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ON A PROBLEM OF GIRSANOV

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Let (Ω, F, P) be a probability space with a non-decreasing right continuous family (F_t) of sub σ-fields of F such that F_0 contains all null sets. Let M_t be a continuous martingale with $M_0 = 0$, and set

$$Z_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right).$$

I. V. Girsanov [1] raised the problem of finding sufficient conditions for the process Z_t to be a martingale. It plays an important role in certain aspects of the theory of stochastic integral equations. Recently, it was proved by A. A. Novikov [2] that if $\exp(M_t/2) \in L^1$ for each $t > 0$, then Z_t is a martingale.

Our aim is to prove

THEOREM. *If* $\exp(M_t/2) \in L^1$ *for each* $t > 0$, *then the process* Z_t *is a martingale.*

PROOF. Generally, Z_t is a positive local martingale, so that $E[Z_t] \leqq 1$ for every t. Therefore, it is a martingale if and only if $E[Z_t] = 1$ for every t. Our proof is a slight modification of Novikov's proof given in [2].

Now let $\mu_t = \inf\{s > 0; \langle M\rangle_s > t\}$. Each μ_t is an F_t-stopping time and we denote by (G_t) the right continuous family (F_{μ_t}). Let (Ω', F', P') be another probability space which carries a one-dimensional Brownian motion (B'_t, F'_t) with $B'_0 = 0$. We denote by $(\hat{\Omega}, \hat{F}, \hat{P})$ the product of (Ω, F, P) and (Ω', F', P') with π, π' the projections of $\hat{\Omega} = \Omega \times \Omega'$ onto Ω and Ω' respectively. Set $\hat{G}_t = G_t \times F'_t$. Then $\langle M\rangle_t \circ \pi$ is a $\hat{G}_t$-stopping time. Let $\hat{F}_t = \hat{G}_{\langle M\rangle_t \circ \pi}$. The system $(\hat{\Omega}, \hat{F}, \hat{F}_t, \hat{P})$ is a lifting of (Ω, F, F_t, P) under π. It is easy to see that $M_{\mu_t} \circ \pi$ and $B'_t \circ \pi'$ are $\hat{G}_t$-continuous local martingales. As is well-known, by a classical result of P. Lévy,

$$B_t = M_{\mu_t} \circ \pi + B'_t \circ \pi' - B'_{t \wedge (\langle M\rangle_\infty \circ \pi)} \circ \pi'$$

is a Brownian motion over $(\hat{G}_t)$. Here $x \wedge y$ is the minimum of x and y. It is clear that $M_t \circ \pi = B_{\langle M\rangle_t \circ \pi}$. Now, set

$$\tau_a = \inf\{t \geqq 0; B_t \leqq t - a\}, \qquad 0 < a < \infty.$$

As the distribution density of this $\hat{G}_t$-stopping time τ_a equals

$$\frac{a}{\sqrt{2\pi t^3}}\exp\left(-\frac{(t-a)^2}{2t}\right),$$

we have $\hat{E}[\exp(\tau_a/2)] = \exp(a)$ (see [2]). Since $B_{\tau_a} - \tau_a/2 = \tau_a/2 - a$, it follows that

$$\hat{E}\left[\exp\left(B_{\tau_a} - \frac{1}{2}\tau_a\right)\right] = 1 .$$

This implies that the process $X_t = \exp(B_{t\wedge\tau_a} - t\wedge\tau_a/2)$ is a $\hat{P}$-uniformly integrable martingale over $(\hat{G}_t)$. Then, as $\langle M\rangle_t \circ \pi$ is a $\hat{G}_t$-stopping time, we have $\hat{E}[X_{\langle M\rangle_t\circ\pi}] = 1$. On the other hand,

$$\hat{E}[X_{\langle M\rangle_t\circ\pi}] = \hat{E}[X_{\langle M\rangle_t\circ\pi};\ \tau_a > \langle M\rangle_t \circ \pi] + \hat{E}[X_{\langle M\rangle_t\circ\pi};\ \tau_a \leqq \langle M\rangle_t\circ\pi] .$$

Since $X_{\langle M\rangle_t\circ\pi} = Z_t\circ\pi$ on $\{\tau_a > \langle M\rangle_t\circ\pi\}$, the first term on the right side is smaller than $\hat{E}[Z_t\circ\pi] = E[Z_t]$. And the second term is smaller than

$$\hat{E}\left[\exp\left(\frac{1}{4}B_{(\langle M\rangle_t\circ\pi)\wedge\tau_a}\right)\exp\left(\frac{3}{4}B_{\tau_a} - \frac{1}{2}\tau_a\right)\right]$$
$$\leqq \left(\hat{E}\left[\exp\left(\frac{1}{2}B_{(\langle M\rangle_t\circ\pi)\wedge\tau_a}\right)\right]\right)^{1/2}\left(\hat{E}\left[\exp\left(\frac{3}{2}B_{\tau_a} - \tau_a\right)\right]\right)^{1/2} .$$

As $B_{\tau_a} = \tau_a - a$, the second term on the right side is

$$\left(\hat{E}\left[\exp\left(\frac{1}{2}\tau_a\right)\right]\exp\left(-\frac{3}{2}a\right)\right)^{1/2} = \exp\left(-\frac{1}{4}a\right)$$

which converges to 0 as $a \to \infty$. To estimate the first term, set

$$T = \inf\{s \geqq 0;\ \langle M\rangle_s\circ\pi \geqq \tau_a\} .$$

For each t, $\{T \leqq t\} = \{\tau_a \leqq \langle M\rangle_t\circ\pi\} \in \hat{G}_{\langle M\rangle_t\circ\pi} = \hat{F}_t$, so that T is an $\hat{F}_t$-stopping time. It follows from the definition of T that

$$\langle M\rangle_{t\wedge T}\circ\pi = (\langle M\rangle_t\circ\pi)\wedge\tau_a .$$

As $M_t\circ\pi = B_{\langle M\rangle_t\circ\pi}$ is a martingale over $(\hat{F}_t)$, by the Doob optional sampling theorem

$$\hat{E}[B_{\langle M\rangle_t\circ\pi}\,|\,\hat{F}_{t\wedge T}] = B_{\langle M\rangle_{t\wedge T}\circ\pi} = B_{(\langle M\rangle_t\circ\pi)\wedge\tau_a} .$$

Thus, by the Jensen inequality,

$$\hat{E}\left[\exp\left(\frac{1}{2}B_{(\langle M\rangle_t\circ\pi)\wedge\tau_a}\right)\right] \leqq \hat{E}\left[\exp\left(\frac{1}{2}B_{\langle M\rangle_t\circ\pi}\right)\right]$$

$$= \hat{E}\Big[\exp\Big(\frac{1}{2}M_t\circ\pi\Big)\Big]$$
$$= E\Big[\exp\Big(\frac{1}{2}M_t\Big)\Big].$$

Consequently, we have

$$1 \leqq E[Z_t] + \Big(E\Big[\exp\Big(\frac{1}{2}M_t\Big)\Big]\Big)^{1/2}\exp\Big(-\frac{a}{4}\Big).$$

The right side converges to $E[Z_t]$ as $a \to \infty$. Thus, $E[Z_t] = 1$ for every t. This completes the proof.

REMARK. If $\exp(\langle M\rangle_t/2) \in L^1$, then $\exp(M_t/2) \in L^1$. Indeed, applying the Schwarz inequality we get

$$\begin{aligned} E\Big[\exp\Big(\frac{1}{2}M_t\Big)\Big] &= E\Big[\exp\Big(\frac{1}{2}M_t - \frac{1}{4}\langle M\rangle_t\Big)\exp\Big(\frac{1}{4}\langle M\rangle_t\Big)\Big] \\ &\leqq (E[Z_t])^{1/2}\Big(E\Big[\exp\Big(\frac{1}{2}\langle M\rangle_t\Big)\Big]\Big)^{1/2} \\ &\leqq \Big(E\Big[\exp\Big(\frac{1}{2}\langle M\rangle_t\Big)\Big]\Big)^{1/2}. \end{aligned}$$

Namely, our result is an improvement of the Novikov theorem.

Finally, we give such a continuous local martingale M_t that $Z_t = \exp(M_t - \langle M\rangle_t/2)$ is a martingale, but $Z'_t = \exp(-M_t - \langle M\rangle_t/2)$ is not a martingale. For that, let (B_t) be a one-dimensional Brownian motion such that $B_0 = 0$. We set

$$\xi = \inf\{t > 0;\ B_t \geqq 1\},$$

which is stopping time such that $0 < \xi < \infty$ a.s. Now let $\alpha\colon [0, 1[\to [0, \infty[$ be an increasing homeomorphic function, and set

$$\theta_t = \begin{cases} \alpha(t) \wedge \xi & \text{if } 0 \leqq t < 1 \\ \xi & \text{if } 1 \leqq t < \infty. \end{cases}$$

Then each θ_t is a stopping time such that $\theta_0 = 0$ and $\theta_1 = \xi$. For a.e. $\omega \in \Omega$ the sample functions $\theta_0(\omega)$ are non-decreasing and continuous, so that $M_t = B_{\theta_t}$ is a continuous local martingale. As $\theta_t \leqq \xi$, we have $M_t \leqq 1$ and $Z_t \leqq e$. Therefore, Z_t is a bounded martingale. On the other hand, as $M_1 = B_\xi = 1$, $E[Z'_1] \leqq E[\exp(-M_1)] = 1/e < 1$. This implies that the process Z'_t is not a martingale. Of course, if $\exp(\langle M\rangle_t/2) \in L^1$ for every t, then Z_t and Z'_t are martingales. But, for every $\delta > 0$, there is a continuous martingale M_t such that $E[\exp((1/2 - \delta)\langle M\rangle_t)]$ is finite and the process Z_t is not a martingale (see [2]). Therefore, as $E[\exp(\alpha M_t)] \leqq (E[\exp(2\alpha^2\langle M\rangle_t)])^{1/2}$ for every α, our condition may not be essentially weakened.

REFERENCES

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