both end points of $(0, 1)$ lie in $\Lambda(\Gamma)$. By a well known fact for a finitely generated Fuchsian group of the second kind, there is a hyperbolic element γ of Γ with the fixed points 0, 1. Let $g = w\gamma w^{-1}$. Then g is a desired loxodromic element of $G_{D_1} \cap G_{D_2} \subset G_{\varDelta_1} \cap G_{\varDelta_2}$. By the same reasoning as before, we see that $\Lambda(G_{\varDelta_1} \cap G_{\varDelta_2} \cap G_{\varDelta_3}) \supset \partial\varDelta_1 \cap \partial\varDelta_2 \cap \partial\varDelta_3$.

Next we shall show that the case, where $D_{ij} = D_i$ $(i = 1, 2, 3)$ and $\partial\varDelta_1 \cap \partial\varDelta_2 \cap \partial\varDelta_3$ consists of one point z_0, does not occur. If $G_{\varDelta_1} \cap G_{\varDelta_2}$ is an elementary group, then it contains a loxodromic or a parabolic element g of $G_{\Delta_1} \cap G_{\Delta_2}$ with z_0 as a fixed point. If g is loxodromic, then, by Theorem B, there is an integer n such that $g^n \in G_{\Delta_3}$. Since $g^n \in G_{\Delta_1} \cap G_{\Delta_2} \cap G_{\Delta_3}$ and $\Lambda(G_{\Delta_1} \cap G_{\Delta_2} \cap G_{\Delta_3}) \subset \partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$, another fixed point of g must lie on $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$. This contradicts our assumption. Hence g is parabolic. By Theorem C, there is a parabolic element $g' \in G_{\Delta_3}$ with the fixed point z_0. Let G_{D_i} $(i = 1, 2, 3)$ be as before. Since G_{D_i} is identical with the component subgroup $G_{\Delta^*_{ij}}$ for a component Δ^*_{ij} of G_{Δ_i} and there is a parabolic element of G_{Δ_i} with z_0 as the fixed point, there is a parabolic element $g_i \in G_{D_i}$ with z_0 as the fixed point by Theorem C, $i = 1, 2, 3$. Since G_{D_i} is a quasi-Fuchsian group of the first kind, z_0 corresponds to a puncture of the Riemann surface D_i/G_{D_i}. Hence there is an open disc in D_i whose boundary passes through z_0. This means that there are three open discs which are mutually disjoint and tangent each other at z_0. This is impossible. Therefore $G_{\Delta_1} \cap G_{\Delta_2}$ is not elementary. Thus as was already shown, there is a loxodromic element $g \in G_{\Delta_1} \cap G_{\Delta_2}$ with z_0 as one fixed point. In the same way as before, we arrive at the same contradiction that $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$ consists of two points. Hence, the case, where $D_{ij} = D_i$ $(i = 1, 2, 3)$ and $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$ consists of one point z_0, does not occur.

Next we assume that there is a triple (i, j, k) such that $D_{ij} \neq D_{ik}$. We may assume $D_{12} \neq D_{13}$. By Proposition, $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$ consists of at most one point and is identical with $\partial D_2 \cap \partial D_3 = \partial\Delta_2 \cap \partial\Delta_3$. If $z_0 = \partial\Delta_2 \cap \partial\Delta_3$, then, by Theorem A, we have $z_0 = \Lambda(G_{\Delta_2} \cap G_{\Delta_3})$. Hence there is a parabolic element $g \in G_{\Delta_2} \cap G_{\Delta_3}$ with z_0 as the fixed point. By Theorem C, there is a parabolic element $g' \in G_{\Delta_1}$ with z_0 as the fixed point. If g and g' do not belong to the same cyclic subgroup of G, then an invariant curve in Δ_2 under g intersects an invariant curve in Δ_1 under g'. This contradicts the fact that Δ_1 and Δ_2 are the distinct components. Hence g and g' belong to the same cyclic subgroup of G and there are two integers m, n such that $g^m = (g')^n \in G_{\Delta_1} \cap G_{\Delta_2} \cap G_{\Delta_3}$. Thus g^m is a parabolic element of $G_{\Delta_1} \cap G_{\Delta_2} \cap G_{\Delta_3}$ with z_0 as the fixed point and we have the proof of theorem in the case I.

The case II where $\{\Delta_i\} = \{\Delta_1, \Delta_2, \cdots, \Delta_p\}$, $p > 3$. Let z_1 and z_2 $(\neq z_1)$ be points of $\bigcap \partial\Delta_i$. Then for any three components of $\{\Delta_i\}$, say Δ_1, Δ_2, Δ_3, $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3 = \{z_1, z_2\}$. By the result in the case I, $\Lambda(G_{\Delta_1} \cap G_{\Delta_2} \cap G_{\Delta_3}) = \{z_1, z_2\}$. Hence there is a loxodromic element $g \in G_{\Delta_1} \cap G_{\Delta_2} \cap G_{\Delta_3}$ with z_1, z_2 as the fixed points. By Theorem B, for each Δ_i there are an integer n_i and a loxodromic element $g_i \in G_{\Delta_i}$ such that $g_i = g^{n_i}$. Let n_0 be a common multiple of $n_4, n_5, \cdots, n_p$. Then g^{n_0} is a loxodromic element of $\bigcap G_{\Delta_i}$ with z_1, z_2 as the fixed points. Hence $\Lambda(\bigcap G_{\Delta_i}) \supset \bigcap \partial\Delta_i$.

Next assume that $\bigcap \partial\Delta_i$ consists of only one point z_0. In the same way as just stated above, we see that there is a parabolic element $g \in G_{\Delta_1} \cap G_{\Delta_2} \cap G_{\Delta_3}$ with z_0 as the fixed point. By Theorem C, for each Δ_i, $i > 3$, there is a parabolic element $g_i \in G_{\Delta_i}$ with z_0 as the fixed point. By the same reasoning as in the last step of the case I, we see that each g_i is an element of a cyclic subgroup of G containing g so that there are two integers m_i, n_i such that $g^{m_i} = g_i^{n_i}$. Let m_0 be a common multiple of $m_4, m_5, \cdots, m_p$. Then g^{m_0} is a parabolic element of $\bigcap G_{\Delta_i}$ with z_0 as the fixed point. Hence we have the required.

The case III where $\{\Delta_i\}$ consists of infinite elements. The proof of this case is somewhat long, so it will be given in a sequence of lemmas.

LEMMA 4. *If $\bigcap \partial\Delta_i$ is not empty, then it consists of one point.*

PROOF. Assume that $\bigcap \partial\Delta_i$ consists of two points z_1 and z_2. By Proposition, for each triple $(\Delta_i, \Delta_j, \Delta_k)$ of $\{\Delta_i\}$, $D_{ij} = D_i$. Hence we can use the notation D_i instead of D_{ij}. Note that $D_i \cap D_j = \phi$ for each $i, j\ (\neq i)$. Conjugating G by a linear transformation, we may assume $z_1 = 0$ and $z_2 = \infty$. Since each G_{D_i} is a finitely generated quasi-Fuchsian group of the first kind with a quasi-circle ∂D_i as the fixed curve and since ∂D_i passes through ∞, there is a positive number C_i depending only on G_{D_i} such that $|\zeta_i - \zeta_i'| \geqq C_i|\zeta_i|$ for any two points ζ_i, ζ_i' on ∂D_i separated by 0 and ∞ (see [1]). Since there are only a finite number of non-conjugate components of G, there are also only a finite number of non-conjugate D_i so that there are only a finite number of distinct C_i's. Let C be the maximum of $\{C_i\}$. Then it holds that $|\zeta_i - \zeta_i'| \geqq C|\zeta_i|$ for each i and for any two points ζ_i, ζ_i' on ∂D_i separated by 0 and ∞. Choose ζ_i and ζ_i' on ∂D_i such that $|\zeta_i| = |\zeta_i'| = 1$ and such that the open arc on the unit circle bounded by ζ_i and ζ_i' lies in D_i. Then $|\zeta_i - \zeta_i'| \geqq C$ for each i. Therefore, there can be only finitely many distinct D_i and hence only finitely many Δ_i. Thus we have our lemma.

LEMMA 5. *Assume that $\bigcap \partial\Delta_i$ consists of one point z_0. Let Δ_i, Δ_j and Δ_k be any three distinct components of $\{\Delta_i\}$. Then $\partial\Delta_i \cap \partial\Delta_j \cap \partial\Delta_k$ consists of the point z_0.*

PROOF. Assume that $\partial\Delta_i \cap \partial\Delta_j \cap \partial\Delta_k$ contains another point $z_1 \neq z_0$. From a result in the case I, $\Lambda(G_{\Delta_i} \cap G_{\Delta_j} \cap G_{\Delta_k}) = \{z_0, z_1\}$. Hence there is a loxodromic element $g \in G_{\Delta_i} \cap G_{\Delta_j} \cap G_{\Delta_k}$ with z_0, z_1 as the fixed points. By Theorem B, for each Δ_i there is an integer n_i such that $g^{n_i} \in G_{\Delta_i}$. Hence $z_1 \in \partial\Delta_i$ for every i. This implies $z_1 \in \bigcap \partial\Delta_i$, a contradiction. Hence we have our lemma.

LEMMA 6. *If* $\bigcap \partial\Delta_i$ *consists of one point* z_0, *then each* G_{Δ_i} *contains a parabolic element* g_i *with* z_0 *as the fixed point.*

PROOF. By Lemma 5 and by a result in the case I, $\Lambda(G_{\Delta_i} \cap G_{\Delta_j} \cap G_{\Delta_k}) = z_0$ for any three distinct components Δ_i, Δ_j, Δ_k. Hence there is a parabolic element $g_i \in G_{\Delta_i} \cap G_{\Delta_j} \cap G_{\Delta_k}$ with z_0 as the fixed point, which is clearly an element of G_{Δ_i}.

Let $E = \{\Delta_1, \cdots, \Delta_n\}$ be a complete list of non-conjugate components of $\{\Delta_i\}$ in G and let E_i be the conjugacy class of $\Delta_i \in E$ in $\{\Delta_i\}$. Then for each $\Delta_j \in E_i$ there is an element $h_{ji} \in G$ such that $h_{ji}(\Delta_j) = \Delta_i$. We can prove the following.

LEMMA 7. *If* $\bigcap \partial\Delta_i$ *consists of one point* z_0, *then the point* $h_{ji}(z_0)$ *corresponds to a puncture of* $\Omega(G_{\Delta_i})/G_{\Delta_i}$.

PROOF. Obviously it suffices to show that z_0 corresponds to a puncture of $\Omega(G_{\Delta_j})/G_{\Delta_j}$. Let Δ_k $(\neq \Delta_j)$ be a component of $\{\Delta_i\}$ and let Δ_{jk}^* be the component of G_{Δ_j} which includes Δ_k. Then by Lemma 1, $D_{kj} \subset \Delta_{jk}^*$. On the other hand, $D_{jk} \cap D_{kj} = \phi$ and $D_{jk} \cap \Delta_{jk}^* = \phi$. Hence, if $\bigcap \partial\Delta_i$ consists of one point z_0, then $z_0 \in \partial\Delta_j \cap \partial\Delta_k = \partial D_{jk} \cap \partial D_{kj}$, so we have $z_0 \in \partial\Delta_{jk}^*$. By Lemma 6, there is a parabolic element of G_{Δ_j} with z_0 as the fixed point. By Theorem C, there is a parabolic element of $G_{\Delta_{jk}^*}$ with z_0 as the fixed point, where $G_{\Delta_{jk}^*}$ is the component subgroup for Δ_{jk}^* of G_{Δ_j}. Since $G_{\Delta_{jk}^*}$ is a quasi-Fuchsian group, z_0 corresponds to a puncture of $\Delta_{jk}^*/G_{\Delta_{jk}^*}$. Since $\Delta_{jk}^*/G_{\Delta_{jk}^*}$ is a component of $\Omega(G_{\Delta_j})/G_{\Delta_j}$, z_0 corresponds to a puncture of $\Omega(G_{\Delta_j})/G_{\Delta_j}$. Thus Lemma 7 is proved.

Now we shall define an equivalence relation between components in E_i as follows: Let Δ_j and Δ_j' be in E_i and let h_{ji} and h_{ji}' be elements of G such that $h_{ji}(\Delta_j) = \Delta_i$ and $h_{ji}'(\Delta_j') = \Delta_i$, respectively. Then we say that Δ_j and Δ_j' are equivalent if $h_{ji}(z_0)$ and $h_{ji}'(z_0)$ correspond to the same puncture of $\Omega(G_{\Delta_i})/G_{\Delta_i}$. This equivalence relation is independent of choice of h_{ji} and h_{ji}'. Denote by $F_i = \{\Delta_{i_1}, \cdots, \Delta_{i_j}\}$ a complete list of non-equivalent components of E_i. Then $\{h_{i_1 i}(z_0), \cdots, h_{i_j i}(z_0)\}$ corresponds to a subset of the (non-conjugate) punctures of $\Omega(G_{\Delta_i})/G_{\Delta_i}$, where $h_{i_l i}(\Delta_{i_l}) = \Delta_i$, $1 \leqq l \leqq j$. Let F be a set of all components of G belonging to F_i for some i $(1 \leqq i \leqq n)$.

LEMMA 8. *Each component of* $\{\Delta_i\}$ *is equivarent to a component of* G *in* F *by an element of* G *with* z_0 *as a fixed point.*

PROOF. Let Δ be a component of $\{\Delta_i\}$ and let $h(\Delta) = \Delta_i \in E$ for some $h \in G$. Clearly $\Delta \in E_i$. By Lemma 7, $h(z_0)$ corresponds to a puncture of

$\Omega(G_{\varDelta_i})/G_{\varDelta_i}$ which corresponds to one of $h_{i_1 i}(z_0), \cdots, h_{i_j i}(z_0)$, say $h_{i_l i}(z_0)$, by an element $g \in G_{\varDelta_i}$. Set $h^* = h_{i_l i}^{-1} g h$. Then $\varDelta$ is equivalent to $\varDelta_{i_l}$ by $h^* \in G$ with $h^*(z_0) = z_0$. Thus Lemma 8 is proved.

LEMMA 9. *There is a parabolic element* $g^* \in \bigcap_{\varDelta \in F} G_{\varDelta}$ *satisfying* $g^*(z_0) = z_0$.

PROOF. Lemma 4 and Lemma 6 imply that for each $\varDelta$ of F there is a parabolic element $g_{\varDelta}$ of $G_{\varDelta}$ with z_0 as the fixed point. By the same reasoning used already in the last step of the case I, we see that $\{g_{\varDelta}\}_{\varDelta \in F}$ are in the same cyclic subgroup G_0 of G. Since F is a finite set of components of G, there is a parabolic element $g^* \in G_0$ which is denoted by $g_{\varDelta}^{k(\varDelta)}$ for some integer $k(\varDelta)$. This element g^* is a desired one.

LEMMA 10. *Let* g^* *be in Lemma* 9. *Then* $g^* \in G_{\varDelta_k}$ *for each component* $\varDelta_k$ *in* $\{\varDelta_i\}$.

PROOF. By Lemma 8, $\varDelta_k$ is equivalent to some $\varDelta \in F$ by an $h \in G$ with $h(z_0) = z_0$. We may assume $\varDelta_k \neq \varDelta$. Then $g = h^{-1} g^* h$ is a parabolic element of $G_{\varDelta_k}$ with $g(z_0) = z_0$. Since g^* is a parabolic element of G with z_0 as the fixed point, h is not loxodromic, for, otherwise G is not Kleinian. If h is parabolic, then it is easy to see $g = g^*$. Next consider the case where h is elliptic. By a suitable conjugation, we may suppose $g^*(z) = z + 1$ and $h(z) = e^{2\pi i/n} z$. Then $g(z) = z + e^{-2\pi i/n}$. If $n \neq 2$, then an invariant curve in $\varDelta$ under g^* intersects an invariant curve in $\varDelta_k$ under g. This contradicts $\varDelta_k \neq \varDelta$. Hence $n = 2$ and $g = (g^*)^{-1}$. In both cases, $g^* \in G_{\varDelta_k}$. Thus Lemma 10 is proved.

Now we can prove the inclusion relation $\Lambda(\bigcap G_{\varDelta_i}) \supset \bigcap \partial \varDelta_i$ in the case III. Namely, by Lemma 10, we see $g^* \in \bigcap G_{\varDelta_i}$ and $z_0 \in \Lambda(\bigcap G_{\varDelta_i})$, which shows $\Lambda(\bigcap G_{\varDelta_i}) \supset \bigcap \partial \varDelta_i$. Thus we have completed the proof of Theorem 2.

4. In the case where $\{\varDelta_i\}$ consists of an infinite number of components, we can also show the following.

THEOREM 3. *Let* G *be a finitely generated Kleinian group and let* $\{\varDelta_i\}$ *be an infinite collection of the components of* G. *If* $\bigcap_{i=1}^{\infty} \partial \varDelta_i \neq \phi$, *then* $\bigcap_{i=1}^{\infty} \partial \varDelta_i$ *consists of one point* z_0. *Moreover, there is a parabolic element* h *of* G *with* z_0 *as the fixed point such that* h *does not keep invariant any component of* G.

PROOF. The first assertion was shown in Lemma 4. In order to show the second assertion, we continue the discussion in the case III of the proof of Theorem 2 under the notation used there.

Since $\{\Delta_i\}$ and F are an infinite set and a finite set, respectively, there is a component $\Delta \in F$ whose equivalence class consists of an infinite number of components $\{\Delta_{i_j}\}$ in $\{\Delta_i\}$. By Lemma 8, for each $\Delta_{i_j} \in \{\Delta_{i_j}\}$ there is an element $h^*_{i_j} \in G$ such that $h^*_{i_j}(\Delta_{i_j}) = \Delta$ and $h^*_{i_j}(z_0) = z_0$. As is seen from the proof of Lemma 10, the set $\{h^*_{i_j}\}$ of those $h^*_{i_j}$ consists of parabolic elements and elliptic elements of order 2. Lemma 9 and Lemma 10 imply the existence of a parabolic element $g^* \in \bigcap_{\Delta \in F} G_\Delta$ such that $g^*(z_0) = z_0$ and such that $g^* \in \bigcap G_{\Delta_i}$. We may assume that $z_0 = \infty$ and that $g^*: z \mapsto z + 1$. First we shall show that G contains a parabolic element h of the form $h: z \mapsto z + a$ with $\operatorname{Im} a \neq 0$.

If $\{h^*_{i_j}\}$ contains an infinite number of the elliptic elements, then each elliptic element $h^*_{i_j}$ in $\{h^*_{i_j}\}$ has the form $h^*_{i_j}: z \mapsto -z + a_{i_j}$. We assert that $\{\operatorname{Im} a_{i_j}\}$ are not all the same. Assume that each a_{i_j} has the same imaginary part. Since for each integer m, we have

$$(g^*)^m h^*_{i_j}(g^*)^{-m}(\Delta_{i_j}) = \Delta \text{ and } (g^*)^m h^*_{i_j}(g^*)^{-m}(\infty) = \infty ,$$

we may assume that $0 \leqq \operatorname{Re} a_{i_j} < 2$. Then the infinite set $\{h^*_{i_j}\}$ has a convergent subsequence of distinct elements, which contradicts that G is Kleinian. Hence we have the assertion that $\{\operatorname{Im} a_{i_j}\}$ are not all the same. Thus there are two elliptic element $h^*_{i_j}: z \mapsto -z + a_{i_j}$ and $h^*_{i_{j'}}: z \mapsto - z + a_{i_{j'}}$, where $\operatorname{Im} a_{i_j} \neq \operatorname{Im} a_{i_{j'}}$. Set $h = h^*_{i_j} h^*_{i_{j'}}: z \mapsto z + a_{i_j} - a_{i_{j'}}$. This is a desired parabolic element of G.

If $\{h^*_{i_j}\}$ contains an infinite number of the parabolic elements, then each parabolic element $h^*_{i_j}$ in $\{h^*_{i_j}\}$ has the form $z \mapsto z + b_{i_j}$. We assert that there is a b_{i_j} with $\operatorname{Im} b_{i_j} \neq 0$. Assume that $\operatorname{Im} b_{i_j} = 0$ for all b_{i_j}. Since $g^* \in G_\Delta$, we see $(g^*)^m h^*_{i_j}(\Delta_{i_j}) = \Delta$ and $(g^*)^m h^*_{i_j}(\infty) = \infty$ for any integer m. Hence we may assume that $0 \leqq \operatorname{Re} b_{i_j} < 1$. In the same manner as above, we arrive at the contradiction that G is not Kleinian. Thus our assertion follows. Hence there is an $h^*_{i_j}$ with $\operatorname{Im} b_{i_j} \neq 0$ and we take this $h^*_{i_j}$ as h.

In both cases we can show that h does not keep any component of G invariant. Assume that there is a component Δ^* of G such that $h(\Delta^*) = \Delta^*$. Choose a component Δ_i in $\{\Delta_i\}$ which is different from Δ^*. Then an invariant curve in Δ_i under g^* intersects an invariant curve in Δ^* under h, which is impossible. Thus the second assertion follows and Theorem 3 is proved.

5. Finally, we shall give a criterion for the intersection of boundaries of the components of G to be one point or two points.

THEOREM 4. *Let $\{\Delta_i\}$ be a collection of more than two components of a finitely generated Kleinian group G and let the intersection of* *their boundaries be not empty. Then the intersection of their boundaries consists of one (or two) point if and only if there is a triple* $(\Delta_i, \Delta_j, \Delta_k)$ *of the components of* $\{\Delta_i\}$ *such that* $D_{ij} \neq D_{ik}$ *(or* $D_{ij} = D_{ik}, D_{jk} = D_{ji}$ *and* $D_{ki} = D_{kj}$*).*

PROOF. Assume that $\bigcap \partial\Delta_i$ consists of one point z_0, where the intersection is taken over all elements of $\{\Delta_i\}$. Then by Theorem 2, $\Lambda(\bigcap G_{\Delta_i}) = z_0$ Hence there is a parabolic element of G with z_0 as the fixed point. Therefore for any triple $(\Delta_i, \Delta_j, \Delta_k)$ it holds that $\partial\Delta_i \cap \partial\Delta_j \cap \partial\Delta_k = z_0$. For, if $\partial\Delta_i \cap \partial\Delta_j \cap \partial\Delta_k$ contains another point z_1, then, by Theorem 2, $\Lambda(G_{\Delta_i} \cap G_{\Delta_j} \cap G_{\Delta_k}) = \{z_0, z_1\}$ and hence there is a loxodromic element of G with z_0, z_1 as the fixed points, which contradicts the assumption $\bigcap \partial\Delta_i = \{z_0\}$. From the case I of the proof of Theorem 2, we see easily that there is a triple $(\Delta_i, \Delta_j, \Delta_k)$ such that $D_{ij} \neq D_{ik}$.

Assume that there is a triple $(\Delta_i, \Delta_j, \Delta_k)$ such that $D_{ij} \neq D_{ik}$. Then, by Proposition, $\partial\Delta_i \cap \partial\Delta_j \cap \partial\Delta_k$ consists of one point. Hence $\bigcap \partial\Delta_i$ consists of one point.

Assume that $\bigcap \partial\Delta_i$ consists of two points. If there is a triple $(\Delta_i, \Delta_j, \Delta_k)$ such that $D_{ij} \neq D_{ik}$, then, from Proposition, $\bigcap \partial\Delta_i$ consists of one point, which contradicts our assumption. Hence for any triple $(\Delta_i, \Delta_j, \Delta_k)$ it holds that $D_{ij} = D_{ik}$, $D_{jk} = D_{ji}$ and $D_{ki} = D_{kj}$.

Assume that there is a triple $(\Delta_i, \Delta_j, \Delta_k)$ such that $D_{ij} = D_{ik}$, $D_{jk} = D_{ji}$ and $D_{ki} = D_{kj}$. Then, by the fact stated in the case I of the proof of Theorem 2, $\partial\Delta_i \cap \partial\Delta_j \cap \partial\Delta_k$ consists of two points, say z_1, z_2. By Theorem 2, there is a loxodromic element in G with z_1, z_2 as the fixed points. On the other hand, if $\bigcap \partial\Delta_i$ consists of one point, say z_1, then, by Theorem 2, $\Lambda(\bigcap G_{\Delta_i}) = z_1$. Hence there is a parabolic element of G with z_1 as the fixed point. Since G is Kleinian, this is not the case. Hence $\bigcap \partial\Delta_i$ consists of two points.

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