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THE FLOW OF WEIGHTS ON FACTORS OF TYPE III

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CONTENTS

Introduction. The recent developments in the theory of operator algebras showed the importance of the study of weights. A weight φ on a von Neumann algebra M is a linear map from M_+ to $[0, +\infty]$; φ is faithful if $\varphi(x)=0$ implies $x=0$, normal if it commutes with the sup operation, semi-finite when $\varphi(x)<+\infty$ on a σ-strongly dense subset of M_+. Throughout this paper, a weight means a semi-finite normal one.

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The origin of this paper lies in the following relation between the authors' previous works [3] and [30]. Let F_∞ be the type I_∞ factor of all bounded operators in $L^2(\boldsymbol{R})$ and ω the weight on F_∞ such that $\omega(x) = \mathrm{Trace}\,(\rho x)$, $x \geqq 0$, where ρ^{it} is for any $t \in \boldsymbol{R}$ the translation by t in $L^2(\boldsymbol{R})$. In [3; Lemma 1.2.5], it was shown that on $M \otimes F_\infty$ the weight $\bar{\omega} = \varphi \otimes \omega$ is independent, up to unitary equivalence, of the choice of the faithful weight φ. In [30; Theorem 8.1] it was proven independently that the crossed product of M by the modular automorphism group σ^φ is unaffected by changing φ. In fact, the centralizer of $\bar{\omega}$ is trivially equal to the above crossed product. If $V_\lambda \in F_\infty$ for any $\lambda \in \boldsymbol{R}_+^*$ is the multiplication by the function: $t \longrightarrow \lambda^{it}$, then $1 \otimes V_\lambda$ implements a unitary equivalence between $\bar{\omega}$ and $\lambda\bar{\omega}$. We shall show that the weight $\bar{\omega}$ is characterized, up to unitary equivalence, as the only faithful weight with properly infinite centralizer, which is unitarily equivalent to $\lambda\bar{\omega}$ for any $\lambda \in \boldsymbol{R}_+^*$, in any properly infinite von Neumann algebra with separable predual, Theorem II.1.1. To understand the meaning of this result, we first develop an elementary comparison theory for weights, analogue to the usual comparison theory of projections. Two weights φ and ψ are by definition equivalent when there exists a partial isometry u with initial projection the support $s(\varphi)$ of φ and final projection $s(\psi)$ such that

$$\varphi(x) = \psi(uxu^*), \quad u \in M_+ .$$

The set of equivalence classes of weights on a properly infinite von Neumann algebra is endowed with the following natural addition:

$$\text{class } \varphi + \text{class } \psi = \text{class } (\varphi + \psi) \text{ provided } s(\varphi) \perp s(\psi) .$$

The class of a weight φ is idempotent: class φ + class φ = class φ if and only if the centralizer of φ is properly infinite. It is then shown that such classes form a Boolean algebra isomorphic to the lattice of all σ-finite projections in a unique abelian von Neumann algebra $\mathfrak{p}_M$. Thus there exists a map p_M from weights to σ-finite projections of $\mathfrak{p}_M$ such that

$$p_M(\varphi + \psi) = p_M(\varphi) \vee p_M(\psi) \text{ if } s(\varphi) \perp s(\psi) ;$$
$$p_M(\varphi) = p_M(\psi) \text{ if } \varphi \sim \psi .$$

Each $\lambda \in \boldsymbol{R}_+^*$ determines a unique automorphism $\mathfrak{F}_\lambda^M$ of $\mathfrak{P}_M$ such that

$$p_M(\lambda\varphi) = \mathfrak{F}_\lambda^M p_M(\varphi) \text{ for any weight } \varphi .$$

We shall call the couple $\{\mathfrak{P}_M, \mathfrak{F}^M\}$ the global flow of weights. The global flow of weights depends functorially on M by its construction.

Let M be a properly infinite von Neumann algebra and $\bar{\omega}$ as above.

Then $d_M = p_M(\bar{\omega})$ is the largest σ-finite projection of $\mathfrak{P}_M$ invariant under $\mathfrak{F}^M$, and moreover the following three conditions are equivalent for any weight φ on M: (i) $p_M(\varphi) \leqq d_M$; (ii) the map: $\lambda \mapsto p_M(\lambda\varphi)$ is σ-strongly continuous; (iii) $\int \sigma_t^\varphi(\cdot)dt$ has a weakly dense domain. The last condition provides the name of integrable weight for φ. In Chapter II, we establish their regularity properties and their density in the set of weights with properly infinite centralizer.

By condition (ii), the restriction of $\mathfrak{F}^M$ to d_M is a σ-strongly continuous flow F^M called the smooth flow of weights. So the smooth flow of weights F^M is just the continuous part of the global one: class $\varphi \longrightarrow$ class $\lambda\varphi$. Hence it depends functorially on M, therefore defining a homomorphism mod of $\mathrm{Aut}(M)$ into $\mathrm{Aut}(F^M)$ which corresponds to the fundamental group of Murray and von Neumann in the semi-finite case. This functor is exactly the analogue of the module of a locally compact group. For instance, let G be a principal virtual group, and $M = U(G)$ be the factor arising from the left regular representation of G on $L^2(G)$, [16]. Then F^M is precisely the closure of the range of the module Δ_G as defined in [16]. This allows us to understand better the meaning of the modular automorphism and to extend it to the whole dual group of F^M considered as a virtual group. This extended modular automorphism group yields a one-to-one homomorphism $\bar{\delta}_M$ of $H^1(F^M)$ into $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M)$, whose range is contained in the kernel of mod.

The work presented here has been undertaken since February 1973, and most of the results wete obtained while the authors stayed at Le Centre de Physique Théorique, CNRS, Marseille, France, from September 1973 through May 1974. The main results were announced in [7], 1974. The authors would like to express their sincere gratitude to Professor D. Kastler and his colleagues for their warm hospitality extended to them, which made this collaboration possible and pleasant. The second named author would like to thank the John Simon Guggenheim Memorial Foundation for a generous support extended to him.

Preliminary. Given a factor M of type III_λ, $\lambda \neq 1$, there exists a von Neumann algebra N of type II_∞ and an automorphism θ of N such that

$$M \cong W^*(N, \theta) ,$$

where $W^*(N, \theta)$ means the crossed product of N by a single automorphism θ. Here if $\lambda > 0$, then N is a factor and $\tau\circ\theta = \lambda\tau$ for a faithful semi-finite normal trace τ on N; if $\lambda = 0$, then N has a non-atomic center C and there exists a faithful semi-finite normal trace τ and $0 < \lambda_0 < 1$ such that $\tau\circ\theta \leqq \lambda_0\tau$. We shall call $W^*(N, \theta)$ or some times the covariant system $\{N, \theta\}$ itself a *discrete decomposition* of M. If M is a von Neumann algebra of type III, then there exists a von Neumann algebra N of type II_∞ and a one parameter automorphism group $\{\theta_t\}$ such that

$$M \cong W^*(N, \boldsymbol{R}, \theta)$$

and $\tau\circ\theta_s = e^{-s}\tau$ for some faithful semi-finite normal trace τ on N, where $W^*(N, \boldsymbol{R}, \theta)$ means the crossed product of N by $\boldsymbol{R}$ the additive group $\boldsymbol{R}$ with respect to the action θ. We shall call this $W^*(N, \boldsymbol{R}, \theta)$ or the covariant system $\{N, \theta\}$ a *continuous decomposition* of M.

In this paper, we consider often an action α of a locally compact group G on a measure space $\{\Gamma, \mu\}$ preserving the family of null sets. Throughout, we consider the action of G only from the left hand side. The action of G on $L^\infty(\Gamma, \mu)$ induced by α of G on $\{\Gamma, \mu\}$ means the one defined by

$$(\alpha_g f)(\gamma) = f(\alpha_g^{-1}\gamma),\ g\in G,\ f\in L^\infty(\Gamma, \mu),\ \gamma\in\Gamma\ .$$

Let M be a von Neumann algebra. By $\mathrm{Aut}(M)$, we denote the group of all automorphisms of M and $\mathrm{Int}(M)$ means the normal subgroup of $\mathrm{Aut}(M)$ consisting of all inner automorphisms. We consider the quotient group

$$\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M)\ .$$

The canonical homomorphism of $\mathrm{Aut}(M)$ onto $\mathrm{Out}(M)$ is denoted by ε_M. Besides the norm topology, we consider the following topology in $\mathrm{Aut}(M)$: a net $\{\alpha_i\}$ in $\mathrm{Aut}(M)$ is said to converge to $\alpha\in\mathrm{Aut}(M)$ if for each $\varphi\in M_*$, $\lim_i \|\varphi\circ\alpha_i - \varphi\circ\alpha\| = 0$. Since $\mathrm{Aut}(M)$, or more precisely its adjoint transformations, is contained in the group of isometries on M_*, $\mathrm{Aut}(M)$ is a topological group under this topology. If M_* is separable, then $\mathrm{Aut}(M)$ is a *topological polish group* under the *two sided uniform structure*. We note, however, that the one sided uniform structure of $\mathrm{Aut}(M)$ is not complete.

CHAPTER I. THE GLOBAL FLOW OF WEIGHTS

I.0. Introduction. Let M be a von Neumann algebra with faithful semi-finite normal trace τ. Then the map $h \longrightarrow \tau(h\cdot) = \psi$ is a bijection between positive self-adjoint operators affiliated with M and weights on M. Each ψ being characterized by the representation: $t \longrightarrow h^{it}$ of the real line, one sees immediately that the study of weights on such M reduces to the study of representations of $\boldsymbol{R}$ in M. In particular, when $M = \mathfrak{L}(\mathfrak{H})$, the algebra of all bounded operators in $\mathfrak{H}$, the study of weights is thus the classical multiplicity theory of positive self adjoint operators. When M is no longer semi-finite, so that such a τ does not exist, we shall establish a comparison theory for weights as a generalization of the comparison theory for representations of $\boldsymbol{R}$. For each faithful φ, we regard the unitary cocycle: $t \longrightarrow u_t = (D\psi : D\varphi)_t$ as the analogue of the above representation: $t \longrightarrow h^{it} = (D\psi : D\tau)_t$. This allows to define equivalence, subequivalence, disjoint sums for weights. Moreover those notions do not depend on the special choice of the reference weight φ. We then show that the set of idempotents for disjoint sum forms a σ-complete boolean lattice and that the map: $(\lambda, \varphi) \longrightarrow \lambda\varphi$ induces on this lattice an action of $\boldsymbol{R}_+^*$, the global flow of weights $\mathfrak{F}^M$ of M. With the help of this construction we then, for factors M which are not of type III_1, give an isomorphism of the global flow of weights of M with the flow $(\mathfrak{F}^N)^\theta$ where $M = W^*(\theta, N)$ is a discrete decomposition of M. (cf. Corollary 2.8. (iii)). This allows to show the normality of arbitrary centralizers in III_0-factors, and the existence of normal states with abelian centralizers in all factors of non-type III_1.

I.1. Elementary comparison of weights. In this section, we develope an elementary dimension theory for weights on a σ-finite properly infinite von Neumann algebra which may be viewed as a generalization of the usual dimension theory for projections.

Throughout this section, M denotes a fixed σ-finite properly infinite von Neumann algebra. By a *weight* on M we mean a normal semi-finite weight on M, and by $\mathfrak{W}_M$ we denote the set of all weights on M. Since we consider weights which are not necessarily faithful, we need some modification in terminologies and definitions which were given to faithful weights.

DEFINITION 1.1. For a weight φ on M, the support of φ, denoted by $s(\varphi)$, means the projection e of M such that $\varphi(1-e) = 0$ and that the restriction of φ to M_e is faithful.

The support $s(\varphi)$ of φ is also characterized as the projection e in M such that $M(1-e) = \{x \in M : \varphi(x^*x) = 0\}$. The modular automorphism group $\{\sigma_t^\varphi\}$ of φ means the modular automorphism group of M_e associated with the restriction of φ to M_e. The centralizer M_φ of φ is the von Neumann subalgebra of M_e which is the fixed point algebra under $\{\sigma_t^\varphi\}$.

For a weight φ on M and a partial isometry u in M with $e = uu^* \in M_\varphi$, we define a new weight $\psi = \varphi_u$ by

$$\psi(x) = \varphi(uxu^*), \quad x \in M_+ .$$

One checks that ψ is a weight with support $s(\psi) = u^*u$ and that ψ is isomorphic to the restriction of φ to M_e disregarding the trivial part where ψ vanishes. For a projection $e \in M_\varphi$, φ_e is a weight with support e, called a *subweight* of φ. Note that φ_e is a weight on M not only on M_e.

We now introduce an equivalence and a partial ordering among all weights as follows:

DEFINITION 1.2. Let φ_1 and φ_2 be weights on M. We say that φ_1 and φ_2 are equivalent and write $\varphi_1 \sim \varphi_2$, if there exists a partial isometry $u \in M$ with $uu^* = s(\varphi_1)$ and $u^*u = s(\varphi_2)$ such that $\varphi_2 = \varphi_{1,u}$. We say that φ_1 is subequivalent to φ_2 and write $\varphi_1 \prec \varphi_2$, if φ_1 is equivalent to a subweight of φ_2.

In other words, $\varphi_1 \prec \varphi_2$ if and only if $\varphi_1 = \varphi_{2,u}$ for some partial isometry u with $uu^* \in M_{\varphi_2}$. It will be seen shortly that the equivalence "$\sim$" is the equivalence relation associated with the partial ordering "$\prec$."

LEMMA 1.3. *Let φ be a weight on M and u a partial isometry with $p = u^*u \in M_\varphi$ and $q = uu^* \in M_\varphi$. We have $\varphi_u = \varphi_p$ if and only if $u \in M_\varphi$.*

PROOF. Let $e = s(\varphi)$. Since u belongs to M_e, we may restrict our attention to M_e, so that we may assume φ faithful.

Suppose that u belongs to M_φ. We have then $\mathfrak{n}_\varphi u \subset \mathfrak{n}_\varphi$ and $\mathfrak{n}_\varphi u^* \subset \mathfrak{n}_\varphi$. It follows then that

$$\mathfrak{n}_{\varphi_u} = \{x \in M: xu^* \in \mathfrak{n}_\varphi\} ;$$
$$\mathfrak{n}_{\varphi_p} = \{x \in M: xp \in \mathfrak{n}_\varphi\} ,$$

which implies that $\mathfrak{n}_{\varphi_u} = \mathfrak{n}_{\varphi_p}$; hence $\mathfrak{m}_{\varphi_u} = \mathfrak{m}_{\varphi_p}$. It follows from [24; Theorem 3.6] that for any $x \in \mathfrak{m}_{\varphi_u}$, $pxp \in \mathfrak{m}_\varphi$ and $\varphi(uxu^*) = \varphi(upxpu^*) = \varphi(pxp)$. Thus $\varphi_u = \varphi_p$.

Suppose $\varphi_u = \varphi_p$. We have for any $x \in M_+$, $\varphi(uxu^*) = \varphi(pxp)$. Replacing x by u^*xu, we get $\varphi(u^*xu) = \varphi(qxq)$. Therefore, we have $\mathfrak{n}_\varphi u^* \subset \mathfrak{n}_\varphi$ and $\mathfrak{n}_\varphi u \subset \mathfrak{n}_\varphi$; hence we get $u\mathfrak{m}_\varphi \subset \mathfrak{m}_\varphi$ and $\mathfrak{m}_\varphi u \subset \mathfrak{m}_\varphi$. We have, for any $x \in \mathfrak{m}_\varphi$,

$$\varphi(ux) = \varphi(qux) = \varphi(uxq) = \varphi(uxuu^*) = \varphi(pxup) = \varphi(xu) ,$$

which implies by [24; Theorem 3.6] that $u \in M_\varphi$. q.e.d.

We now extend the notion of the cocycle Radon-Nikodym derivative $(D\varphi : D\psi)$, [3; Lemma 1.2.1], to the case where ψ is faithful and φ is not necessarily faithful. Let $P = M \otimes F_2$, and set

$$\theta\Big(\sum_{i,j=1}^{2} x_{ij} \otimes e_{ij}\Big) = \psi(x_{11}) + \varphi(x_{22}) .$$

We have $s(\theta) = 1 \otimes e_{11} + s(\varphi) \otimes e_{22}$, hence $s(\varphi) \otimes e_{21} \in P_{s(\theta)}$ and there exists a unique one parameter family of partial isometries $\{u_t\}$ such that

$$u_t u_t^* = s(\varphi),\ u_t^* u_t = \sigma_t^\psi(s(\varphi))\ ;$$
$$\sigma_t^\theta(s(\varphi) \otimes e_{21}) = u_t \otimes e_{21}\ , \qquad t \in \boldsymbol{R}\ .$$

We denote by $(D\varphi\colon D\psi)_t$ this u_t, $t \in \boldsymbol{R}$.

LEMMA 1.4. *Let φ_1, φ_2 and ψ be weights on M with ψ faithful. Let $P = M \otimes F_2$ and put*

$$\varphi\left(\sum_{i,j=1}^{2} x_{ij} \otimes e_{ij}\right) = \varphi_1(x_{11}) + \varphi_2(x_{22})\ .$$

(a) *For a partial isometry $w \in M$ with $w^*w = s(\varphi_2)$ and $ww^* \in M_{\varphi_1}$, the following statements are equivalent:*

$$\varphi_2 = \varphi_{1,w} \Longleftrightarrow w \otimes e_{12} \in P_\varphi \Longleftrightarrow (D\varphi_2\colon D\psi)_t = w^*(D\varphi_1\colon D\psi)_t \sigma_t^\psi(w)\ , \qquad t \in \boldsymbol{R}\ .$$

(b) $\varphi_2 \prec \varphi_1 \Leftrightarrow s(\varphi_2) \otimes e_{22} \prec s(\varphi_1) \otimes e_{11}$ *relative to* P_φ.

(c) $\varphi_1 \sim \varphi_2 \Leftrightarrow s(\varphi_1) \otimes e_{11} \sim s(\varphi_2) \otimes e_{22}$ *relative to* P_φ.

(d) *For partial isometries $u, v \in M$ with $uu^* \in M_{\varphi_1}$ and $vv^* \in M_{\varphi_1}$,*

$$\varphi_{1,u} \prec \varphi_{1,v} \Longleftrightarrow uu^* \prec vv^* \text{ relative to } M_{\varphi_1}\ .$$

PROOF. (a) We have $s(\varphi) = s(\varphi_1) \otimes e_{11} + s(\varphi_2) \otimes e_{22}$, so that $w \otimes e_{12} \in P_{s(\varphi)}$. As $(w \otimes e_{12})^*(w \otimes e_{12}) = s(\varphi_2) \otimes e_{22}$ and $(w \otimes e_{12})(w \otimes e_{12})^* = ww^* \otimes e_{11}$ both belong to P_φ, it follows from Lemma 1.3 that $w \otimes e_{12} \in P_\varphi$ if and only if for any $x \in M$

$$\varphi((w \otimes e_{12})(x \otimes e_{22})(w \otimes e_{12})^*) = \varphi((s(\varphi_2) \otimes e_{22})(x \otimes e_{22}(s(\varphi_2) \otimes e_{22}))$$

if and only if $\varphi_{1,w} = \varphi_2$. Put now $Q = M \otimes F_3$ and

$$\theta\left(\sum_{i,j=1}^{3} x_{ij} \otimes e_{ij}\right) = \varphi_1(x_{11}) + \varphi_2(x_{22}) + \psi(x_{33})\ .$$

We note that

$$\sigma_t^\theta(s(\varphi_j) \otimes e_{j3}) = (D\varphi_j\colon D\psi)_t \otimes e_{j3}\ , \qquad t \in \boldsymbol{R},\ j = 1, 2\ .$$

We have

$$\begin{aligned}\sigma_t^\theta(w \oplus e_{12}) &= \sigma_t^\theta((s(\varphi_1) \otimes e_{13})(w \otimes e_{33})(s(\varphi_2) \otimes e_{32})) \\ &= (D\varphi_1\colon D\psi)_t \sigma_t^\psi(w)(D\varphi_2\colon D\psi)_t^* \otimes e_{12}\ , \qquad t \in \boldsymbol{R}\ ,\end{aligned}$$

so that $w \otimes e_{12} \in P_\varphi$ if and only if $w \otimes e_{12} \in Q_\theta$, if and only if

$$(D\varphi_2\colon D\psi)_t = w^*(D\varphi_1\colon D\psi)_t \sigma_t^\psi(w)\ , \qquad t \in \boldsymbol{R}\ ,$$

where we consider P as the reduced von Neumann algebra $Q_{(1 \otimes e_{11} + 1 \otimes e_{22})}$.

(b) By definition, $\varphi_2 \prec \varphi_1$ if and only if $\varphi_2 = \varphi_{1,w}$ with a partial isometry $w \in M$ satisfying the condition in (a). Hence it follows from

(a) that $\varphi_2 \prec \varphi_1$ implies $s(\varphi_2) \otimes e_{22} \prec s(\varphi_1) \otimes e_{11}$ relative to P_φ. Suppose $s(\varphi_2) \otimes e_{22} \prec s(\varphi_1) \otimes e_{11}$ in P_φ. Let u be a partial isometry in P_φ with $u^*u = s(\varphi_2) \otimes e_{22}$ and $uu^* \leqq s(\varphi_1) \otimes e_{11}$. Then there exists a partial isometry $w \in M$ such that $u = w \otimes e_{12}$. By (a) we have $\varphi_2 = \varphi_{1,w}$; hence $\varphi_2 \prec \varphi_1$.

(c) The same arguments as (b) work.

(d) Put $p = uu^*$ and $q = vv^*$. It follows that $\varphi_{1,u} \sim \varphi_{1,p}$ and $\varphi_{1,v} \sim \varphi_{1,q}$. Suppose $p \prec q$ relative to M_{φ_1}. Let w be a partial isometry in M_{φ_1} with $p = w^*w$ and $ww^* \leqq q$. We have then by Lemma 1.3

$$\varphi_{1,p} = \varphi_{1,w} \prec \varphi_{1,q} \sim \varphi_{1,v} .$$

Conversely suppose $\varphi_{1,p} \prec \varphi_{1,q}$. There exists a partial isometry w in M with $w^*w = p$ and $ww^* \leqq q$ such that $ww^* \in M_{\varphi_{1,q}}$ and $\varphi_{1,p} = \varphi_{1,w}$. By Lemma 1.3, w belongs to M_{φ_1}, so that $p \prec q$ relative to M_{φ_1}. q.e.d.

Recalling the fact that for any pair e, f of projections in a von Neumann algebra, $c = c(e)c(f)$, the product of the central support $c(e)$ of e and $c(f)$ of f, is the largest central projection such that ce and cf are quasi-equivalent, and $(1 - c)e$ and $(1 - c)f$ are centrally orthogonal, we give the following definition which measures the "quasiequivalent" piece of a given pair of weights.

DEFINITION 1.5. Let φ_1 and φ_2 be weights on M, and $P = M \otimes F_2$. Put

$$\varphi(\sum_{ij=1,2}^{2} x_{ij} \otimes e_{ij}) = \varphi_1(x_{11}) + \varphi_2(x_{22}) .$$

We define $c_{\varphi_1}(\varphi_2)$ as the unique projection c in the center of M_{φ_1} such that

$$c \otimes e_{11} = (\text{Central support of } s(\varphi_2) \otimes e_{22} \text{ in } P_\varphi)(s(\varphi_1) \otimes e_{11}) .$$

LEMMA 1.6. *Let ψ, ψ_1 and ψ_2 be weights on M.*

(a) *For any partial isometry $w \in M$ with $ww^* \in M_\psi$, $c_\psi(\psi_w)$ is the central support of ww^* in M_ψ.*

(b) *For any partial isometry $w \in M$ with $ww^* \in M_\psi$,*

$$c_{\psi_w}(\psi_1) = w^* c_\psi(\psi_1) w .$$

(c) $\psi_1 \prec \psi_2 \Rightarrow c_\psi(\psi_1) \leqq c_\psi(\psi_2)$.

(d) *If $\{\psi_n\}$ is a sequence of weights on M with pairwise orthogonal supports and $\psi = \sum_{n=1}^\infty \psi_n$, then*

$$c_\psi(\varphi) = \sum_{n=1}^{\infty} c_{\psi_n}(\varphi) ;$$

$$c_\varphi(\psi) = \bigvee_{n=1}^{\infty} c_\varphi(\psi_n)$$

for any other weight φ on M.

PROOF. (a) Putting $\varphi_1 = \psi$ and $\varphi_2 = \psi_w$, we consider $P = M \otimes F_2$ and φ as in Lemma 1.4. It follows then that $s(\varphi_2) \otimes e_{22} \sim ww^* \otimes e_{11}$ in P_φ by Lemma 1.4 (d).

(b) Putting $\varphi_1 = \psi$, $\varphi_2 = \psi_w$ and $\varphi_3 = \psi_1$, we define $Q = M \otimes F_3$ and θ as in the proof of Lemma 1.4. Let c be the central support of $s(\varphi_3) \otimes e_{33}$ in Q_θ. We have then

$$\begin{aligned} c(s(\varphi_2) \otimes e_{22}) &= c(w^*w \otimes e_{22}) = c(w^* \otimes e_{21})(w \otimes e_{12}) \\ &= (w \otimes e_{12})^* c(w \otimes e_{12}) \qquad \text{by Lemma 1.4.(a)} \\ &= (w \otimes e_{12})^* c(s(\varphi_1) \otimes e_{11})(w \otimes e_{12}) \\ &= (w \otimes e_{12})^* (c_{\varphi_1}(\varphi_3) \otimes e_{11})(w \otimes e_{12}) \\ &= w^* c_\psi(\psi_1) w \otimes e_{22} \, . \end{aligned}$$

(c) With $\varphi_1 = \psi_1$, $\varphi_2 = \psi_2$ and $\varphi_3 = \psi$, Let Q and θ be as before. We have then $s(\varphi_1) \otimes e_{11} \prec s(\varphi_2) \otimes e_{22}$ in Q_θ by Lemma 1.4(b); so $c_{\varphi_3}(\varphi_1) \leqq c_{\varphi_3}(\varphi_1)$.

(d) Put $P = M \otimes F_2$ and

$$\theta(\sum_{i,j=1}^{2} x_{ij} \otimes e_{ij}) = \varphi(x_{11}) + \psi(x_{22}) \, .$$

Let c be the central support of $s(\varphi) \otimes e_{11}$ in P_θ. Since $s(\psi) = \sum_{n=1}^{\infty} s(\psi_n)$, we get

$$c(s(\psi) \otimes e_{22}) = \sum_{n=1}^{\infty} (s(\psi_n) \otimes e_{22})c = \sum_{n=1}^{\infty} c_{\psi_n}(\varphi) \otimes e_{22} \, .$$

For the second equality, let P and θ be as above, and d be the central support of $s(\psi) \otimes e_{22}$ in P_θ. As $s(\psi) = \sum_{n=1}^{\infty} s(\psi_n)$, d is the supremum of the central supports of the $s(\psi_n) \otimes e_{22}$, which proves our assertion.

q.e.d.

LEMMA 1.7. *Let φ be a faithful weight on M. If N is a factor of type I contained in M_φ, then the tensor product decomposition $M = (N' \cap M) \otimes N$ splits φ into the tensor product weight $\varphi = \psi \otimes \mathrm{Tr}$ with ψ a faithful weight on $N' \cap M$ and* Tr *the usual trace on N.*

PROOF. Put $Q = N' \cap M$. It follows then that M is identified with $Q \otimes N$. Let $\{e_n\}$ be a sequence of orthogonal minimal projections in N with $\sum_n e_n = 1$. Let $\{u_n\}$ be a sequence of partial isometries in N with $e_1 = u_n^* u_n$ and $e_n = u_n u_n^*$, $n = 1, 2 \cdots$. Choose an $h \in \mathfrak{m}_\varphi^+$ with $e_1 h e_1 \neq 0$, which is always possible due to the density of $\mathfrak{m}_\varphi$ in M. Put

$a = \sum_{n=1}^{\infty} u_n h u_n^* \in Q$. Multiplying a by a scalar, we may assume that $\varphi(ae_1) = 1$. Consider the function τ on N given by $\tau(x) = \varphi(ax)$. By [24; Theorem 3.6], τ is a normal trace on N and $\tau(e_1) = 1$. Hence τ is the usual trace Tr on N. Put $\psi(b) = \varphi(be_1)$, $b \in Q$. It follows then that ψ is faithful and normal on Q. Let $\{h_i\}$ be an approximate identity in $\mathfrak{m}_\varphi^+$ with respect to the σ-strong* topology. Putting

$$k_i = \sum_{n=1}^{\infty} u_n h_i u_n^* \in Q ,$$

we obtain an approximate identity $\{k_i\}$ of Q with respect to the σ-strong* topology. Since $\psi(k_i) = \varphi(k_i e_1) = \varphi(h_i e_1) < \infty$, we have $k_i \in \mathfrak{m}_\psi$; so ψ is semi-finite on Q.

For each $b \in Q_+$, put $\tau_b(x) = \varphi(bx)$, $x \in N$. Since τ_b is a faithful normal trace, which may be purely infinite though, there is a scaler $\lambda_b \geqq 0$ (with possibility of $+\infty$) such that $\tau_b(x) = \lambda_b \mathrm{Tr}(x)$ and $\lambda_b = \tau_b(e_1)$. Thus we get for any $x \in N$

$$\varphi(bx) = \tau_b(x) = \tau_b(e_1)\mathrm{Tr}(x) = \varphi(be_1)\mathrm{Tr}(x) = \psi(b)\mathrm{Tr}(x) .$$

Therefore, we have the decomposition $\varphi = \psi \otimes \mathrm{Tr}$, using [24; Prop. 5.9]. q.e.d.

DEFINITION 1.8. A weight φ on M is said to be of infinite multiplicity if the centralizer M_φ is properly infinite.

LEMMA 1.9. *Suppose $M = Q \otimes F_\infty$ with Q isomorphic to M.*

(a) *For any weight ψ on Q, the weight $\psi \otimes \mathrm{Tr}$ on M is of infinite multiplicity.*

(b) *Any faithful weight φ of infinite multiplicity on M is equivalent to a weight of the form $\psi \otimes \mathrm{Tr}$ for some faithful weight ψ on Q which is isomorphic to φ.*

(c) *If φ is a weight on M, then there exists a sequence $\{\varphi_n\}$ of weights with pairwise orthogonal supports such that $\varphi_n \sim \varphi$ and $\check{\varphi} = \sum_{n=1}^{\infty} \varphi_n$ is isomorphic to the weight $\varphi \otimes \mathrm{Tr}$ on $M \otimes F_\infty$.*

(d) *If $\{\psi_n\}$ is a sequence of weights on M, then the weight $\psi' = \sum_{n=1}^{\infty} \psi'_n$ is independent, up to equivalence, of the choice of a sequence $\{\psi'_n\}$ of weights on M with pairwise orthogonal supports such that $\psi'_n \sim \psi_n$, $n = 1, 2, \cdots$. Moreover, we have $\psi_n \prec \psi'$, $n = 1, 2, \cdots$.*

PROOF. (a) We have $s(\psi \otimes \mathrm{Tr}) = s(\psi) \otimes 1$ and $M_{\psi\otimes\mathrm{Tr}} = Q_\psi \otimes F_\infty$.

(b) Choose a type I_∞ subfactor $N \subset M_\varphi$ such that $N' \cap M_\varphi$ is properly infinite. Let u be a unitary in M such that $u^*Nu = \boldsymbol{C} \otimes F_\infty$. We have then

$$M_{\varphi_u} = u^* M_\varphi u \supset \boldsymbol{C} \otimes F_\infty .$$

Replacing φ by φ_u, we may assume that $\boldsymbol{C} \otimes F_\infty$, say N, is contained in M_φ and that $N' \cap M_\varphi$ is properly infinite. By Lemma 1.7, we have $\varphi = \psi \otimes \mathrm{Tr}$ with ψ a faithful weight on Q. Thus we must show that $\{M, \varphi\} \cong \{Q, \psi\}$. With the notions of the proof of Lemma 1.7, the porperly infiniteness of $N' \cap M_\varphi$ entails the existence of an isometry $v \in M_\varphi$ such that $vv^* = e_1$. Let π be the isomorphism of M onto $Q \otimes \boldsymbol{C} = N' \cap M$ defined by

$$\pi(x) = \sum_{n=1}^{\infty} u_n v x v^* u_n^*, \qquad x \in M .$$

We have then for any $x \in M_+$,

$$\psi \circ \pi(x) = \varphi(\pi(x)e_1) = \varphi\Big(\sum_{n=1}^{\infty} u_n v x v^* u_n^* e_1\Big) = \varphi(vxv^*) = \varphi(x) .$$

Thus $\psi \circ \pi = \varphi$.

(c) Let $\{w_n\}$ be a sequence of isometries in M with pairwise orthogonal ranges such that $\Sigma w_n w_n^* = 1$. For each n, $w_n s(\varphi)$ is a partial isometry with initial projection $s(\varphi)$, so that $\varphi_n = \varphi_{(w_n s(\varphi))^*}$ makes sense as well as $\sum_{n=1}^{\infty} \varphi_n = \check{\varphi}$ because $\{s(\varphi_n)\}$ are pairwise orthogonal. Let $\{e_{j,k}\}$ be a system of matrix units in F_∞, and put, for each $x \in M$,

$$\pi(x) = \sum_{j,k} (w_j^* x w_k) \otimes e_{j,k} \in M \otimes F_\infty .$$

It follows that π is an isomorphism of M onto $M \otimes F_\infty$ and that $(\varphi \otimes \mathrm{Tr}) \circ \pi = \check{\varphi}$.

(d) Putting $\psi_n' = \psi_{n,(w_n s(\psi_n))^*}$ with w_n as in (c), we obtain a sequence $\{\psi_n'\}$. Then the rest is trivial. q.e.d.

For each weight φ on M, we denote by $\check{\varphi}$ the weight of infinite multiplicity on M, unique up to equivalence, determined by Lemma 1.9 (c).

LEMMA 1.10. *Let φ be a weight on M. The map c_φ of Definition 1.5 is an order isomorphism of the set of equivalence classes of weights ψ of infinite multiplicity with $\psi \prec \check{\varphi}$ onto the set of all projections of the center C_φ of M_φ.*

PROOF. With the notations in Lemma 1.9 (c) and $\check{\varphi} = \sum_{n=1}^{\infty} \varphi_n$, we have, by Lemma 1.6 (b) and (d),

$$c_{\check{\varphi}}(\psi) = \sum_{n=1}^{\infty} c_{\varphi_n}(\psi) = \sum_{n=1}^{\infty} w_n c_\varphi(\psi) w_n^*$$

for every weight ψ. Hence $\pi(c_{\check{\varphi}}(\psi)) = c_\varphi(\psi) \otimes 1$, so that we may assume that φ is of infinite multiplicity. Suppose $\varphi_1 \prec \varphi$, $\varphi_2 \prec \varphi$ and $\varphi_j = \varphi_{w_j}$ with w_j a partial isometry such that $w_j w_j^* \in M_\varphi$ for $j = 1, 2$. If φ_j is of infinite multiplicity, then $w_j w_j^* = e_j$ is properly infinite in M_φ, and hence equivalent to its central support in M_φ which is $c_\varphi(\varphi_j)$ by Lemma 1.6 (a). Thus it follows that $c_\varphi(\varphi_1) \leqq c_\varphi(\varphi_2)$ if and only if $\varphi_1 \prec \varphi_2$ for φ_1 and φ_2 as above.

As the surjectivity of c_φ is obvious, we get the assertion. q.e.d.

The next result tells us that the set of all equivalence classes of weights with infinite multiplicity form a σ-complete Boolean lattice.

THEOREM 1.11. *There exists a couple* $(p_M, \mathfrak{p}_M)$ *of an abelian von Neumann algebra* $\mathfrak{P}_M$ *and a surjection* p_M *from the set of weights on* M *to the set of all* σ*-finite projections of* $\mathfrak{P}_M$ *with the following properties*:

(i) *For any weight* φ, φ_1 *and* φ_2 *on* M,

$$p_M(\varphi) = p_M(\check{\varphi}) ;$$
$$p_M(\varphi_1) \leqq p_M(\varphi_2) \Leftrightarrow \check{\varphi}_1 \prec \check{\varphi}_2 .$$

(ii) *For each* φ *on* M *if* p_φ *is the map from the central projections* e *of* M_φ *to* $(\mathfrak{P}_M)_{p_M(\varphi)}$ *defined by*

$$p_\varphi(e) = p_M(\varphi_e) ,$$

then p_φ *is extended uniquely to an isomorphism of the center* C_φ *of* M_φ *onto* $(\mathfrak{P}_M)_{p_M(\varphi)}$ *and we have*

$$p_\varphi(c_\varphi(\psi)) = p_M(\varphi)p_M(\psi)$$

for any weight ψ *on* M.

(iii) *For any sequence of weights* $\{\varphi_n\}$ *on* M, *with pairwise othogonal supports, we have*

$$p_M(\sum_{n=1}^{\infty} \varphi_n) = \bigvee_{n=1}^{\infty} p_M(\varphi_n) .$$

The couple $(p_M, \mathfrak{P}_M)$ *is uniquely determined by property* (i).

PROOF. (i) Consider the von Neumann algebra $Q = M \otimes \mathfrak{L}(l^2(\mathfrak{W}_M))$, where $\mathfrak{W}_M$ means the set of all weights on M as mentioned at the beginning of this section. Let $\{e_{\varphi,\psi}: \varphi, \psi \in \mathfrak{W}_M\}$ be the canonical matrix units in $\mathfrak{L}(L^2(\mathfrak{W}_M))$. We define a weight Φ on Q as follows:

$$\Phi(\sum x_{\varphi,\psi} \otimes e_{\varphi,\psi}) = \sum \varphi(x_{\varphi,\varphi}) .$$

Then, for any $\varphi \in \mathfrak{W}_M$, we have

$$\sum_{\psi \in \mathfrak{W}_M} c_\psi(\varphi) \otimes e_{\psi,\psi} = \text{Central support of } s(\varphi) \otimes e_{\varphi,\varphi} \text{ in } Q_\Phi .$$

We denote by $\mathfrak{P}_M$ the center of Q_Φ, and for each $\varphi \in \mathfrak{W}_M$, by $p_M(\varphi)$ the central support of $s(\varphi) \otimes e_{\varphi,\varphi}$ in Q_Φ. We have then, by Lemma 1.6(d),

$p_M(\varphi) = p_M(\check{\varphi})$, $\varphi \in \mathfrak{W}_M$. For any pair φ_1, φ_2 in $\mathfrak{W}_M$, there exists, by Lemma 1.9(d), a $\varphi \in \mathfrak{W}_M$ with $\varphi_j \prec \varphi$, $j = 1, 2$, which in turn shows by Lemma 1.10 that:

$$\check{\varphi}_1 \prec \check{\varphi}_2 \Leftrightarrow p_M(\varphi_1) \leqq p_M(\varphi_2) .$$

We now show that $p_M(\varphi)$, $\varphi \in \mathfrak{W}_M$, is σ-finite in $\mathfrak{P}_M$. Let $\{c_i\}_{i \in I}$ be an orthogonal family of non-zero projections in $\mathfrak{P}_M$ with $c_i \leqq p_M(\varphi)$. For each $i \in I$, $c_i(s(\varphi) \otimes e_{\varphi,\varphi}) \neq 0$, whence I is countable because M_φ is σ-finite.

We now show that given a σ-finite projection $c \in \mathfrak{p}_M$ there exists a $\varphi \in \mathfrak{W}_M$ with $p_M(\varphi) = c$. Since $s(\Phi) = \sum s(\varphi) \otimes e_{\varphi,\varphi}$, and $c \leqq s(\Phi)$, we have $c \leqq \bigvee_\varphi p_M(\varphi)$. Hence there is a sequence $\{\varphi_n\}$ in $\mathfrak{W}_M$ such that $c \leqq \bigvee_{n=1}^\infty p_M(\varphi_n)$, because c is σ-finite. We choose a weight ψ of infinite multiplicity by Lemma 1.9(d) such that $\varphi_n \prec \psi$, $n = 1, 2, \cdots$. It follows that $c \leqq p_M(\psi)$. By Lemma 1.10, there exists a weight $\varphi \prec \psi$ such that $c(s(\psi) \otimes e_{\psi,\psi}) = c_\psi(\varphi) \otimes e_{\psi,\psi}$. Now c and $p_M(\varphi)$ are both projections in the center of Q_Φ dominated by the central support $p_M(\psi)$ of $s(\psi) \otimes e_{\psi,\psi}$ and such that $c(s(\psi) \otimes e_{\psi,\psi}) = p_M(\varphi)(s(\psi) \otimes e_{\psi,\psi})$. Hence $c = p_M(\varphi)$.

(ii) Let $(p_M, \mathfrak{P}_M)$ be as above. For each $x \in C_\varphi$, ($C_\varphi =$ the center of M_φ), let $p_\varphi(x) = y$ be the unique element in $(\mathfrak{P}_M)_{p_M(\varphi)}$ such that $y(s(\varphi) \otimes e_{\varphi,\varphi}) = x \otimes e_{\varphi,\varphi}$. Clearly p_φ is an isomorphism of C_φ onto $(\mathfrak{P}_M)_{p_M(\varphi)}$. For any weight ψ on M, we have

$$c_\varphi(\psi) \otimes e_{\varphi,\varphi} = p_M(\psi)(s(\varphi) \otimes e_{\varphi,\varphi}) = p_M(\varphi)p_M(\psi)(s(\varphi) \otimes e_{\varphi,\varphi}) ;$$

hence $p_\varphi(c_\varphi(\psi)) = p_M(\varphi)p_M(\psi)$.

(iii) This follows from Lemma 1.6(d).

The uniqueness of $(p_M, \mathfrak{P}_M)$ follows from the fact that an isomorphism of the lattices of σ-finite projections of two abelian von Neumann algebras extends uniquely to an isomorphism of the algebras. q.e.d.

DEFINITION 1.12.[1] The global flow of weights on M, denoted by $(\mathfrak{P}_M, \mathfrak{F}^M)$, is the couple of the abelian von Neumann algebra $\mathfrak{P}_M$ defined in Theorem 1.11 and the action $\mathfrak{F}^M$ of the multiplicative group $\boldsymbol{R}_+^*$ of positive real numbers on $\mathfrak{P}_M$ determined by

$$\mathfrak{F}_\lambda^M p_M(\varphi) = p_M(\lambda\varphi) , \qquad \varphi \in \mathfrak{W}_M, \lambda \in \boldsymbol{R}_+^* .$$

Note that the construction of $\{\mathfrak{P}_M, \mathfrak{F}^M\}$ is functorial in the sense that to each $\alpha \in \mathrm{Aut}(M)$ there corresponds a unique $\bar{\alpha} \in \mathrm{Aut}(\mathfrak{P}_M)$ commuting with $\mathfrak{F}_\lambda^M$, $\lambda \in \boldsymbol{R}_+^*$, defined by the condition:

$$\bar{\alpha} p_M(\varphi) = p_M(\varphi \cdot \alpha^{-1}) , \qquad \varphi \in \mathfrak{W}_M .$$

[1] We shall define later the smooth flow of weights as the "continuous" part of the global flow of weights.

We quote now a rather formal consequences of Theorem 1.11.

COROLLARY 1.13. (i) *For each $\varphi \in M_*^+$, there exists a normal positive linear functional μ_φ on $\mathfrak{P}_M$ uniquely determined by the equality*:

$$\mu_\varphi(p_M(\psi)) = \varphi(c_\varphi(\psi)) , \qquad \psi \in \mathfrak{W}_M .$$

(ii) *If $\varphi_1, \varphi_2 \in M_*^+$, then*

$$\varphi_1 \prec \varphi_2 \Leftrightarrow \mu_{\varphi_1} \leqq \mu_{\varphi_2} ;$$
$$\mu_{\varphi_1+\varphi_2} = \mu_{\varphi_1} + \mu_{\varphi_2} \quad \textit{if} \quad s(\varphi_1) \perp s(\varphi_2) ;$$
$$\mu_{\lambda\varphi} = \lambda\mu_\varphi \cdot \mathfrak{F}^M_{\lambda^{-1}} , \qquad \lambda \in \boldsymbol{R}_+^* .$$

PROOF. (i) The restriction of φ to C_φ, the center of M_φ, is mapped by p_φ^{-1} to a functional μ_φ on $(\mathfrak{P}_M)_{p_M(\varphi)}$ satisfying the required condition.

(ii) By construction, μ_φ only depends on the equivalence class of φ. Hence, to prove the first equivalence, we can assume that $\varphi_j = \psi_{e_j}$ for some $\psi \in M_*^+$, where e_j, $j = 1, 2$, are projections in M_ψ. For any projection $e \in C_\psi$, we have

$$\begin{aligned} c_{\varphi_j}(\psi_e) &= c_{\psi_{e_j}}(\psi_e) = e_j c_\psi(\psi_e) e_j && \text{by Lemma 1.6(b)} \\ &= e_j e && \text{by Lemma 1.6(a);} \end{aligned}$$

$$\varphi_j(c_{\varphi_j}(\psi_e)) = \psi(e_j e) ;$$

hence

$$\mu_{\varphi_j}(p_M(\psi_e)) = \psi(e_j e) .$$

As ψ is a faithful finite normal trace on M_ψ, this proves the first equivalence by making use of the center valued trace in M_ψ. The same computation, with $e_1 \perp e_2$, proves the second equality. Noticing that $c_{\lambda\varphi}(\lambda\psi) = c_\varphi(\psi)$ for any $\psi \in \mathfrak{W}_M$ and $\lambda > 0$, we have

$$\begin{aligned} \mu_{\lambda\varphi}(p_M(\psi)) &= \lambda\varphi(c_{\lambda\varphi}(\psi)) = \lambda\varphi(c_\varphi(\lambda^{-1}\psi)) \\ &= \lambda\mu_\varphi \cdot \mathfrak{F}^M_{\lambda^{-1}}(p_M(\psi)) \end{aligned}$$

for every $\lambda \in \boldsymbol{R}_+^*$ and $\psi \in \mathfrak{W}_M$. q.e.d.

I.2. The global flow of weights for factors of type III_λ, $\lambda \neq 1$. In this section, we examine the flow of weights on a factor of type III_λ, $\lambda \neq 1$, and describe it in terms of the flow of weights on the associated von Neumann algebra of type II_∞ and its automorphism.

Throughout this section, we denote by M a fixed σ-finite factor of type III_λ, $0 \leqq \lambda < 1$.

DEFINITION 2.1. A faithful weight φ on M is said to be lacunary if 1 is an isolated point in the spectrum of the modular operator Δ_φ.

Let φ be a faithful lacunary weight of infinite multiplicity on M. By [3; p. 238] there exists a unitary $U \in M(\sigma^{\varphi},]1, \infty[)$ such that $UM_{\varphi}U^* = M_{\varphi}$, and that M_{φ} and U together generate M. Moreover, by [3; p. 241] this unitary U is unique modulo M_{φ} as well as the element ρ of the center C_{φ} of M_{φ} such that

$$\varphi_U = \varphi(\rho \cdot) .$$

We remark that $0 \leqq \rho \leqq \lambda_0 < 1$ for some λ_0.

We state here the main result of this section and prove it in several steps.

THEOREM 2.2. *Let M, φ, U and ρ be as above, and let E be the unique conditional expectation of M onto $N = M_{\varphi}$. Let τ denote the restriction of φ to N.*

(i) *For each $h \in N_+$ such that $\rho s(h) \leqq h < 1$ and $1-h$ is non-singular, putting $\omega_h = \tau(h \cdot)$, we have*

$$M_{\omega_h \circ E} = N_{\omega_h} .$$

(ii) *For each weight ψ on M, there exists an $h \in N_+$ satisfying the conditions in* (i) *such that*

$$\omega_h \circ E \sim \psi .$$

(iii) *The weight ω_h defined in* (ii) *is unique up to equivalence on N.*

This theorem reduces the problem of comparison of weights on M to the problem of comparison of weights on the von Neumann algebra N of type II_∞. More precisely, the space of equivalence classes of weights on M is isomorphic to the space of equivalence classes of weights on N of the form ω_h with h described above, which means that this space is determined only by N and ρ, and independent of the automorphism on N induced by U.

LEMMA 2.3. *Let M and φ be as above. If ψ be a weight of infinite multiplicity, then there exists a positive $h \in M_{\varphi}$ such that $\varphi(h \cdot) \prec \psi$.*

PROOF. Making use of [3; Corollary 3.2.5], we see that the spectrum of the modular operator of a suitable subweight of ψ will not intersect $\exp(]-2\varepsilon_0, -\varepsilon_0[\cup [\varepsilon_0, 2\varepsilon_0])$ for some $\varepsilon_0 > 0$. We hence assume that ψ is faithful and

$$\mathrm{Sp}(\mathrm{Log}\, \Delta_{\psi}) \cap ([-2\varepsilon_0, -\varepsilon_0] \cup [\varepsilon_0, 2\varepsilon_0])) = \varnothing .$$

By [3; Lemma 5.2.3], there exists an $H \in M_{\psi}$ with $-\varepsilon_0/2 \leqq H \leqq \varepsilon_0/2$ such that the weight $\psi_1 = \psi(e^{-H} \cdot)$ is lacunary. As ψ is of infinite multiplicity, we can choose ψ_1 to be of infinite multiplicity because ψ and ψ_1 both can be replaced by $\psi' \otimes \mathrm{Tr}$ and $\psi_1' \otimes \mathrm{Tr}$ respectively by identifying M and $M \otimes F_\infty$. By [3; Lemma 5.4.6], there exists a non-zero projection e in the center C_φ of M_φ, a non-zero projection f in the center C_{ψ_1} of M_{ψ_1} and a partial isometry $v \in M$ such that

$$e = v^*v\ , \qquad vv^* = f\ ;$$
$$vM_{\varphi_e}v^* = M_{\psi_{1,f}}\ .$$

We have $\psi = \psi_1(e^H\cdot)$; hence $f \in M_\psi$ and

$$\psi_f(x) = \psi(fx) = \psi_1(e^H fx)\ , \qquad x \in M_+\ .$$

Put $k = e^H f \geqq 0$. We have then

$$k \in M_{\psi_{1,f}} \quad \text{and} \quad \psi_f = \psi_{1,f}(k\cdot)\ .$$

We then define a new weight ψ_2 on M by

$$\psi_2(x) = \psi_f(vxv^*)\ , \qquad x \in M_+\ .$$

It follows then that $\psi_2 \prec \psi$, and that

$$\psi_2(x) = \psi_{1,f}(kvxv^*) = \psi_{1,v}(v^*kvx)\ , \qquad x \in M_+\ .$$

Now $\psi_{1,v}$ is lacunary and has the same centralizer as the lacunary weight φ_e, so that by [3; Theorème 5.2.1.b] there exists a positive operator h_1 affiliated with the center C_{φ_e} of M_{φ_e} such that $\psi_{1,v} = \varphi_e(h_1\cdot)$. We have now $v^*kv \in M_{\varphi_e}$; so putting $h = h_1 v^*kv$, which is affiliated to M_φ, we get $\psi_2 = \varphi(h\cdot)$. Cutting h by a spectral projection so that the reduced h is bounded, and reducing ψ_2 further by the same projection, we complete the proof because $\psi_2 \prec \psi$. q.e.d.

Before stating the next lemma, we need some explanation on a notation: Throughout this section, the symbol $k_1 < k_2$ between two positive operators k_1 and k_2 means that $k_1 \leqq k_2$ and $k_2 - k_1$ is non-singular on the support of k_2.

LEMMA 2.4. *Let N be a properly infinite von Neumann algebra with a faithful semi-finite normal trace τ. For each weight φ on N, let h_φ denote the positive operator affiliated with N such that $\tau(h_\varphi\cdot) = \varphi$. If δ_1 and δ_2 are positive operators affiliated with the center of N such that $\delta_1 < \delta_2$, then there exists a unique projection $[\delta_1, \delta_2[\in \mathfrak{P}_N$ such that for any weight ψ of infinite multiplicity on N*

$$p_N(\psi) \leqq [\delta_1, \delta_2[\Leftrightarrow s(h_\psi)\delta_1 \leqq h_\psi < s(h_\psi)\delta_2\ .$$

PROOF. Let $\mathscr{E}$ be the set of weights of infinite multiplicity on N

which satisfy the condition on the right. Suppose that ψ_1 is a weight in $\mathscr{E}$ and $\psi_2 \sim \psi_1$. We have $\psi_2 = \psi_{1,w}$, $ww^* = s(\psi_1) = s(h_{\psi_1})$ and $w^*w = s(\psi_2) = s(h_{\psi_2})$ where $h_{\psi_2} = w^*h_{\psi_1}w$. It follows then that

$$s(\psi_2)\delta_1 = w^*w\delta_1 = w^*\delta_1 w \leqq w^*h_{\psi_1}w = h_{\psi_2} ;$$
$$h_{\psi_2} = w^*h_{\psi} w \leqq w^*s(h_{\psi_1})\delta_2 w = s(h_{\psi_2})\delta_2 .$$

Since $s(h_{\psi_2})\delta_2 - h_{\psi_2} = w^*(s(h_{\psi_1})\delta_2 - h_{\psi_1})w$, $s(h_{\psi_2})\delta_2 - h_{\psi_2}$ is non-singular by the assumption on h_{ψ_1}; hence $h_{\psi_2} < s(h_{\psi_2})\delta_2$. Let $\psi_1 \in \mathscr{E}$ and e be a projection in M_{ψ_1}. Put $\psi_2 = \psi_{1,e}$. It follows that

$$h_{\psi_2} = h_{\psi_1}e = eh_{\psi_1},\ s(\psi_2) = es(\psi_1) ;$$

hence $s(\psi_2) = e$ and

$$s(\psi_2)\delta_1 = es(\psi_1)\delta_1 \leqq eh_{\psi_1}e \leqq s(\psi_2)\delta_2$$

with $\delta_2 - h_{\psi_1}$ non-singular on $e = s(\psi_2)$. Therefore, we get $h_{\psi_2} < s(\psi_2)\delta_2$.

Now let $\{\psi_n\}$ be a sequence of elements of $\mathscr{E}$ with pairwise orthogonal supports. Putting $\psi = \sum_{k=1}^{\infty} \psi_k$, we get

$$h_{\psi} = \sum_{n=1}^{\infty} h_{\psi_n},\ s(\psi) = \sum_{n=1}^{\infty} s(\psi_n) ;$$
$$s(\psi)\delta_1 = \sum_{n=1}^{\infty} s(\psi_n)\delta_1 \leqq \sum_{n=1}^{\infty} h_{\psi_n} < \sum_{n=1}^{\infty} s(\psi_n)\delta_2 = s(\psi)\delta_2 ,$$

where we understand naturally the sum of infinitely many positive self-adjoint operators $\{h_{\psi_n}\}$ with pairwise orthogonal supports. Thus ψ belongs to $\mathscr{E}$. Therefore, the usual exhaustion arguments show that the set of $p_N(\psi)$, $\psi \in \mathscr{E}$, is precisely the set of all σ-finite subprojections of $\vee\{p_N(\psi): \psi \in \mathscr{E}\} = [\delta_1, \delta_2[\in \mathfrak{P}_N$. q.e.d.

LEMMA 2.5. *Let M be a general von Neumann algebra (not necessarily a factor of type* III$_\lambda$) *and E a faithful normal conditional expectation of M onto a von Neumann subalgebra N.*

(a) *For any weight ψ on N, $\psi \circ E$ is a weight on M with the same support as ψ.*

(b) *If ψ_1 and ψ_2 are weights on N with $\psi_1 \prec \psi_2$, then $\psi_1 \circ E \prec \psi_2 \circ E$.*

(c) *If $\{\psi_j\}$ is a sequence of weights on N with pairwise orthogonal supports, then $\sum_{j=1}^{\infty} (\psi_j \circ E) = (\sum_{j=1}^{\infty} \psi_j) \circ E$.*

PROOF. (a) Let $e = s(\psi)$. We have then $E(e) = e$ and $\psi \circ E(1-e) = 0$. Moreover if $x \in M_e^+$, then $E(x) \in N_e^+$; hence $\psi \circ E$ is faithful on M_e; so $s(\psi \circ E) = e$.

(b) Let u be a partial isometry in N with $uu^* \in N_{\psi_2}$ such that $\psi_{2,u} = \psi_1$ on N. Since the modular automorphism group of $\psi_2 \circ E$ agrees on N with that of ψ_2 by [ ], we have $N_{\psi_2} = M_{\psi_2 \circ E} \cap N$; hence $uu^* \in M_{\psi_2 \circ E}$. For each $x \in M_+$, we get

$$\psi_2 \circ E(uxu^*) = \psi_2(uE(x)u^*) = \psi_1 \circ E(x) ;$$

thus $\psi_1 \circ E \prec \psi_2 \circ E$.

(c) By (a), $\{\psi_j \circ E\}$ have pairwise orthogonal supports, and for each $x \in M_+$ we have

$$\sum_{j=1}^{\infty} (\psi_j \circ E)(x) = (\sum_{j=1}^{\infty} \psi_j)(E(x)) . \qquad \text{q.e.d.}$$

LEMMA 2.6. *Let M, φ, U and ρ be as in Theorem 2.2.*

(a) *If an $h \in M_\varphi^+$ satisfies the condition $\rho s(h) \leqq h < 1$, then the centralizer of the weight $\psi = \varphi(h \cdot)$ satisfies $M_\psi \subset M_\varphi$.*

(b) *If an $h \in M_\varphi^+$ satisfies the condition in (a), then any subweight of $\psi = \varphi(h \cdot)$ is of the form $\varphi(k \cdot)$ for some $k \in M_\varphi^+$ with $\rho s(k) \leqq k < 1$.*

(c) *Let $\psi_j = \varphi(h_j \cdot)$, $j = 1, 2$, with $h_1, h_2 \in M_\varphi^+$ satisfying the condition in (a). If $\psi_2 = \psi_{1,u}$ for a partial isometry $u \in M$ with $uu^* = s(\psi_1)$ and $u^*u = s(\psi_2)$ then we have $u \in M_\varphi$.*

PROOF. (c) Put $k_j = \rho(1 - s(h_j)) + h_j$, $j = 1, 2$. We have then $p \leqq k_j < 1$. By Lemma 1.4(a), we have

$$\begin{aligned} uk_2^{it} = us(h_2)k^{it} &= u(D\psi_2 : D\varphi)_t = uu^*(D\psi_1 : D\varphi)_t \sigma_t^\varphi(u) \\ &= s(h_1)h_1^{it}\sigma_t^\varphi(u) = k_1^{it}s(h_1)\sigma_t^\varphi(u) = k_1^{it}\sigma_t^\varphi(u) , \qquad t \in \boldsymbol{R} . \end{aligned}$$

Hence we get

$$uk_2^{it} = k_1^{it}\sigma_t^\varphi(u) , \qquad t \in \boldsymbol{R} .$$

Let $\theta = \mathrm{Ad}\, U$ be the automorphism of M induced by the unitary U. We have then, as mentioned before Theorem 2.2,

$$\varphi \circ \theta = \varphi(\rho \cdot) .$$

For each $n \in \boldsymbol{Z}$, we define ρ_n as the positive operator affiliated with the center C_φ of M_φ satisfying $\varphi \circ \theta^n = \varphi(\rho_n \cdot)$. We then have, for any $x \in M_+$,

$$\begin{aligned} \varphi \circ \theta_{n+m}(x) &= \varphi(\rho_n \theta^m(x)) = \varphi \circ \theta^m(\theta^{-m}(\rho_n)x) \\ &= \varphi(\rho_m \theta^{-n}(\rho_n)x) , \end{aligned}$$

hence $\rho_{m+n} = \rho_m \theta^{-m}(\rho_n)$. Hence $\sigma_t^\varphi(U^n) = U^n \rho_n^{it}$ since $\varphi_{U^n} = \varphi(\rho_n \cdot)$, and $\rho_n \leqq \rho = \rho_1$ for $n > 0$.

Recalling that M is the crossed product of M_φ by the automorphism $\theta|_{M_\varphi}$, we choose a sequence $\{x_n\}$ in M_φ such that $u = \sum_{m=-\infty}^{+\infty} x_n U^n$. We have, for each n and t,

$$k_1^{it}\sigma_t^\varphi(u) = k_1^{it}\sum_{n=-\infty}^{\infty} x_n U^n \rho_n^{it} = \sum_{n=-\infty}^{\infty} k_1^{it} x_n \theta^n(\rho_n^{it}) U^n ;$$
$$uk_2^{it} = \sum_{n=-\infty}^{+\infty} x_n U^n k_2^{it} = \sum_{n=-\infty}^{\infty} x_n \theta^n(k_2^{it}) U^n .$$

Thus, we get

$$k_1^{it} x_n \theta^n(\rho_n^{it}) = x_n \theta^n(k_2^{it}) , \qquad t \in \boldsymbol{R} ;$$
$$k_1^{it} x_n = x_n \theta^n(k_2^{it}\rho_n^{-it}) .$$

Fixing $n \neq 0$, we show that $k_1^{it}y = y\theta^n(k_2^{it}\rho_n^{-it})$, $t \in \boldsymbol{R}$, implies $y = 0$ for any $y \in M_\varphi$. If this were done, then $u = x_0 \in M_\varphi$; hence the conclusion. Making use of the polar decomposition of y, we can assume that y is a partial isometry such that yy^* commutes with k_1^{it}, $t \in \boldsymbol{R}$, and y^*y commutes with $\theta^n(k_2^{it}\rho_n^{-it})$, $t \in \boldsymbol{R}$. Let $n > 0$. We have

$$k_1^{it} yy^* = y\theta^n(k_2^{it}\rho_n^{-it})y^* , \qquad t \in \boldsymbol{R} .$$

Since $\rho_n \leqq \rho \leqq k_2$, the right hand side of the above equality extends to an analytic function in the upper half plane:
$z \longrightarrow F(z) = y\theta^n(k_2^{iz}\rho_n^{-iz})y^*$ with

$$||F(z)|| \leqq ||k_2^{iz}\rho_n^{-iz}|| = ||(k_2^{-1}\rho)^{-iz}|| \leqq 1 ,$$

for $\mathrm{Im}\, z \geqq 0$. But $k_1 < 1$, so that the left hand side $k_1^{it}yy^*$ extends to a bounded analytic function in the lower half plane. By the Liouville theorem, the function: $t \mapsto k_1^{it}yy^*$ must be constant, which is possible only in the case that $yy^* = 0$. Next let $n < 0$. We use the equality $k_1^{it}\theta^n(\rho_n^{it})y = y\theta^n(k_2^{it})$, $t \in \boldsymbol{R}$. As y^*y commutes with $\theta^n(k_2^{it})$, the same arguments as above applies, provided that $k_1\theta^n(\rho_n) \geqq 1$. But we have $\theta^n(\rho_n) = \rho_{-n}^{-1}$ by the cocycle identity $1 = \rho_0 = \rho_{-n}\theta^n(\rho_n)$. Hence the inequality $k_1\theta^n(\rho_n) \geqq 1$ follows from the inequality $\rho_{-n} \leqq \rho_1 \leqq k_1$.

(a) If u is a partial isometry in M_ψ with $uu^* = s(\psi) = u^*u$, then $\psi_u = \psi$; hence above (c) implies $u \in M_\varphi$, which shows that $M_\psi \subset M_\varphi$.

(b) A subweight of ψ is of the form ψ_e with e a projection in M_ψ. As $M_\psi \subset M_\varphi$, e belongs to M_φ and e commutes with h. It shows that $\psi_e = \varphi(eh\cdot)$ and $\rho s(eh) \leqq eh \leqq h < 1$. q.e.d.

PROOF OF THEOREM 2.2. Let θ and ρ_n be as above, $N = M_\varphi$ and τ be as in the theorem. We first claim that $\sum_{n=-\infty}^{\infty}[\rho_n, \rho_{n-1}[=1$ in $\mathfrak{P}_N$. Let ψ be $\tau(h\cdot)$ with h a positive operator affiliated to N. Then all $\rho_n's$ and h commute, and $\rho_n = \rho_{n-1}\theta^{-(n-1)}(\rho) \leqq \lambda_0\rho_{n-1}$ with $\lambda_0 < 1$, so that there exists an orthogonal sequence $\{e_n\}$ or projections such that $\sum_{n=-\infty}^{+\infty} e_n = 1$, $\{e_n\}$ commute with h and $\rho_n's$, and $e_n\rho_n \leqq he_n < \rho_{n-1}e_n$. Thus any weight ψ on N is decomposed as a sum $\psi = \sum_{n=-\infty}^{\infty} \psi_n$ such that $\{\psi_n\}_{n\in Z}$ is a sequence of weights with orthogonal supports, and $\psi_n = \tau(h_n\cdot)$ with

$s(h_n)\rho_n \leqq h_n < \rho_{n-1}$. Hence $p_N(\psi) = \bigvee_{n\in Z} p_N(\psi_n) \leqq \bigvee_{n\in Z}[\rho_n, \rho_{n-1}[$. Thus we have $\bigvee_{n\in Z}[\rho_n, \rho_{n-1}[=1$. Suppose $n\neq m$. If h_1 and h_2 are positive elements in N such that $s(h_1)\rho_n \leqq h_1 < \rho_{n-1}$ and $s(h_2)\rho_m \leqq h_2 < \rho_{m-1}$, then no non-zero subweights of $\psi_1 = \tau(h_1\cdot)$ and $\tau(h_2\cdot)$ are equivalent; hence $p_N(\psi_1)$ and $p_N(\psi_2)$ are orthogonal. Therefore, $[\rho_n, \rho_{n-1}[$ and $[\rho_m, \rho_{m-1}[$ are orthogonal and our claim follows.

For each σ-finite projection $e \in \mathfrak{P}_N$, put $I(e) = p_M(\psi\circ E)$ where ψ is an arbitrary weight of infinite multiplicity on N with $p_N(\psi) = e$. By Lemma 2.5, $I(e)$ does not depend on the choice of ψ and we have

$$e_1 \leqq e_2 \Rightarrow I(e_1) \leqq I(e_2)\ , \quad I(\bigvee_{j=1}^{\infty} e_j) = \bigvee_{j=1}^{\infty} I(e_j)\ .$$

for any sequence $\{e_j\}$ of σ-finite projections of $\mathfrak{P}_N$ because we can choose ψ_j on N with $p_N(\psi_j) = e_j$ and $s(\psi_j)s(\psi_k) = 0$ for $j \neq k$. We claim that any σ-finite projection in $\mathfrak{P}_M$ of the form $I(e)$ with $e \leqq [\rho_n, \rho_{n-1}[$ is also of the form $I(f)$ with $f \leqq [\rho_1, 1[$. We have

$$I(e) = p_M(\varphi(h\cdot))$$

for some $h \in N_+$ with $\rho_n s(h) \leqq h < \rho_{n-1}$ by hypothesis. For every $x \in N_+$, we get

$$\varphi(hU^k x U^{*k}) = \varphi_{U^k}(\theta^{-k}(h)x) = \varphi(\rho_k\theta^{-k}(h)x)\ .$$

It follows that $I(e) = I(f_k)$ with $f_k = p_N(\tau(\rho_k\theta^{-k}(h)\cdot))$ for $k \in \boldsymbol{Z}$. From the inequality $\rho_n s(h) \leqq h < \rho_{n-1})$ it follows that

$$\theta^{-k}(\rho_n)\theta^{-k}(s(h)) \leqq \theta^{-k}(h) < \theta^{-k}(\rho_{n-1})\ ;$$

hence

$$\rho_{k+n}\theta^{-k}(s(h)) = \rho_k\theta^{-k}(\rho_n)\theta^{-k}(s(h)) \leqq \rho_k\theta^{-k}(h) < \rho_k\theta^{-k}(\rho_{n-1}) = \rho_{n+k-1}\ .$$

Hence, taking $k = 1-n$, we get $f = f_{1-n} \leqq [\rho_1, 1[$ and $I(e) = I(f)$. An application of Lemma 2.3 shows that for any σ-finite non-zero projection $g \in \mathfrak{P}_M$ there exists a non-zero σ-finite projection $e \in \mathfrak{P}_N$ with $I(e) = g_1 \leqq g$. Let $n \in \boldsymbol{Z}$ be such that $e_1 = e[\rho_n, \rho_{n-1}[\neq 0$, and apply Lemma 2.5(b) to show that $g_2 = I(e_1) \leqq g_1$, and hence $g_2 = I(f)$ for some $f \leqq [\rho_1, 1[$. Therefore, the usual exaustion arguments show that any σ-finite projection g in $\mathfrak{p}_M$ is a sum $g = \sum_{n=1}^{\infty} I(f_n)$ where $\{f_n\}$ is an orthogonal sequence of σ-finite projections in $\mathfrak{P}_N$ with $f_n \leqq [\rho_1, 1[$. Putting $f = \sum_{n=1}^{\infty} f_n$, we get $I(f) = g$ and $f \leqq [\rho_1, 1[$. This proves the existence part for weights ψ of infinite multiplicity. Thus Lemma 2.6 assures the rest of the claim in the theorem. q.e.d.

REMARK 2.7. In Theorem 2.2, the condition, $\rho s(h) \leqq h < 1$, can be replaced, for any $n \in \boldsymbol{Z}$, by the condition $\rho_n s(h) \leqq h < \rho_{n-1}$; in particular, for $n = 0$, by $s(h) \leqq h < \rho_{-1}$.

This follows from:

(i) For any weight ω on N and $k \in \boldsymbol{Z}$, $\omega \circ E \sim \omega \circ \theta^k \circ E$;

(ii) If $\omega = \tau(h\cdot)$ on N, then $\omega \circ \theta^k = \tau(\rho_k \theta^{-k}(h)\cdot)$, $k \in \boldsymbol{Z}$;

(iii) $\rho_{n-1}\theta^{1-n}(\rho_1) = \rho_n$ and $\rho_{n-1}\theta^{1-n}(1) = \rho_{n-1}$.

To state the next consequence of Theorem 2.2, we must extend the mapping I defined in the proof of Theorem 2.2. We put, for any projection $e \in \mathfrak{p}_N$,

$I(e) = \bigvee\{I(f): f$ is a σ-finite projection in $\mathfrak{P}_N$ with $f \leqq e\}$. It follows easily that $I(\bigvee_{\alpha \in A} e_\alpha) = \bigvee_{\alpha \in A} I(e_\alpha)$ for any family $\{e_\alpha: \alpha \in A\}$ of projections in $\mathfrak{p}_N$.

COROLLARY 2.8. *Let M, φ, U, ρ and θ be as in Theorem 2.2.*

(i) *For each $n \in \boldsymbol{Z}$, the mapping I is an isomorphism of $(\mathfrak{P}_N)_{[\rho_n, \rho_{n-1}[}$ onto $\mathfrak{p}_M$.*

(ii) *Denoting by $\bar{\theta}$ the automorphism of $\mathfrak{P}_N$ corresponding to θ, we have*

$$I \circ \bar{\theta}(e) = I(e) \text{ for any projection } e \in \mathfrak{P}_N\,;$$
$$\bar{\theta}([\rho_n, \rho_{n-1}[) = [\rho_{n-1}, \rho_{n-2}[\,.$$

(iii) *The map I induces an isomorphism denoted by I again, of $(\mathfrak{P}_N)^{\bar{\theta}}$ onto $\mathfrak{P}_M$ intertwining the action of $\boldsymbol{R}_+^*$:*

$$I\mathfrak{F}_\lambda^N I^{-1} = \mathfrak{F}_\lambda^M\,, \qquad \lambda \in \boldsymbol{R}_+^*\,.$$

PROOF. (i) It follows from Theorem 2.2, its proof, Lemma 2.6 and Remark 2.7 that I is an isomorphism of the lattice of σ-finite projections of $(\mathfrak{P}_N)_{[\rho_n, \rho_{n-1}[}$ onto that of σ-finite projections of $\mathfrak{P}_M$; hence the conclusion.

(ii) By definition, $\bar{\theta}\mathfrak{P}_N(w) = \mathfrak{P}_N(w \circ \theta^{-1})$; hence (i) and (ii) following Remark 2.7 entail the conclusion.

(iii) By (b), the fixed point algebra $(\mathfrak{P}_N)^{\bar{\theta}}$ is isomorphic to $(\mathfrak{P}_N)_{[\rho_n, \rho_{n-1}[}$ under the map: $x \in (\mathfrak{P}_N)^{\bar{\theta}} \longrightarrow x[\rho_n, \rho_{n-1}[\in (\mathfrak{P}_N)_{[\rho_n, \rho_{n-1}[}$, whose inverse is given by the map: $y \in (\mathfrak{P}_N)_{[\rho_n, \rho_{n-1}[} \longrightarrow \sum_{n \in Z} \theta^n(y) \in (\mathfrak{P}_N)^{\bar{\theta}}$. The intertwining property follows from the simple computation:

$$\begin{aligned}\mathfrak{F}_\lambda^M I P_N(\psi) &= \mathfrak{F}_\lambda p_M(\psi \circ E) = p_M(\lambda\psi \circ E) \\ &= I p_N(\lambda\psi) = I\mathfrak{F}_\lambda^N p_N(\psi)\end{aligned}$$

for any weight ψ on N. q.e.d.

COROLLARY 2.9. *On a σ-finite factor M of type III_λ, $\lambda \neq 1$, there exists a faithful normal state ψ such that M_ψ is a maximal abelian subalgebra of M.*

PROOF. With the same notations as above, choose a projection $e \in N$ with $\tau(e) < +\infty$ and an $h \in N_+$ such that $\rho e \leqq h < e$ and N_{ω_h} is a maximal abelian subalgebra of eNe. Put $\psi = \omega_h \circ E$. By Theorem 2.2, we have $M_\psi = N_{\omega_h}$. Since eNe is the centralizer of φ_e, it follows from [3; Lemma 4.2.3] that M_ψ is maximal abelian in eMe. Clearly ψ is a faithful normal functional on $eMe \cong M$. q.e.d.

COROLLARY 2.10. *In a σ-finite factor M of type* III_0, *the centralizer M_ψ of any faithful weight ψ is the relative commutant $C'_\psi \cap M$ of its center C_ψ; namely*

$$M_\psi = C'_\psi \cap M \quad \textit{and} \quad C_\psi = M'_\psi \cap M .$$

PROOF. Since the automorphism $\theta = \mathrm{Ad}\, U$ acts freely on the center C of N, [3; 5.3], we have $N = C' \cap M$, which, together with Theorem 2.2, yields the conclusion. q.e.d.

COROLLARY 2.11. *If M is a σ-finite factor of type* III_0, *for any faithful weights φ, ψ on M there exists a non-singular positive self-adjoint operator h affiliated with the center C_φ of the centralizer M_φ such that*

$$\mathrm{Sp}(\Delta_{\varphi(h\cdot)}) \subset \mathrm{Sp}(\Delta_\psi) .$$

CHAPTER II. INTEGRABLE WEIGHTS ON FACTORS OF TYPE III

II.0. Introduction. The aim here is to introduce and study a very manageable class of faithful semi-finite normal weights (faithful weights for short) on factors of type III. Those weights are called integrable (φ is integrable by definition if the integral $\int_{-\infty}^{\infty} \sigma_t^\varphi(\cdot)dt$ has a weakly dense domain in the von Neumann algebra in question). They play the role of a substitute for almost periodic weights which may fail to exist on factors of type III_1. Like for almost periodic weights M is spanned by the eigenelements for the modular automorphism group, which will be shown using the Fourier transform of the function: $t \longrightarrow \sigma_t^\varphi(x)$. Though no integrable weight is strictly semi-finite, there is still an unbounded normal conditional expectation E_φ from M to the centralizer M_φ, and a semi-finite normal trace τ_φ on M_φ such that $\tau_\varphi \circ E_\varphi = \varphi$. Moreover, the relative commutant of M_φ in M is the center C_φ of M_φ. Unlike almost periodic weights, the integrable weights exist on *any* properly infinite von Neumann algebra and even form a dense subset, for a very strict topology, of the set of faithful weights of infinite multiplicity. In fact, any faithful weights of infinite multiplicity is well approximated by integrable weights commuting with it. Also unlike almost periodic weights, the integrable ones are easily classified, and among them there is a largest one in the sense of the ordering defined in the preliminary section I.1, called the dominant weight. The dominant weight is uniquely characterized, up to the conjugacy under the inner antomorphism group, by the fact that it is invariant within equivalence under the multiplication by *any* positive number.

The dominant weights appeared in fact already in the authors' previous works [3] and [30]. It is shown in [3] that if M is an arbitrary von Neumann algebra, and ω is a faithful weight on $\mathfrak{L}(L^2(\boldsymbol{R}))$ such that $(D\omega: D\mathrm{Tr})_t$, $t \in \boldsymbol{R}$, is the translation in $L^2(\boldsymbol{R})$ by t, then the weight $\bar{\omega} = \varphi \otimes \omega$ on $M \otimes \mathfrak{L}(L^2(\boldsymbol{R}))$ does not depend, within equivalence, on the choice of a faithful weight φ on M. It turns out that $\varphi \otimes \omega$ is dominant. In [30], the weight $\tilde{\tau}$ dual to a trace τ on a semi-finite von Neumann algebra N (yielding the continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$) was studied. In fact, $\tilde{\tau}$ is dominant also.

The integrable weights are characterized by the σ-strong continuity of the mapping: $\lambda \in \boldsymbol{R}_+^* \rightarrow \mathfrak{F}_\lambda^M(p_M(\varphi)) \in \mathfrak{P}_M$. Their study enables us to determine the smooth flow of weights, i.e., the restriction on the flow of weights $\mathfrak{F}^M$ to its σ-strongly continuous part. Integrable weights are then classified, up to equivalence, by their multiplicity: a normal weight, not necessarily semi-finite, on the smooth flow of weights. The smooth flow of weights (P_M, F^M) is isomorphic, under a trivial change of scales, to the restriction of the $\{\theta_s\}$ to the center C_N of N in an arbitrary continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$. When M is a factor, it is ergodic and its kernel is precisely the invariant $S(M) \cap \boldsymbol{R}_+^*$. In particular, it is trivial when M is of type III_1, which has striking consequences on faithful weights φ of infinite multiplicity: for example their domain $\mathfrak{m}_\varphi$ is, up to conjugacy under inner automorphisms, independent of φ. When M is a factor of type III_0, the smooth flow of weights is isomorphic to the flow built on the restriction of θ to $C =$ the center of N under the ceiling function $d\tau \circ \theta^{-1}/d\tau$, in an arbitrary discrete decomposition $M = W^*(N, \theta)$.

II.1. Dominant weights. Throughout this chapter, we shall keep the following notations:

The Plancherel measure on $\boldsymbol{R}$ is denoted by $dr, ds, dp, dq, \cdots$, i.e., $\dfrac{1}{\sqrt{2\pi}} \times$ the Lebesgue measure;

The unitary of $\mathfrak{L}(L^2(\boldsymbol{R})) = F_\infty$ defining the Fourier transformation is denoted by F, i.e.,

$$Ff(p) = \int e^{-isp} f(s)ds, f \in L^1(\boldsymbol{R}) \cap L^2(\boldsymbol{R}) ;$$

For each $s \in \boldsymbol{R}$, V_s is the unitary of F_∞ such that

$$(V_s f)(p) = e^{isp} f(p) , \qquad f \in L^2(\boldsymbol{R}) , \ p \in \boldsymbol{R} ;$$

For each $t \in \boldsymbol{R}$, U_t is the unitary of F_∞ such that

$$(U_t f)(p) = f(p + t) , \qquad f \in L^2(\boldsymbol{R}) , \ p \in \boldsymbol{R} ;$$

Also, V (resp. U) is the isomorphism of $L^\infty(\boldsymbol{R})$ into F_∞ such that

$$V(\exp(it\cdot)) = V_t , \qquad t \in \boldsymbol{R} ,$$
$$(\text{resp. } U(\exp(it\cdot)) = U_t) ;$$

The usual trace on F_∞ is denoted by Tr, and ω is the weight on F_∞ such that

$$(D\omega : D\mathrm{Tr})_t = U_t , \qquad t \in \boldsymbol{R} .$$

THEOREM 1.1. *Let M be a properly infinite von Neumann algebra with separable predual.*

(i) *There exists a faithful weight $\bar{\omega}$ of infinite multiplicity on M such that*

$$\bar{\omega} \sim \lambda\bar{\omega} . \qquad \lambda > 0 .$$

(ii) *The weight $\bar{\omega}$ satisfying* (a) *is unique up to equivalence.*

PROOF. (i) Looking at ω on F_∞, we have

$$(*) \qquad (D\omega_{V_s} : D\omega)_t = V_s^* U_t V_s U_t^* = e^{ist} ,$$

so that $\omega_{V_s} = e^{-s}\omega$, $s \in \boldsymbol{R}$. By the proper infiniteness of M, we identify M with $P \otimes F_\infty$, where P is a properly infinite von Neumann algebra isomorphic to M. For any weight φ of infinite multiplicity on P, $\bar{\omega} = \varphi \otimes \omega$ satisfies condition (i) on M since

$$\lambda\bar{\omega} = \varphi \otimes \lambda\omega \sim \varphi \otimes \omega = \bar{\omega} , \qquad \lambda > 0 .$$

(ii) Suppose $\bar{\omega}_1$ and $\bar{\omega}_2$ satisfy condition (i) on $M = P \otimes F_\infty$. By Lemma 1.9b, we choose weights φ_1 and φ_2 on P satisfying condition (i) and such that

$$\bar{\omega}_j \sim \varphi_j \otimes \mathrm{Tr} , \qquad j = 1, 2 .$$

For each $s \in \boldsymbol{R}$, there exists a unitary $X_j(s) \in P$ such that

$$(**) \qquad \varphi_{j, X_j(s)} = e^s \varphi_j \ \text{ i.e.,} \ \ \sigma_t^{\varphi_j}(X_j(s)) = e^{ist} X_j(s) .$$

By the separability of the predual P_* of P, we can select a Borel map: $s \in \boldsymbol{R} \to X_j(s) \in P$, $j = 1, 2$. Let X_j be the element of $P \otimes U(L^\infty(\boldsymbol{R})) = (1 \otimes U)(L^\infty(\boldsymbol{R}, P))$ corresponding to the above map: $s \to X_j(s)$. We have then

$$\sigma_t^{\varphi_j \otimes \mathrm{Tr}}(X_j) = (\sigma_t^{\varphi_j} \otimes \iota)(X_j) = X_j(1 \otimes U_t) .$$

Hence we get

$$X_j^* \sigma_t^{\varphi_j \otimes \mathrm{Tr}}(X_j) = (D(\varphi_j \otimes \omega) : D(\varphi_j \otimes \mathrm{Tr}))_t , \qquad t \in \boldsymbol{R} .$$

Thus we have

$$\varphi_j \otimes \omega \sim \varphi_j \otimes \mathrm{Tr} \sim \bar{\omega}_j , \qquad j = 1, 2 .$$

By [3: Lemma 1.2.5], we conclude that

$$\bar{\omega}_1 \sim \varphi_1 \otimes \omega \sim \varphi_2 \otimes \omega \sim \bar{\omega}_2 .$$

q.e.d.

DEFINITION 1.2. A dominant weight on a properly infinite von Neumann algebra with separable predual is a faithful weight of infinite multiplicity satisfying condition (i) in Theorem 1.1.

THEOREM 1.3. *For a faithful weight $\bar{\omega}$ on a properly infinite von Neumann algebra M with separable predual, consider the conditions*:

(i) *$\bar{\omega}$ is dominant;*

(ii) *$\bar{\omega} = \varphi \otimes \omega$ in some factorization $M = P \otimes F_\infty$, where $P \cong M$;*

(iii) *$\bar{\omega}$ is dual to the trace τ on N in a continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$ where $\tau \circ \theta_s = e^{-s}\tau$, $s \in \boldsymbol{R}$;*

(iv) *There exists a faithful weight $\bar{\omega}'$ on M which does not commute with $\bar{\omega}$ but such that*

$$\sigma_{t_1}^{\bar{\omega}'} \sigma_{t_2}^{\omega} = \sigma_{t_2}^{\bar{\omega}} \sigma_{t_1}^{\bar{\omega}'} , \qquad t_1, t_2 \in \boldsymbol{R} ;$$

(v) $\bar{\omega} \sim \lambda\bar{\omega}$, $\lambda > 0$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv). *If M is a factor, then* (iv) $\Rightarrow$ (v). *Moreover if M is of type* III, *then* (v) $\Rightarrow$ (i).

LEMMA 1.4. *Let P be a properly infinite von Neumann algebra acting on the separable Hilbert space $\mathfrak{H}_\varphi$ corresponding to a given faithful weight φ on P.*

(a) *The weight $\bar{\omega} = \varphi \otimes \omega$ is dominant on $M = P \otimes F_\infty$.*

(b) *$M_{\bar{\omega}} = W^*(P, \boldsymbol{R}, \sigma^\varphi)$ on $\mathfrak{H}_\varphi \otimes L^2(\boldsymbol{R})$.*

(c) *$M_{\bar{\omega}}$ is generated by the $1 \otimes U_t$, $t \in \boldsymbol{R}$, and the element of $P \otimes V(L^\infty(\boldsymbol{R}))$ corresponding to the function: $t \in \boldsymbol{R} \to \sigma_t^\varphi(x) \in P$, $x \in P$.*

(d) *The action θ^φ of $\boldsymbol{R}$ on $W^*(P, \boldsymbol{R}, \sigma^\varphi) = M_{\bar{\omega}}$ dual to σ^φ is given by:*

$$\theta_s^\varphi(x) = (1 \otimes V_s)x(1 \otimes V_s)^* , \qquad x \in M_{\bar{\omega}},\ s \in \boldsymbol{R} .$$

(e) *There exists a faithful semi-finite normal trace τ on $M_{\bar{\omega}}$ such that for any unitary $w \in M$ with $\bar{\omega}_w = \lambda\bar{\omega}$ we have*

$$\tau_w = \lambda\tau \ ,$$

where τ_w should be naturally understood since $wM_{\bar{\omega}}w^* = M_{\bar{\omega}}$.

(f) *For any $\lambda > 0$, there is a unique automorphism $F_\lambda^{\bar{\omega}}$ of the center $C_{\bar{\omega}}$ of $M_{\bar{\omega}}$ such that*

$$F_\lambda^{\bar{\omega}}(x) = wxw^* \ , \qquad x \in C_{\bar{\omega}} \ ,$$

with any unitary $w \in M$ such that $\bar{\omega}_w = \lambda\bar{\omega}$. Moreover, $F_\lambda^{\bar{\omega}}$ is the restriction of $\theta_{-\mathrm{Log}\lambda}^{\varphi}$ to $C_{\bar{\omega}}$.

PROOF. Assertion (a) follows from the proof of Theorem 1.1. (i) and the definition of a dominant weight.

(b) Let Δ_φ be the modular operator on $\mathfrak{H}_\varphi$ associated with φ. By definition, we have

$$M_{\bar{\omega}} = M \cap \{\Delta_\varphi^{it} \otimes U_t : t \in \boldsymbol{R}\}' \ ,$$

so that the commutant $M_{\bar{\omega}}'$ of $M_{\bar{\omega}}$ in $\mathfrak{H}_\varphi \otimes L^2(\boldsymbol{R})$ is the von Neumann algebra generated by $M' = P' \otimes \boldsymbol{C}$ and $\Delta_\varphi^{it} \otimes U_t$, $t \in \boldsymbol{R}$. It follows from [30; Corollary 5.13] that $M_{\bar{\omega}} = W^*(P, \boldsymbol{R}, \sigma^\varphi)$ on $\mathfrak{H}_\varphi \otimes L^2(\boldsymbol{R})$.

(c) This is an immediate consequence of (b) and the definition of $W^*(P, \boldsymbol{R}, \sigma^\varphi)$.

(d) This is easily seen by checking directly that θ_s^φ and $\mathrm{Ad}(1 \otimes V_s)$ agree on the generators considered in (c).

(e) If w_1 and w_2 are unitaries such that $\bar{\omega}_{w_1} = \bar{\omega}_{w_2}$ then $w_2 w_1^* \in M_{\bar{\omega}}$; so it is enough to find τ with $\tau \circ \theta_s^\varphi = e^{-s}\tau$, $s \in \boldsymbol{R}$, which follows from [30; Lemma 8.2].

(f) As in (e), we see that $\mathrm{Ad}\, w$ restricted to $C_{\bar{\omega}}$ is independent, for any $\lambda > 0$, of the choice of the unitary $w \in M$ with $\bar{\omega}_w = \lambda\bar{\omega}$. Choosing $w = 1 \otimes V_{-\mathrm{Log}\lambda}$, we complete the proof. q.e.d.

PROOF OF THEOREM 1.3. Both weights of the form $\varphi \otimes \omega$ and dual weights $\tilde{\tau}$ to a trace τ on N such that $\tau \circ \theta_s = e^{-s}\tau$, $s \in \boldsymbol{R}$, are dominant on $M = P \otimes F_\infty$ and $M = W^*(N, \boldsymbol{R}, \theta)$ respectively. Hence the equivalence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follows easily from the uniqueness of dominant weights.

Let ω' be the weight on F_∞ such that $(D\omega' : D\mathrm{Tr})_t = V_t$, $t \in \boldsymbol{R}$. It is easy to check that $(D(\varphi \otimes \omega') : D(\varphi \otimes \omega))_t = 1 \otimes (D\omega' : D\omega)_t$ is not a one parameter unitary group; hence for each weight of the form $\varphi \otimes \omega$ there exists a weight $\varphi \otimes \omega' = \bar{\omega}'$ which does not commute with $\varphi \otimes \omega$ but whose modular automorphism group $\sigma_t^{\varphi\otimes\omega'}$ commutes with $\sigma_s^{\varphi\otimes\omega}$, $s, t \in \boldsymbol{R}$. Hence we have shown the implication (i) $\Rightarrow$ (iv).

Now, we show the implication (iv) $\Rightarrow$ (v) when M is a factor. Let $u_t = (D\bar{\omega}' : D\bar{\omega})_t$, $t \in \boldsymbol{R}$. For each $s, t \in \boldsymbol{R}$ and $x \in M$, we have, by hypothesis,

$$\begin{aligned} u_t \sigma^{\bar{\omega}}_{s+t}(x) u_t^* &= \sigma_t^{\bar{\omega}'} \circ \sigma_s^{\bar{\omega}}(x) = \sigma_s^{\bar{\omega}} \circ \sigma_t^{\bar{\omega}'}(x) \\ &= \sigma_s^{\bar{\omega}}(u_t \sigma_t^{\bar{\omega}}(x) u_t^*) \\ &= \sigma_s^{\bar{\omega}}(u_t) \sigma^{\bar{\omega}}_{s+t}(x) \sigma_s^{\bar{\omega}}(u_t^*) \, . \end{aligned}$$

Hence u_t and $\sigma_s^{\bar{\omega}}(u_t)$ give rise to the same inner automorphism of M, so that $\sigma_s^{\bar{\omega}}(u_t)$ is of the form $f(s, t)u_t$, (M being a factor). Namely, u_t is an eigen operator of $\{\sigma_s^{\bar{\omega}}\}$; hence $f(s_1 + s_2, t) = f(s_1, t)f(s_2, t)$ for each $s_1, s_2, t \in \boldsymbol{R}$. As $u_{t_1+t_2} = u_{t_1}\sigma^{\bar{\omega}}_{t_1}(u_{t_2})$, we get

$$f(s, t_1 + t_2) = f(s, t_1)f(s, t_2) \qquad s, t_1, t_2 \in \boldsymbol{R} \, .$$

Since f is a continuous function on $\boldsymbol{R} \times \boldsymbol{R}$ with modulus one, there exists a real number $\alpha \in \boldsymbol{R}$ such that $f(s, t) = e^{i\alpha st}$. As $\bar{\omega}$ does not commute with $\bar{\omega}'$, we have $\alpha \neq 0$. For $\lambda > 0$, let $s = (1/\alpha) \operatorname{Log} \lambda$. We have then

$$\sigma_t^{\bar{\omega}}(u_s) = \lambda^{it} u_s \, , \qquad t \in \boldsymbol{R} \, ;$$

therefore $\bar{\omega}_{u_s} = \lambda \bar{\omega}$.

We postpone the proof of the implication: (v) $\Rightarrow$ (i) for a von Neumann algebra of type III until the end of the next section. q.e.d.

Given two weights φ and ψ on factor M with commuting modular automorphism groups σ^{φ} and σ^{ψ}, we have seen in the above arguments that there exists a constant $\alpha \in \boldsymbol{R}$ such that

$$\sigma_t^{\varphi}((D\psi : D\varphi)_s) = e^{i\alpha st}(D\psi : D\varphi)_s \, , \qquad s, t \in \boldsymbol{R} \, .$$

DEFINITION 1.5. The constant α is called the Stone-von Neumann constant of the pair φ and ψ, and denoted by $\alpha(\varphi, \psi)$. It is clear that φ and ψ commute if $\alpha(\varphi, \psi) = 0$. When $\alpha(\varphi, \psi) \neq 0$, φ and ψ are said to be quasi-commuting.

THEOREM 1.6. *Let M be an infinite factor with separable predual. Let $\{\bar{\omega}_1, \bar{\omega}_2\}$ and $\{\bar{\omega}_1', \bar{\omega}_2'\}$ be two pairs of quasi-commuting dominant weights on M. Then the following two conditions are equivalent*:

(i) *There exists a unitary $u \in M$ such that*

$$\bar{\omega}_{1,u} = \bar{\omega}_1' \text{ and } \bar{\omega}_{2,u} = \bar{\omega}_2' \, ;$$

(ii) $$\alpha(\bar{\omega}_1, \bar{\omega}_2) = \alpha(\bar{\omega}_1', \bar{\omega}_2') \, .$$

The implication: (i) $\Rightarrow$ (ii) is trivial. The proof of the reversed implication requires further analysis of dominant weights, so we postpone it until the end of III.5.

II.2. Integrable weights and the smooth flow of weights. In the sequel we shall meet expressions of the type $\int_{-\infty}^{\infty} x(t)dt$ for a σ-strongly continuous function $x(\cdot)$ on $\boldsymbol{R}$ with values in the positive part of a von Neumann algebra M. We write $\int_{-\infty}^{\infty} x(t)\,dt \in M$ to mean that the increasing net $\int_K x(t)dt$, K compact subset of $\boldsymbol{R}$, is bounded above in M_+, and of course $\int_{-\infty}^{\infty} x(t)dt$ stands for the least upper bound.

DEFINITION 2.1. Let φ be a weight on a von Neumann algebra M with support $e = s(\varphi)$. We say that φ is integrable if

$$\mathfrak{q}_\varphi = \left\{x \in M_e\colon \int_{-\infty}^{\infty} \sigma_t^\varphi(x^*x)dt \quad \text{exists}\right\}$$

is dense in M_e for the σ-weak topology.

As in the case of weights, we have the following properties:

(1°) $\mathfrak{q}_\varphi$ is a left ideal;

(2°) $\mathfrak{p}_\varphi = \mathfrak{q}_\varphi^*\mathfrak{q}_\varphi$ is a hereditary *-subalgebra of M_e;

(3°) $\mathfrak{p}_\varphi$ is linearly spanned by its positive part and

$$\mathfrak{p}_\varphi^+ = \left\{x \in M_e^+\colon \int_{-\infty}^{\infty} \sigma_t^\varphi(x)dt \quad \text{exists}\right\} ;$$

(4°) The integral

$$E_\varphi(x) = \int_{-\infty}^{\infty} \sigma_t^\varphi(x)dt$$

makes sense for every $x \in \mathfrak{p}_\varphi$ and takes values in the centralizer M_φ of φ;

(5°) $\mathfrak{p}_\varphi$ is a two sided M_φ-module and

$$E_\varphi(axb) = aE_\varphi(x)b,\ x \in \mathfrak{p}_\varphi,\ a,\ b \in M_\varphi ;$$

(6°) For any bounded increasing net $\{x_i\}$ in $\mathfrak{p}_\varphi^+$ we have

$$E_\varphi(\sup x_i) = \sup E_\varphi(x_i) ,$$

where $\sup x_i$ is not necessarily in $\mathfrak{p}_\varphi^+$, and so we understand the above equality in the extended sense allowing $+\infty$ as its value;

(7°) φ is integrable if and only if $\mathfrak{p}_\varphi^+$ contains an increasing net of projections converging σ-strongly to 1.

THEOREM 2.2. *For a weight φ on a properly infinite von Neumann algebra M with separable predual, the following three statements are equivalent*:

(i) *φ is integrable;*

(ii) *The map: $\lambda \in \boldsymbol{R}_+^* \mapsto \mathfrak{F}_\lambda^M p_M(\varphi) \in \mathfrak{P}_M$ is σ-strongly continuous;*

(iii) $\varphi \prec \bar{\omega}$ *for some, and hence all, dominant weight* $\bar{\omega}$ *on* M.

The proof of the implication: (i) $\Rightarrow$ (iii) relies on the following:

LEMMA 2.3. *For an integrable weight* φ *on a von Neumann algebra* M, *the following statement hold*:

(a) *For all* $y \in \mathfrak{p}_\varphi$ *and* $\lambda \in \boldsymbol{R}_+^*$, *the integral*

$$\hat{y}_\lambda = \int_{-\infty}^{\infty} \sigma_t^\varphi(y) \lambda^{-it} dt$$

belongs to $M(\sigma^\varphi, \{\lambda\})$, i.e., $\sigma_t^\varphi(\hat{y}_\lambda) = \lambda^{it}\hat{y}_\lambda$, $t \in \boldsymbol{R}$;

(b) $\mathfrak{p}_\varphi$ *is a two sided module over* M_φ *with*

$$(xyz)\hat{}_\lambda = x\hat{y}_\lambda z, \ x, z \in M_\varphi \quad \textit{and} \quad y \in \mathfrak{p}_\varphi, \lambda \in \boldsymbol{R}_+^* ;$$

(c) *Let* $\mathfrak{A}_\varphi$ *be the algebra of analytic*, (entire), *elements for* σ^φ. *Then* $\mathfrak{A}_\varphi \cap \mathfrak{p}_\varphi$ *is a* σ*-weakly dense *-subalgebra of* M *and*

$$y = \frac{1}{2\pi} \int_0^\infty \hat{y}_\lambda \frac{d\lambda}{\lambda} , \quad y \in \mathfrak{A}_\varphi \cap \mathfrak{p}_\varphi ;$$

(d) *For any pair* e_1, e_2 *of projections in* M_φ *which are not centrally orthogonal in* M, *there exist a* $\lambda > 0$ *and a partial isometry* $u \in M(\sigma^\varphi, \{\lambda\})$ *such that*

$$u^*u \leqq e_1 \quad \textit{and} \quad uu^* \leqq e_2 .$$

PROOF. (a) This follows immediately from the observation that for each $\psi \in M_*$, the function: $t \in \boldsymbol{R} \mapsto \psi(\sigma_t^\varphi(y))$ is integrable if y belongs to $\mathfrak{p}_\varphi$.

(b) If $x \in \mathfrak{H}_\varphi$ and $y \in M_\varphi$, then we have $\sigma_t^\varphi(y^*x^*xy) = y^*\sigma_t^\varphi(x^*x)y$, $t \in \boldsymbol{R}$. Hence our assertion follows.

(c) For any $x \in \mathfrak{p}_\varphi^+$ and non-negative $f \in L^1(\boldsymbol{R})$, we have

$$\sigma_t^\varphi(\sigma_f^\varphi(x)) = \int_{-\infty}^{\infty} f(s)\sigma_{s+t}^\varphi(x) ds .$$

Hence a direct application of Fubini's theorem shows that $\sigma_f^\varphi(\mathfrak{p}_\varphi^+) \subset \mathfrak{p}_\varphi^+$; hence $\sigma_f^\varphi(\mathfrak{p}_\varphi) \subset \mathfrak{p}_\varphi$ for any $f \in L^1(\boldsymbol{R})$. Choosing f to be $n^{1/2}\pi^{-1/2} \exp(-nt^2)$, we conclude the density of $\mathfrak{A}_\varphi \cap \mathfrak{p}_\varphi$. For any $y \in \mathfrak{A}_\varphi \cap \mathfrak{p}_\varphi$ and $\psi \in M_*$, the function: $t \in \boldsymbol{R} \mapsto \psi(\sigma_t^\varphi(y))$ is analytic and integrable, so that the Fourier inversion formula applies.

(d) There exists a $y \in M$ with $e_2ye_1 \neq 0$. It follows from (c) that there exists an $x \in \mathfrak{A}_\varphi \cap \mathfrak{p}_\varphi$ with $e_2xe_1 \neq 0$. By (c) once again, $z = (e_2xe_1)\hat{}_\lambda \neq 0$ for some $\lambda \in \boldsymbol{R}_+^*$. Now let $z = uh$ be the polar decomposition of z. We have then $h \in M_\varphi$ and $u \in M(\sigma^\varphi, \{\lambda\})$, and also $e_2u = ue_1 = u$ by construction. q.e.d.

LEMMA 2.4. *If φ is an integrable weight on a von Neumann algebra M, then we conclude the following*:

(a) *Any other weight ψ on M with $\psi \prec \varphi$ is integrable;*

(b) *The tensor product weight $\varphi \otimes \psi$ on $M \otimes N$ with ψ an arbitrary weight on another von Neumann algebra N is integrable.*

PROOF. (a) It is clear that any weight equivalent to an integrable one is integrable, and it follows from property (5°) fop $\mathfrak{p}_\varphi$ that any subweight of an integrable weight is integrable. Thus the assertion follows.

(b) Use that the algebraic tensor product $\mathfrak{p}_\varphi \otimes N$ of $\mathfrak{p}_\varphi$ and N is contained in $\mathfrak{p}_{\varphi\otimes\psi}$. q.e.d.

PROOF OF THEOREM 2.2. (ii) $\Rightarrow$ (iii): Let $e = p_M(\varphi)$ and $f = \bigvee_{\lambda \in \boldsymbol{Q}} \mathfrak{F}_\lambda^M(e)$. By assumption, each $\mathfrak{F}_\lambda^M(e)$, $\lambda > 0$, is a σ-strong limit of $\mathfrak{F}_{\lambda_n}^M(e)$ for some sequence $\{\lambda_n\}$ in $\boldsymbol{Q}$, so that $f = \bigvee_{\lambda>0} \mathfrak{F}_\lambda^M(e)$ is a σ-finite invariant projection. By Theorem 1.1, we have $f = p_M(\bar{\omega})$ with $\bar{\omega}$ a dominant weight. Hence $e \leqq p_M(\bar{\omega})$. Therefore we get $\varphi \prec \check{\varphi} \prec \bar{\omega}$.

(iii) $\Rightarrow$ (i): Let ω be the weight on $F_\infty = \mathfrak{L}(L^2(\boldsymbol{R}))$ defined above. It follows that for each $f \in L^\infty(\boldsymbol{R})$, $\sigma_t^\omega(V(f)) = V(f_t)$ where $f_t(s) = f(s+t)$, so that we have $V(L^\infty(\boldsymbol{R}) \cap L^1(\boldsymbol{R})) \subset \mathfrak{p}_\omega$. Hence ω is integrable. Therefore the integrability of a dominant weight $\bar{\omega}$ follows from Theorem 1.3.iii and Lemma 2.4.b. Hence Lemma 2.4.a entails the integrability of any φ with $\varphi \prec \bar{\omega}$.

(i) $\Rightarrow$ (iii): We first observe that given a dominant weight $\bar{\omega}$ on M, the weight ψ on $P = M \otimes F_2$ defined by $\psi(\sum x_{ij} \otimes e_{ij}) = \varphi(x_{11}) + \bar{\omega}(x_{22})$ is integrable if φ is. Let $e \otimes e_{11}$, $e \in M_\varphi$, be a non-zero subprojection of $s(\varphi) \otimes e_{11}$ in P_ψ. Since $e \otimes e_{11}$ and $1 \otimes e_{22}$ are not centrally orthogonal in P, there exists, by Lemma 2.3.d, a partial isometry v in P belonging to $P(\sigma^\varphi, \{\lambda\})$ for some $\lambda > 0$ such that $v^*v \leqq e \otimes e_{11}$ and $vv^* \leqq 1 \otimes e_{22}$. Let w be a unitary in $M(\sigma^{\bar{\omega}}, \{\lambda^{-1}\})$, where the existence of w is granted by Theorem 1.3. Then $(w \otimes e_{22})v$ belongs to P_ψ. Thus we have shown that any nonzero subprojection of $s(\varphi) \otimes e_{11}$ in P_ψ is not disjoint from $1 \otimes e_{22}$, with respect to P_ψ. As $\bar{\omega}$ is of infinite multiplicity $1 \otimes e_{22}$ is a properly infinite projection of P_ψ, so that $s(\varphi) \otimes e_{11} \prec 1 \otimes e_{22}$ in P_ψ, which means that $\varphi \prec \bar{\omega}$, cf(1.1.4.b).

(iii) $\Rightarrow$ (ii): Since $p_M(\varphi) = p_M(\check{\varphi})$ for any weight φ, we may assume that φ is of infinite multiplicity. It follows then that $\varphi \sim \bar{\omega}_p$ for some projection p in the center $C_{\bar{\omega}}$ of the centralizer $M_{\bar{\omega}}$. By Theorem 1.3, with a continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$ of M, $\bar{\omega}$ is dual to a faithful semi-finite normal trace τ on N such that $\tau \circ \theta_s = e^{-s}\tau$. Denoting the one parameter unitary group generating M together with N by $\{u(s): s \in \boldsymbol{R}\}$, we have $\bar{\omega}_{u(s)} = e^{-s}\bar{\omega}$. We have then

$$e^{-s}\varphi = e^{-s}\bar{\omega}_p = (e^{-s}\bar{\omega})p = (\bar{\omega}_{u(s)})_p$$
$$= [\bar{\omega}_{\theta_s(p)}]_{u(s)} .$$

Hence we have

$$\mathfrak{F}^M_{e^{-s}}(p_M(\varphi)) = p_M(e^{-s}\varphi) = p_M([\bar{\omega}_{\theta_s(p)}]_{u(s)})$$
$$= p_M(\bar{\omega}_{\theta_s(p)}) = p_{\bar{\omega}}(\theta_s(p)) ,$$

where $p_{\bar{\omega}}$ is the isomorphism of $C_{\bar{\omega}}$ into $\mathfrak{P}_M$ defined in Theorem I.1.11. Since the map: $s \in \boldsymbol{R} \mapsto \theta_s(p) \in C_{\bar{\omega}}$ is σ-strongly continuous, the map: $\lambda \in \boldsymbol{R}^*_+ \mapsto \mathfrak{F}^M_\lambda(p_M(\varphi))$ is σ-strongly continuous. q.e.d.

COROLLARY 2.5. *For a properly infinite von Neumann algebra M with separable predual, the following three statements hold:*

(i) *There exists a largest σ-finite projection $d_M \in \mathfrak{P}_M$ such that the map: $\lambda \in \boldsymbol{R}^*_+ \mapsto \mathfrak{F}^M_\lambda(d)$ is σ-strongly continuous;*

(ii) $$p_M(\bar{\omega}) = d_M , \quad \textit{and}$$
$$\mathfrak{F}^M_\lambda | (\mathfrak{P}_M)_{d_M} = p_{\bar{\omega}} \circ F^{\bar{\omega}}_\lambda \circ p^{-1}_{\bar{\omega}} , \quad \lambda \in \boldsymbol{R}^*_+ ,$$

where $F^{\bar{\omega}}_\lambda$ is the automorphism of $C_{\bar{\omega}}$ defined in Lemma 1.4;

(iii) *For each continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$, the restriction of the action: $\lambda \longrightarrow \theta_{-\log\lambda}$ of $\boldsymbol{R}^*_+$ to the center C_N of N is isomorphic to the restriction of $\mathfrak{F}^M$ to $(\mathfrak{P}_M)_{d_M}$.*

PROOF. With $d_M = p_M(\bar{\omega})$, our assertions follow immediately from Theorem 2.2 or its proof. q.e.d.

DEFINITION 2.6 For a properly infinite von Neumann algebra M with separable predual, the couple $\{P_M, F^M\}$ consisting of the reduced von Neumann algebra $(\mathfrak{P}_M)_{d_M}$ and the restriction F^M of the action $\mathfrak{F}^M$ of $\boldsymbol{R}^*_+$ is called the (smooth) flow of weights of M.

The map p_M is clearly an order preserving bijection from the set of equivalence classes of integrable weights of infinite multiplicity to the set of projections of P_M. We now describe the set of equivalence classes of integrable weights of arbitrary multiplicity. To this end, we need the following regularity property of integrable weights.

LEMMA 2.7. *For an integrable weight φ on a properly infinite von Neumann algebra M with separable predual, there exist a σ-weakly dense *-subalgebra $\mathfrak{c}_\varphi$ of M contained in $\mathfrak{m}_\varphi \cap \mathfrak{p}_\varphi$, which is a two sided M_φ-module, and a unique faithful semi-finite normal trace τ_φ on M_φ such that*

$$\tau_\varphi \circ E_\varphi(x) = \varphi(x)\ , \quad x \in \mathfrak{c}_\varphi\ ,$$

where E_φ is given by the integration:

$$E_\varphi(x) = \int_{-\infty}^{\infty} \sigma_t^\varphi(x)dt\ .$$

PROOF. By Theorem 2.2, we may assume that φ is dominant, because if the conclusion of this lemma holds for an integrable weight, then it is also true for all subweight of this weight. By Theorem 1.3, a dominant weight $\bar{\omega}$ is dual to the trace τ on N in a continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$ where $\tau \circ \theta_s = e^{-s}\tau$, $s \in \boldsymbol{R}$. Therefore, we must show that τ and $\bar{\omega}$ are connected by the formula

$$\tau \circ E_{\bar{\omega}}(x) = \bar{\omega}(x)$$

for each x in some σ-weakly dense *-subalgebra $\mathfrak{c}_{\bar{\omega}}$ of M, which is a two sided N-module. This is not entirely trivial; but it can be shown by a routine rearrangement of the arguments in [30; Lemma 5.19]. Thus we leave it to the reader. q.e.d.

THEOREM 2.8. *Let M be a properly infinite von Neumann algebra with separable predual and no type I component.*

(i) *For each integrable weight φ on M, there exists a unique normal, but not necessarily semi-finite, weight ν_φ on P_M such that*

$$\nu_\varphi(p_M(\psi)) = \tau_\varphi(c_\varphi(\psi))$$

for every integrable weight ψ on M.

(ii) *The map: $\varphi \mapsto \nu_\varphi$ is a bijection from the set of equivalence classes of integrable weights on M onto the set of normal, but not necessarily semi-finite weights on P_M, which enjoys the properties:*

$$\varphi_1 \prec \varphi_2 \Leftrightarrow \nu_{\varphi_1} \leqq \nu_{\varphi_2}\ ;$$
$$\nu_{\varphi_1 \oplus \varphi_2} = \nu_{\varphi_1} + \nu_{\varphi_2}\ ;$$
$$\nu_{\lambda\varphi} = \lambda\nu_\varphi \circ (F_\lambda^M)^{-1}\ , \quad \lambda \in \boldsymbol{R}_+^*\ .$$

PROOF. (i) For each $x \in (P_M)^+$, we put

$$\nu_\varphi(x) = \tau_\phi \circ p_\varphi^{-1}(x p_M(\varphi))\ .$$

Since $p_\varphi(c_\varphi(\psi)) = p_M(\varphi)p_M(\psi)$ by Theorem I.1.11, we get

$$\nu_\varphi(p_M(\psi)) = \tau_\varphi(c_\varphi(\psi))\ .$$

(ii) By construction ν_φ only depends on the equivalence class of φ. Let $\bar{\omega}$ be a dominant weight on M. Making use of $p_{\bar{\omega}}$, we identify P^M with the center $C_{\bar{\omega}}$ of $M_{\bar{\omega}}$. Every integrable weight is equivalent to a weight of the form $\bar{\omega}_e$ for a projection $e \in M_{\bar{\omega}}$ by Theorem 2.2. Since M has no type I component, $M_{\bar{\omega}}$ is of type II_∞ by [30; §8], so that any normal weight ν on the center $C_{\bar{\omega}}$ of $M_{\bar{\omega}}$ is of the form

$$\nu(x) = \tau_{\bar{\omega}}(ex)$$

with e a projection in $M_{\bar{\omega}}$. Hence the weight ν is of the form $\nu_{\bar{\omega}_e}$. Thus the map: $\varphi \mapsto \nu_\varphi$ is surjective. Let $\varphi_j = \bar{\omega}_{e_j}$, $j = 1, 2$, where e_1 and e_2 are projections in $M_{\bar{\omega}}$. Then we have

$$\nu_j(x) = \tau_{\bar{\omega}}(xe_j) , \quad x \in C_{\bar{\omega}}^{\pm} .$$

It follows then that $e_1 \prec e_2$ in $M_{\bar{\omega}}$ if and only if $\nu_1 \leqq \nu_2$, and that if e_1 and e_2 are orthogonal, then $\nu_1 + \nu_2 = \nu_{\varphi_1+\varphi_2}$. Finally, for each $\lambda \in \boldsymbol{R}_+^*$, integrable weights φ and ψ, we have

$$\begin{aligned}\nu_{\lambda\varphi}(p_M(\psi)) &= \tau_{\lambda\varphi}(c_{\lambda\varphi}(\psi)) = \lambda\tau_\varphi(c_\varphi(\lambda^{-1}\psi)) \\ &= \lambda\nu_\phi(F^M_{\lambda^{-1}}p_M(\psi)) .\end{aligned}$$

q.e.d.

We now finish the proof of the implication: (v) $\Rightarrow$ (i) in Theorem 1.3 for a von Neumann algebra M of type III. Suppose φ is a faithful weight on M such that $\varphi \sim \lambda\varphi$, $\lambda > 0$. By definition, $\check{\varphi}$ is dominant, so that φ is integrable. By the last equality in Theorem 2.8.(ii), we have $\nu_\varphi \circ F^M_\lambda = \lambda\nu_\phi$. Identifying P_M with the center $C_{\bar{\omega}}$ of the centralizer $M_{\bar{\omega}}$ of a dominant weight, the largest projection $e \in C_{\bar{\omega}}$, such that ν_ϕ is semi-finite on $C_{\bar{\omega}}e$, is invariant under F^M_λ, $\lambda > 0$. Hence it follows from Lemma 1.4(f) and [30; Theorem 8.5] that e is a central projection of M. By [30; Lemmas 8.9 and 8.10] and [30; Theorem 8.6], M_e must be semi-finite. Hence we have $e = 0$. Therefore, ν_ϕ has no semi-finite portion, which means that φ is of infinite multiplicity. Thus φ is dominant by definition. This completes the proof.

II.3 **Computation of the smooth flow of weights** (1). First of all, we state a consequence of §§8 and 9 of [30] in terms of the smooth flow of weights as follows:

THEOREM 3.1. *For a properly infinite von Neumann algebra M with separable predual, the smooth flow of weights on M is ergodic if and only if M is a factor. Moreover, $F^M_\lambda = 1$ if and only if $\lambda \in S(M) \cap \boldsymbol{R}_+^*$.*

Therefore, we conclude another immediate result as follows.

COROLLARY 3.2. *Let M be a factor of type* III_1 *with separable predual.*
(i) *Two integrable weights φ_1 and φ_2 are equivalent if and only if $\tau_{\varphi_1}(1) = \tau_{\varphi_2}(1)$;*
(ii) *Any integrable faithful weight of infinite multiplicity is dominant.*

Thus, the smooth flow of weights on a factor of III_1 with separable predual is trivial. We now compute the smooth flow of weights in the other cases. First, let N be a semi-finite properly infinite von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, and τ a faithful semi-finite normal trace on N. We identify the center C of N with $L^\infty(\Omega, \mu)$, where Ω is a compact metrizable space and μ a positive Radon measure on Ω. Let

$$\mathfrak{H} = \int_\Omega^\oplus \mathfrak{H}(\alpha)d\mu(\alpha),\ N = \int_\Omega^\oplus N(\alpha)d\mu(\alpha),\ \tau = \int_\Omega^\oplus \tau_\alpha d\mu(\alpha)$$

be the direct integral decompositions with respect to C. For a weight ω on N, there exists a unique positive self-adjoint h affiliated with N such that

$$h^{it} = (D\omega : D\tau)_t\ , \quad t \in \boldsymbol{R}\ .$$

Also, this gives rise to a measurable field $h(\alpha)^{it}$ of continuous one parameter unitary groups on Ω such that

$$\int_\Omega^\oplus h(\alpha)^{it}d\mu(\alpha) = h^{it}\ , \quad t \in \boldsymbol{R}\ .$$

LEMMA 3.3. *In the above situation, if ω is integrable, then for almost every $\alpha \in \Omega$, the weight ω_α on $N(\alpha)$ determined by $(D\omega_\alpha : D\tau_\alpha)_t = h(\alpha)^{it}$ is integrable.*

PROOF. Let x be an element of $\mathfrak{p}_\omega^+$ with

$$x = \int_\Omega^\oplus x(\alpha)d\mu(\alpha) \quad \text{and} \quad E_\omega(x) = \int_\Omega^\oplus y(\alpha)d\mu(\alpha)\ .$$

It follows then that

$$\int_{-n}^{n} h^{it}xh^{-it}dt \leqq E_\omega(x) \quad \text{for} \quad n > 0\ .$$

Hence we have, for almost every $\alpha \in \Omega$,

$$\int_{-n}^{n} h(\alpha)^{it}x(\alpha)h(\alpha)^{-it}dt \leqq y(\alpha)\ .$$

Therefore, $E_{\omega_\alpha}(x(\alpha)) = \int_{-\infty}^{\infty} h(\alpha)^{it}x(\alpha)h(\alpha)^{-it}dt$ exists for almost every $\alpha \in \Omega$. Let A be a countable σ-strongly dense subset of $\mathfrak{p}_\omega^+$. Then we can choose a null set E in Ω such that

$$\int_{-\infty}^{\infty} h(\alpha)^{it}x(\alpha)h(\alpha)^{-it}dt = E_{\omega_\alpha}(x(\alpha))$$

exists for every $x \in A$ and $\alpha \notin E$. Since we can choose another null set F in Ω such that $\{x(\alpha) : x \in A\}$ is σ-strongly dense in $N(\alpha)_+$ for every

$\alpha \notin F$, we conclude that $\mathfrak{p}^{+}_{\omega_\alpha}$ is σ-strongly dense in $N(\alpha)_+$ for every $\alpha \notin E \cup F$. Thus almost every ω_α is integrable. q.e.d.

Therefore, we may assume, deleting a suitable null set from Ω, that each ω_α is integrable. By Theorem 2.2, ω_α is subequivalent to a dominant weight $\bar{\omega}_\alpha$ on $N(\alpha)$. By construction, see Theorem 1.3, the Radon-Nikodym derivative of $\bar{\omega}_\alpha$ with respect to the trace τ_α has a spectral measure which is equivalent to the Lebesgue measure $d\lambda$ on $\boldsymbol{R}^*_+$ in the sense of absolute continuity. Therefore, $h(\alpha)$ has a spectral measure which is absolutely continuous with respect to $d\lambda$, and so equivalent to a measure of the form $e_\alpha d\lambda$ with e_α the characteristic function of a measurable subset E'_α of $\boldsymbol{R}^*_+$. We define a map p from the set of integral weights on N to the set of projections in $L^\infty(\Omega \times \boldsymbol{R}^*_+, d\mu \otimes d\lambda)$ by

$$p(\omega) = \int_\Omega^\oplus e_\alpha d\mu(\alpha) \in L^\infty(\Omega \times \boldsymbol{R}^*_+, d\mu \otimes d\lambda) .$$

We observe then that p is an isomorphism of the smooth flow of weights on N onto $L^\infty(\Omega \times \boldsymbol{R}^*_+, d\mu \otimes d\lambda)$ equipped with the flow defined by the action: $(\alpha, \nu) \longrightarrow (\alpha, \lambda^{-1}\nu)$, $\lambda \in \boldsymbol{R}^*_+$, of $\boldsymbol{R}^*_+$ on $\Omega \times \boldsymbol{R}^*_+$.

THEOREM 3.4. *Let M be a factor of type III_λ, $\lambda \neq 1$, and φ, N, θ, $\{\rho_n\}_{n \in Z}$, E and I be as in I.2. As above put $L^\infty(\Omega, \mu) = C =$ Center of N and let θ_0 be a non-singular transformation of $\{\Omega, \mu\}$ corresponding to the restriction of θ to C, that is, $\theta(f)(\alpha) = f(\theta_0^{-1}\alpha)$, $\alpha \in \Omega$, for every $f \in C$.*

(i) *For any weight ω on N, $\omega \circ E$ is integrable if and only if ω is integrable.*

(ii) *If ψ is an integrable weight on M, then $\psi \sim \omega \circ E$ for some integrable weight ω on N such that $\omega = \tau(h\cdot)$ and $s(h) \leqq h < \rho_{-1}s(h)$.*

(iii) *In statement* (ii), *$p(\omega)$ only depends on ψ and the mapping: $\psi \longmapsto p(\omega)$ is an isomorphism of the smooth flow of weights on M onto the flow built on the transformation θ_0 under the ceiling function ρ^{-1}.*

PROOF. (i) Let $\psi = \omega \circ E$. It follows then that the restriction of σ_t^ψ to N is nothing but σ_t^ω. Hence if ω is integrable, then $\mathfrak{p}_\omega$ contains a sequence of projections converging σ-strongly to the identity 1. But $\mathfrak{p}_\omega \subset \mathfrak{p}_\psi$. Hence ψ is integrable. Conversely, suppose ψ is integrable. Put $e = p_N(\omega)$ and $\bar{e} = \bigvee_{n=-\infty}^{\infty} \bar{\theta}^n(e)$, where $\bar{\theta}$ is the automorphism of $\mathfrak{P}_N$ corresponding to θ. By Corollary I.2.8, we have $I(e) = I(\bar{e})$. By assumption, we have $I(e) = p_M(\psi) \leqq d_M$. Since I is an isomorphism of $(\mathfrak{P}_N)^{\bar{\theta}}$ onto $\mathfrak{P}_M$ intertwining the flows $\mathfrak{F}^N|_{(\mathfrak{P}_N)^{\bar{\theta}}}$ and $\mathfrak{F}^M$, the map: $\lambda \in \boldsymbol{R}^*_+ \longrightarrow \mathfrak{F}^N_\lambda(\bar{e})$ is σ-strongly continuous by the σ-strong continuity of the map: $\lambda \in \boldsymbol{R}^*_+ \longmapsto \mathfrak{F}^M_\lambda I(e) \in \mathfrak{P}^M$. Hence we get $\bar{e} \leqq d_N$, and so $e \leqq d_N$. Thus ω is integrable.

(ii) This is a direct consequence of Remark I.2.7 and (i).

(iii) By virtue of Theorem I.2.2 and Remark I.2.7, the equivalence class of ω is uniquely determined by the equivalence class of $\psi = \omega \circ E$ under the condition that $s(h) \leqq h < \rho_{-1}s(h)$ and $\omega = \tau(h\cdot)$. Hence $p(\omega)$ is uniquely determined by the equivalence class of ψ. Denoting by d the projection in $L^\infty(\Omega \times \boldsymbol{R}_+^*, d\mu \otimes d\lambda)$ corresponding to the set $\Gamma = \{(\alpha, \lambda) \in \Omega \times \boldsymbol{R}_+^*: 1 \leqq \lambda < \rho^{-1}(\alpha)\}$, the condition, $s(h) \leqq h < \rho_{-1}s(h)$, is equivalent to the condition, $p(\omega) \leqq d$. We identify, by means of p, the smooth flow of weights on N with $L^\infty(\Omega \times \boldsymbol{R}_+^*, d\mu \otimes d\lambda)$ equipped with the flow given by the multiplication of $\boldsymbol{R}_+^*$ on the second component. We have then $\bar{\theta}p(\omega) = p(\omega \circ \theta^{-1})$ for every integrable weight ω on N. Since $\omega_h \circ \theta^{-1} = \omega_{\rho_{-1}\theta(h)}$ with $\omega_h = \tau(h\cdot)$, $\bar{\theta}$ corresponds to the non-singular transformation: $(\alpha, \lambda) \in \Omega \times \boldsymbol{R}_+^* \rightarrow (\theta_0(\alpha), \rho_{-1}(\theta_0(\alpha))\lambda) = (\theta_0(\alpha), \rho^{-1}(\alpha)\lambda) \in \Omega \times \boldsymbol{R}_+^*$. Thus, our assertion follows from Corollary I.2.8. (iii). q.e.d.

We now summarize what we know about the smooth flow of weights for infinite factors with separable predual:

Type II_∞: The flow (P_M, F^M) is isomorphic to the flow coming from the action (by multiplication) of $\boldsymbol{R}_+^*$ on $\boldsymbol{R}_+^*$.

Type III_0: For any continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$, (P_M, F^M) is isomorphic to ($C =$ the center of N, $\theta_{\mathrm{Log}\lambda}$ restricted to C). For any discrete decomposition $M = W^*(N, \theta)$, (P_M, F^M) is isomorphic to the flow built on the restriction of θ^{-1} to $C =$ the center of N, under the ceiling function $d\tau \circ \theta^{-1}/d\tau$.

Type III_λ, $\lambda \neq 0$: The flow (P_M, F^M) is isomorphic to the flow coming from the action, by multiplication, of $\boldsymbol{R}_+^*$ on $\boldsymbol{R}_+^*/S(M) \cap \boldsymbol{R}_+^*$.

II.4. **Regularization of weights of infinite multiplicity.** We show in this section how to approximate, by integrable weights, an arbitrary weight φ of infinite multiplicity on a fixed von Neumann algebra M with separable predual. In fact, the approximation will take place in a very strict topology on the set of weights that we shall discuss first.

DEFINITION 4.1. (One Parameter Family of Orderings) For a positive real number $\lambda > 0$ and a couple φ_1, φ_2 of weights on a von Nummann algebra M, we write

$$\varphi_1 \leqq \varphi_2(\lambda)$$

if the map: $t \in \boldsymbol{R} \rightarrow (D\varphi_2: D\varphi_1)_t = u_t$ is extendable to a continuous function u_z on the horizontal ssrip $\bar{D}_{-\lambda} = \{z \in \boldsymbol{C}: -\lambda \leqq \operatorname{Im} z \leqq 0\}$ which is holomorphic inside $D_{-\lambda} = \{z \in \boldsymbol{C}: -\lambda < \operatorname{Im} z < 0\}$ and $\|u_z\| \leqq 1$, $z \in \bar{D}_{-\lambda}$.

LEMMA 4.2. *The relation "$\varphi_1 \leqq \varphi_2(\lambda)$" on the set of faithful weights is indeed an order relation.*

PROOF. Trivially $\varphi \leqq \varphi(\lambda)$. Suppose $\varphi_1 \leqq \varphi_2(\lambda)$ and $\varphi_2 \leqq \varphi_3(\lambda)$. Let $u_t^{i,j} = (D\varphi_i : D\varphi_j)_t$, $i, j = 1, 2, 3$. By assumption, $u_t^{2,1}$ and $u_t^{3,2}$ have extensions $u_z^{2,1}$ and $u_z^{3,2}$ on $\bar{D}_{-\lambda}$. It follows then that $u_z = u_z^{3,2}u_z^{2,1}$ is a continuous function on $\bar{D}_{-\lambda}$ holomorphic in $D_{-\lambda}$, and $u_t = u_t^{3,1}$ by the chain rule on the cocycle Radon-Nikodym derivatives. Obviously, $\|u_z\| \leqq \|u_z^{3,2}\| \|u_z^{2,1}\| \leqq 1$. Thus $\varphi_1 \leqq \varphi_3(\lambda)$.

Suppose $\varphi_1 \leqq \varphi_2(\lambda)$ and $\varphi_2 \leqq \varphi_1(\lambda)$. Then $u_t^{2,1}$ and $u_t^{1,2}$ have extensions $u_z^{2,1}$ and $u_z^{1,2}$ on $\bar{D}_{-\lambda}$. But we have $u_t^{2,1} = (u_t^{1,2})^*$. Let ψ be a normal state on M, and put $f(z) = \psi(u_z^{2,1})$ and $g(z) = \psi(u_z^{1,2})$, $z \in \bar{D}_{-\lambda}$. Since $f(t) = \overline{g(t)}$ for real t, f is extended to a continuous function on

$$\bar{D}_\lambda = \{z \in \boldsymbol{C} : 0 \leqq \operatorname{Im} z \leqq \lambda\}$$

which is holomorphic inside the strip D_λ. Hence f is holomorphic in $D_\lambda \cup D_{-\lambda} \cup \boldsymbol{R}$. But we have $f(0) = 1 \geqq f(z)$ for every z. Hence by the maximal modulus principle, we have $f(z) = 1$ for every z. Hence $u_z^{2,1} = 1$, that is, $\varphi_1 = \varphi_2$. q.e.d.

The ordering corresponding to $\lambda = 1/4$ was analyzed in [5; Lemma 3.13] for states on M, and the one corresponding to $\lambda = 1/2$ was shown, in [6], to be the usual ordering on weights: $\varphi_1 \leqq \varphi_2(1/2)$ if and only if $\varphi_1(x) \leqq \varphi_2(x)$, $x \in M_+$.

PROPOSITION 4.3. *Let M be a von Neumann algebra and let $P = M \otimes F_2$. For faithful weights φ_1 and φ_2 on M, put*

$$\varphi(\sum x_{ij} \otimes e_{ij}) = \varphi_1(x_{11}) + \varphi_2(x_{22}) .$$

The condition, $\varphi_1 \leqq \varphi_2(\infty)$, is equivalent to the condition that $1 \otimes e_{21} \in P(\sigma^\varphi, [0, \infty))$.

PROOF. Our assertion follows from a more general one. With an arbitrary continuous action α of $\boldsymbol{R}$ on P, x belongs to $P(\alpha, [0, \infty))$ if and only if for each $\psi \in P_*$, the Fourier transform, in the sense of tempered distributions, of the function: $t \in \boldsymbol{R} \mapsto \psi(\alpha_t(x))$ has its support contained in $[0, \infty)$; hence by the Paley-Wiener theorem, if and only if the function: $t \in \boldsymbol{R} \mapsto \alpha_t(x) \in P$ is extended to a bounded holomorphic function on the upper half plane D_∞. Taking $\sigma_t = \sigma_t^\varphi$ and $x = 1 \otimes e_{21}$, our assertion follows. q.e.d.

DEFINITION 4.4. For a pair of faithful weights φ_1 and φ_2 on M, we put

$$d(\varphi_1, \varphi_2) = \inf\{\alpha > 0 \colon \varphi_1 \leqq e^{\alpha}\varphi_2(\infty) \quad \text{and} \quad \varphi_2 \leqq e^{\alpha}\varphi_1(\infty)\} .$$

From Lemma 4.2, it follows that d is a distance function on the set $\mathfrak{W}_M^0$ of faithful weights on M with values in the extended positive reals $[0, \infty]$. We note that if $d(\varphi_1, \varphi_2) < +\infty$, then the function $t \in \boldsymbol{R} \mapsto (D\varphi_2 \colon D\varphi_1)_t$ extends to an entire function. The topology on $\mathfrak{W}_M^0$ associated with d will be called the *uniform topology.*

PROPOSITION 4.5. (i) *The set $\mathfrak{W}_M^0$ of all faithful weights on a von Neumann algebra M with metric d is complete, and the function: $\varphi \in \mathfrak{W}_M^0 \to \varphi(x) \in \boldsymbol{R}_+$ is continuous for every $x \in M_+$.*

(ii) *For any normal states φ_1 and φ_2 on M,*

$$\|\varphi_1 - \varphi_2\| \leqq 4d(\varphi_1, \varphi_2) .$$

We need the following lemma in order to prove the above result:

LEMMA 4.6. *For any $\varepsilon > 0$ and $R > 0$ there exists $\delta = \delta(\varepsilon, R) > 0$ such that if f is a Banach space valued entire function such that $\|f(z)\| \leqq e^{\delta|\mathrm{Im}\,z|}$, $z \in \boldsymbol{C}$, then we have*

$$\|f(z) - f(0)\| \leqq \varepsilon , \quad |z| \leqq R .$$

PROOF. Let H be the space of all entire functions equipped with the compact open topology. Let D_λ, $\lambda > 0$, be the set of all entire functions f with $|f(z)| \leqq e^{\lambda|\mathrm{Im}\,z|}$, $z \in \boldsymbol{C}$. It follows then that D_λ is a compact subset of H for each $\lambda > 0$, and $\bigcap_{\lambda>0} D_\lambda = \{\alpha 1 \colon |\alpha| \leqq 1, \alpha \in \boldsymbol{C}\} = D_0$. Put

$$G_{\varepsilon,R} = \{f \in H \colon |f(z) - f(0)| < \varepsilon, |z| \leqq R\} .$$

Clearly $G_{\varepsilon,R}$ is an open subset of H and $G_{\varepsilon,R} \supset D_0$. Hence, by compactness, there exists $\delta = \delta(\varepsilon, R)$ such that $D_\delta \subset G_{\varepsilon,R}$.

Now, let E be a Banach space and f be an E-valued entire function such that $\|f(z)\| \leqq e^{\delta|\mathrm{Im}\,z|}$, $z \in \boldsymbol{C}$. For any $\varphi \in E^*$, $\|\varphi\| \leqq 1$, the function: $z \in \boldsymbol{C} \to \varphi \circ f(z) \in \boldsymbol{C}$ belongs to D_δ; hence to $G_{\varepsilon,R}$, so that

$$|\varphi(f(z)) - \varphi(f(0))| \leqq \varepsilon , \quad |z| \leqq R .$$

Thus we get, for any $|z| \leqq R$,

$$\|f(z) - f(0)\| = \sup\{|\varphi(f(z)) - \varphi(f(0))| \colon \varphi \in E^*, \|\varphi\| \leqq 1\} < \varepsilon .$$

q.e.d

THE PROOF OF PROPOSITION 4.5. (i) Let φ_1 and φ_2 be faithful weights on M and $\delta > 0$. Assume that $\varphi_1 \leqq e^{\delta}\varphi_2(\infty)$ and $\varphi_2 \leqq e^{\delta}\varphi_1(\infty)$. Let $\{u_z, z \in \boldsymbol{C}\}$ be an entire function such that $u_t = (D\varphi_2 \colon D\varphi_1)_t$, $t \in \boldsymbol{R}$. Since

$$(D\varphi_2 \colon D(e^{\delta}\varphi_1))_t = e^{-i\delta t}u_t$$

and $(D\varphi_1: D(e^\delta\varphi_2))_t = e^{-i\delta t}u_t^*$ we have

$$\|u_z\| \leqq e^{\delta|\mathrm{Im}\,z|}, \quad z \in \boldsymbol{C}.$$

Hence, if $\delta = \delta(\varepsilon, R)$ for any $\varepsilon > 0$, then

$$\|u_z - 1\| \leqq \varepsilon, \quad |z| \leqq R,$$

by Lemma 4.6. Therefore, with a fixed faithful weight φ on M such that $d(\varphi_1, \varphi) = \alpha < +\infty$, we have

$$\begin{aligned}\|(D\varphi_2: D\varphi)_z - (D\varphi_1: D\varphi)_z\| &= \|\{(D\varphi_2: D\varphi)_z(D\varphi_1: D\varphi)_z^{-1} - 1\}(D\varphi_1: D\varphi)_z\| \\ &\leqq \|(D\varphi_2: D\varphi_1)_z - 1\|e^{\alpha|\mathrm{Im}\,z|} \\ &\leqq \varepsilon e^{\alpha|\mathrm{Im}\,z|}, \quad |z| \leqq R.\end{aligned}$$

This shows that if $\{\varphi_n\}$ is a Cauchy sequence in $\mathfrak{W}_M^0$ with respect to the metric d, then $(D\varphi_n: D\varphi)_z$ converges to v_z uniformly on every bounded part of $\boldsymbol{C}$; hence the function: $z \in \boldsymbol{C} \mapsto v_z \in M$ is entire, $v_{s+t} = v_s\sigma_s^\varphi(v_t)$, $s, t \in \boldsymbol{R}$, and each v_s is a unitary. By [3; Théorème 1.2.4], there exists a faithful weight ψ on M such that $(D\psi: D\varphi)_t = v_t \in \boldsymbol{R}$. In other words, for any faithful weight φ on M, $(D\varphi_n: D\varphi)_z$ converges to $(D\psi: D\varphi)_z$ uniformly for $|z| \leqq R$. For any $\varepsilon > 0$, choose $m > 0$ such that $\|(D\varphi_n: D\varphi_m)_z\| \leqq e^{\varepsilon|\mathrm{Im}\,z|}$ for any $n \geqq m$. It follows then that $\|(D\psi: D\varphi_m)_z\| \leqq e^{\varepsilon|\mathrm{Im}\,z|}$, $z \in \boldsymbol{C}$. Hence we have $d(\psi, \varphi_m) \leqq \varepsilon$. Thus ψ is the limit of $\{\varphi_n\}$, that is, the metric d is complete.

The continuity of the function: $\varphi \mapsto \varphi(x)$, $x \in M_+$ follows from the observation:

$$e^{-\varepsilon}\varphi_1(x) \leqq \varphi_2(x) \leqq e^\varepsilon\varphi_1(x), \qquad x \in M_+,$$

if $d(\varphi_1, \varphi_2) \leqq \varepsilon$.

(ii) If $d(\varphi_1, \varphi_2) \leqq \varepsilon$, then we have

$$(e^{-\varepsilon} - 1)\varphi_1 \leqq \varphi_2 - \varphi_1 \leqq (e^\varepsilon - 1)\varphi_1 \quad \text{in} \quad M_*^+;$$

hence for faithful normal states φ_1 and φ_2, we get

$$\begin{aligned}\|\varphi_2 - \varphi_1\| &\leqq 2\sup\{|\varphi_2(x) - \varphi_1(x)|: x \in M_+, \|x\| \leqq 1\} \\ &\leqq 2\max\{e^\varepsilon - 1, 1 - e^{-\varepsilon}\} \leqq 4\varepsilon.\end{aligned}$$

q.e.d.

THEOREM 4.7. *If φ is a faithful weight of infinite multiplicity on a von Neumann algebra M, then for any $\varepsilon > 0$ there exists an integrable weight ψ of infinite multiplicity commuting with φ such that $d(\varphi, \psi) < \varepsilon$.*

PROOF. Let $\varepsilon > 0$ and φ be given. Let F_∞ be a type I_∞ subfactor of M_φ and h, $1 - \varepsilon \leqq h \leqq 1 + \varepsilon$, be an element of F_∞ which has absolutely continuous spectrum only and such that $\{h\}' \cap F_\infty$ is properly infinite.

We claim that the weight $\mathrm{Tr}(h\cdot)$ on F_∞ is integrable, that is, there exists an increasing sequence $\{e_n\}$ of projections in F_∞ such that $\int_{-\infty}^{\infty} h^{it}e_nh^{-it}dt$ exists for each $n = 1, 2, \cdots$ and $\lim e_n = 1$. If this is the case, then $\psi = \varphi(h\cdot)$ is an integrable weight because $(D\psi : D\varphi)_t = h^{it}$ and $\sigma_t^{\psi}(x) = h^{it}\sigma_t^{\varphi}(x)h^{-it}$, hence $\sigma_t^{\psi}(e_n) = h^{it}e_nh^{-it}$; thus $e_n \in \mathfrak{p}_\psi$, and we have $d(\varphi, \psi) \leqq \varepsilon$. Therefore, we must show the above claim. Since the quasi-equivalence class of a unitary representation of $\boldsymbol{R}$ is completely determined by the equivalence class of the spectral measure in the sense of absolute continuity of measures, the one parameter unitary group h^{it}, $t \in \boldsymbol{R}$, is quasi-equivalent to a subrepresentation of the regular representation U_t of $\boldsymbol{R}$. This means that the weight $\mathrm{Tr}(h\cdot)$ is quasi-equivalent to a subweight of a dominant weight on F_∞. Therefore, our claim follows. q.e.d.

COROLLARY 4.8. *If M is a factor of type* III_1 *with separable predual, then for any pair φ_1, φ_2 of faithful weights of infinite multiplicity on M and any $\varepsilon > 0$ there exists a unitary u such that $d(\varphi_{1,u}, \varphi_2) < \varepsilon$.*

PROOF. Let ψ_j be integrable weights of infinite multiplicity such that $d(\varphi_j, \psi_j) < \varepsilon/2$, $j = 1, 2$. By Corollary 3.2, (ii), there exists a unitary $u \in M$ such that $\psi_{1,u} = \psi_2$. Now, we have $d(\varphi_{1,u}, \psi_{1,u}) = d(\varphi_1, \psi_1)$, and hence

$$d(\varphi_{1,u}, \varphi_2) \leqq d(\varphi_1, \psi_1) + d(\psi_2, \varphi_2) < \varepsilon\,. \qquad \text{q.e.d.}$$

REMARK 4.9. In the same situation as above, for any $\varepsilon > 0$ there exist $h_i \in M_{\varphi_i}$, $1 - \varepsilon \leqq h_i \leqq 1 + \varepsilon$ and a unitary $u \in M$ such that

$$\varphi_2(x) = \varphi_1(h_2uh_1xh_1u^*h_2)\,, \qquad x \in M_+\,.$$

COROLLARY 4.10. *If M is a factor of type* III_λ, $\lambda > 0$, *with separable predual, then for any pair φ_1, φ_2 of weights of infinite multiplicity on M* ($\varphi_j(1) = +\infty$, $j = 1, 2$, *is enough when* $\lambda \neq 1$) *there exists a unitary $u \in M$ such that*

$$u\mathfrak{m}_{\varphi_1}u^* = \mathfrak{m}_{\varphi_2}\,,$$

where $\mathfrak{m}_{\varphi_j}$ means, of course, the domain of φ_j.

PROOF. The case $\lambda = 1$. In the proof of Corollary 4.8, ψ_j was taken to be $\varphi_j(h_j\cdot)$ with $h_j \in M_{\varphi_j}$, $1 - \varepsilon \leqq h_j \leqq 1 + \varepsilon$, $j = 1, 2$. By [24; Prop. 3.3. ii], we have $\mathfrak{m}_{\psi_j} = \mathfrak{m}_{\varphi_j}$, $j = 1, 2$. Since $\psi_2 = \psi_{1,u}$ for some unitary $u \in M$, we have

$$\mathfrak{m}_{\varphi_1} = \mathfrak{m}_{\psi_1} = u\mathfrak{m}_{\psi_2}u^* = u\mathfrak{m}_{\varphi_2}u^*\,.$$

The case $\lambda \neq 1$, and $\varphi_1(1) = \varphi_2(1) = +\infty$. Let $T = -2\pi/\log\lambda$, and h_1, h_2, $\lambda \leqq h_j \leqq 1$, be elements of M_{φ_j} such that

$$\sigma_T^{\varphi_j}(x) = h_j^{iT} x h_j^{-iT}\ , \qquad x \in M\ . \quad \text{(cf. [24; Chapter III].)}$$

Then $\psi_j = \varphi_j(h_j^{-1}\cdot)$ is a generalized trace on M for $j = 1, 2$, because $\psi_j(1) = \varphi_j(h_j^{-1}) = +\infty$. Since $\mathfrak{m}_{\varphi_j} = \mathfrak{m}_{\psi_j}$, our assertion follows from [3; Théorème 4.3.2]. q.e.d.

CONJECTURE. A factor M is of type III_1 if and only if the orbit $\{\varphi_u: u \in \mathfrak{U}_M\}$ of any normal state φ is dense in the set of all normal states in the norm topology; more precisely if and only if for any pair φ, ψ of normal faithful state and $\varepsilon > 0$ there exists a unitary $u \in M$ such that $(1 - \varepsilon)\psi \leqq \varphi_u \leqq (1 + \varepsilon)\psi$.

II.5. Relative commutant theorem.

THEOREM 5.1. *If ψ is an integrable faithful weight on a von Neumann algebra M with separable predual, then*

$$M'_\psi \cap M \subset M_\psi\ .$$

PROOF. We first note that the existence of an integrable faithful weight implies the proper infiniteness of M. We shall use the notations and the conventions established at the beginning of II.1. Let $M = P \otimes F_\infty$ where P is isomorphic to M and fix a faithful normal state φ. Considering the cyclic representation of P induced by φ, we assume that P acts on a Hilbert space $\mathfrak{H}$ containing a cyclic and separating vector ξ_0 such that $\varphi = \omega_{\xi_0}$. We then represent $M = P \otimes F_\infty$ on $L^2(\mathfrak{H}, \boldsymbol{R}) = \mathfrak{H} \otimes L^2(\boldsymbol{R})$. Let ω be the weight on F_∞ such that $(D\omega: D\,\mathrm{Tr})_t = U_t$, $t \in \boldsymbol{R}$, and $\bar{\omega} = \varphi \otimes \omega$.

We observe next that, replacing ψ by $\check{\varphi}$, one can assume that ψ is of infinite multiplicity. As $\varphi \otimes \omega$ is a dominant weight on M, Theorem 2.2 shows that ψ is isomorphic to a subweight of $\varphi \otimes \omega$. It is hence enough to prove the theorem for $\varphi \otimes \omega$. By Lemma 1.4, $M_{\bar{\omega}}$ is generated by $1 \otimes U(L^\infty(\boldsymbol{R})) = 1 \otimes F_\omega$ and $\pi(P)$, where π is the faithful normal representation of P on $L^2(\mathfrak{H}; \boldsymbol{R})$ defined by

$$\pi(x)\xi(s) = \sigma_{-s}^\varphi(x)\xi(s),\ \xi \in L^2(\mathfrak{H}; \boldsymbol{R})\ , \quad x \in P\ .$$

Since F_ω is maximal abelian in F_∞, it is sufficient to prove the following inclusion:

$$(*) \qquad P \otimes U(L^\infty(\boldsymbol{R})) \cap \pi(P)' \subset P_\varphi \otimes U(L^\infty(\boldsymbol{R}))\ .$$

We denote by $\tilde{U}$ the isomorphism $1 \otimes U$ of $P \otimes L^\infty(\boldsymbol{R}) = L^\infty(P; \boldsymbol{R})$ onto $P \otimes U(L^\infty(\boldsymbol{R}))$ and put $\theta_t = \mathrm{Ad}(1 \otimes V_t)$, $t \in \boldsymbol{R}$. Since $V_t U(f) V_t^* = U(f_t)$, $f \in L^\infty(\boldsymbol{R})$, where $f_t(s) = f(s - t)$, we have

$$\theta_t(\tilde{U}(a)) = \tilde{U}(a_t)\ , \qquad a \in L^\infty(P; \boldsymbol{R})\ ,$$

where $a_t(s) = a(s-t)$. For every continuous function f on $\boldsymbol{R}$ with compact support, we have, for each $a \in L^\infty(P; \boldsymbol{R})$,

$$\theta_f(\tilde{U}(a)) = \int f(t)\theta_t(\tilde{U}(a))dt = \tilde{U}(a_f) ,$$

where $a_f(s) = \int f(t)a(s-t)dt$. Since $1 \otimes V_t$ and $\pi(P)$ commute, θ_t leaves $\pi(P)$ pointwise fixed; hence θ_t leaves $\pi(P)'$ globally invariant, and hence

$$\theta_t(\tilde{U}(L^\infty(P: \boldsymbol{R})) \cap \pi(P)') \subset \tilde{U}(L^\infty(P: \boldsymbol{R})) \cap \pi(P)' .$$

Therefore, if $\tilde{U}(a)$ belongs to $\tilde{U}(L^\infty(P; \boldsymbol{R})) \cap \pi(P)'$, $\tilde{U}(a_f)$ belongs to

$$\tilde{U}(L^\infty(P; \boldsymbol{R})) \cap \pi(P)'$$

for each continuous function f with compact support. Since a_f approximate a arbitrarily well in the σ-strong* topology, in order to prove (*) it suffices to verify that $\tilde{U}(C(P; \boldsymbol{R})) \cap \pi(P)' \subset \tilde{U}(C(P_\varphi; \boldsymbol{R}))$, where $C(P; \boldsymbol{R})$ (resp. $\mathfrak{C}(P_\varphi; \boldsymbol{R})$) denotes the *-algebra of all σ-strong* continuous bounded P-valued (resp. P_φ-valued) functions on $\boldsymbol{R}$. Since $FU_tF^* = V_t$, $t \in \boldsymbol{R}$, we have $(1 \otimes F)\tilde{U}(C(P; \boldsymbol{R}))(1 \otimes F)^* = C(P; \boldsymbol{R})$, where $C(P; \boldsymbol{R})$ is represented on $L^2(\mathfrak{H}; \boldsymbol{R})$ by

$$a\xi(s) = a(s)\xi(s),\ a \in C(P; \boldsymbol{R}),\ \xi \in L^2(\mathfrak{H}; \boldsymbol{R}) .$$

Therefore, putting $\tilde{F} = 1 \otimes F$, we must show the following inclusion:

(**) $$C(P; \boldsymbol{R}) \cap \tilde{F}\pi(P)'\tilde{F} \subset C(P_\varphi; \boldsymbol{R}) .$$

The proof of (**) follows from the next two lemmas:

LEMMA 5.2. *Let $a \in C(P; \boldsymbol{R})$, $x \in P$ and $f, g \in C_c^\infty$, where $C_c^\infty = C_c^\infty(\boldsymbol{R})$, the space of all C^∞-functions on $\boldsymbol{R}$ with compact support.*

(a) $(\tilde{F}\pi(x)\tilde{F}(\xi_0 \otimes f) \mid a^*(\xi_0 \otimes g)) = \iiint e^{it(p-q)}f(p)\overline{g(q)}\varphi(a(q)\sigma_t^\varphi(x))\, dpdqdt,$ *where for p and q the order of integrations is irrelevant.*

(b) *If a commutes with $\tilde{F}\pi(x)\tilde{F}^*$, then we have*

$$\iiint e^{i(s+i)(p-q)}f(p)g(q)\varphi(a(p)\sigma_{-s}^\varphi(x))dpdqds$$
$$= \iiint e^{is(p-q)}f(p)g(q)\varphi(a(q)\sigma_{-s}^\varphi(x))dpdqds .$$

PROOF. (a) We have

$$[\pi(x)\tilde{F}^*(\xi_0 \otimes f)](t) = \sigma_{-t}^\varphi(x)(F^*f)(t)\xi_0 ;$$

$$[\tilde{F}\pi(x)\tilde{F}^*(\xi_0 \otimes f)](q) = \int e^{-itq}\sigma_{-t}^\varphi(x)(F^*f)(t)\xi_0 dt$$
$$= \iint e^{-itq}e^{itp}f(p)\sigma_{-t}^\varphi(x)\xi_0 dpdt ,$$

where the order of integration does matter. Then assertions (a) follows from the definition of the scalar product in $L^2(\mathfrak{H}; \boldsymbol{R})$ and the equality:

$$(\sigma^{\varphi}_{-t}(x)\xi_0 | a^*(q)\xi_0) = \varphi(a(q)\sigma^{\varphi}_{-t}(x)) , \qquad q, t \in \boldsymbol{R} .$$

(b) By hypothesis, we have

$$\begin{aligned}&\left(\iiint e^{is(p-q)} f(p)g(q)\varphi(a(q)\sigma^{\varphi}_{-s}(x))dpdqds\right)^{-}\\ &\quad = (\tilde{F}\pi(x)\tilde{F}^*(\xi_0 \otimes f) | a^*(\xi_0 \otimes \bar{g}))^{-} = (\tilde{F}\pi(x^*)\tilde{F}^*(\xi_0 \otimes \bar{g}) | (\xi_0 \otimes \bar{f}))\\ &\quad = \iiint e^{is(q-p)}\bar{g}(q)\overline{f(p)}\varphi(a^*(p)\sigma^{\varphi}_{-s}(x^*))dpdqds .\end{aligned}$$

But σ^{φ} is the modular automorphism group for φ, so that for each $p \in \boldsymbol{R}$, there exists a bounded holomorphic function $G(z, p)$ on the strip, $0 \leqq \operatorname{Im} z \leqq 1$, such that

$$G(s, p) = \varphi(\sigma^{\varphi}_{-s}(x)a(p)),\ G(s + i, p) = \varphi(a(p)\sigma^{\varphi}_{-s}(x)) .$$

We have then

$$\begin{aligned}&\iiint e^{is(p-q)} f(p)g(q)\varphi(\sigma^{\varphi}_{-s}(x)a(p))dpdqds\\ &\quad = \iiint e^{is(p-q)} f(p)g(q)G(s, p)dpdqds\\ &\quad = \iint e^{isp}(Fg)(s)f(p)G(s, p)dpds\\ &\quad = \iint e^{isp}(Fg)(s)f(p)G(s, p)dsdp \qquad \text{by Fubini's theorem} .\end{aligned}$$

Since the function: $z \longrightarrow e^{izp}(Fg)(z)G(z, p)$ is holomorphic in the strip, $0 \leqq \operatorname{Im} z < 1$, and decays exponentially along horizontal line, the above integral becomes

$$\begin{aligned}&\iint e^{i(s+i)p}(Fg)(s + i)f(p)G(s + i, p)dsdp\\ &\quad = \iint e^{i(s+i)p}(Fg)(s + i)f(p)G(s + i, p)dpds\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{by Fubini's theorem} ,\\ &\quad = \iiint e^{i(s+i)(p-q)} f(p)g(q)\varphi(a(p)\sigma^{\varphi}_{-s}(x))dpdqds\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{by Fubini's theorem.} \quad \text{q.e.d.}\end{aligned}$$

LEMMA 5.3. *If $H(q, s)$ is a bounded continuous function of two real variables $(q, s) \in \boldsymbol{R} \times \boldsymbol{R}$ such that for each $f, g \in C^{\infty}_c$,*

$$\iiint e^{is(p-q)} f(p)g(q)H(p, s)dpdqds = \iiint e^{i(s+i)(p-q)} f(p)g(q)H(q, s)dpdqds ,$$

then $H(q, s) = H(q, 0)$, $(q, s) \in \boldsymbol{R} \times \boldsymbol{R}$.

PROOF. Let $\tilde{H}_1$ and $\tilde{H}_2$ be the distributions on $\boldsymbol{R} \times \boldsymbol{R}$ defined by

$$\langle f, \tilde{H}_1 \rangle = \iiint e^{is(p-q)} f(p, q) H(q, s) dp dq ds , \qquad f \in C_c^\infty(\boldsymbol{R}^2) ;$$
$$\langle f, \tilde{H}_2 \rangle = \iiint e^{i(s+i)(p-q)} f(p, q) H(p, s) dp dq ds .$$

By hypothesis, we have $\langle f \otimes g, \tilde{H}_1 \rangle = \langle f \otimes g, \tilde{H}_2 \rangle$, $f, g \in C_c^\infty(\boldsymbol{R})$, so that $\tilde{H}_1 = \tilde{H}_2$ by the density of $C_c^\infty(\boldsymbol{R}) \otimes C_c^\infty(\boldsymbol{R})$ in $C_c^\infty(\boldsymbol{R}^2)$. Hence we have, for any $f \in C_c^\infty(\boldsymbol{R}^2)$,

$$\iiint e^{isr} f(q + r, q) H(q, s) dr dq ds = \iiint e^{isr} e^{-r} f(q + r, q) H(q + r, s) dr dq ds .$$

Since the functions: $(r, q) \mapsto f(q + r, q)$ exhaust all of $C_c^\infty(\boldsymbol{R}^2)$, we have

$$\iiint e^{isr} g(r, q) H(q, s) dr dq ds = \iiint e^{isr} e^{-r} g(r, t - r) H(t, s) dr dt ds$$

for every $g \in C_c^\infty(\boldsymbol{R}^2)$. Let $\tilde{H}$ be the distribution on $\boldsymbol{R}^2$ given by

$$\langle g, \tilde{H} \rangle = \iiint e^{isp} g(p, q) H(q, s) dp dq ds , \qquad g \in C_c^\infty(\boldsymbol{R}^2) .$$

Being the partial Fourier transform of the bounded continuous function H, the distribution $\tilde{H}$ is tempered. We define a linear transformation T on $C_c^\infty(\boldsymbol{R}^2)$ by

$$(Tg)(p, q) = e^{-p} g(p, q - p) , \qquad g \in C_c^\infty(\boldsymbol{R}^2) .$$

We have then $\langle (1 - T)g, \tilde{H} \rangle = 0$, $g \in C_c^\infty(\boldsymbol{R}^2)$. For an $f \in C_c^\infty(\boldsymbol{R}^2)$ with supp $f \cap \{0\} \times \boldsymbol{R} = \emptyset$, we define a sequence $\{g_n\}$ in $C_c^\infty(\boldsymbol{R}^2)$ by

$$g_n(p, q) = \begin{cases} \displaystyle\sum_{k=0}^{n} e^{-kp} f(p, q - kp) & \text{if} \quad p \geqq 0 , \\ \displaystyle -\sum_{k=0}^{n} e^{(k+1)p} f(p, q + (k + 1)p) & \text{if} \quad p < 0 . \end{cases}$$

It follows then that

$$[(1 - T)g_n](p, q) = f(p, q) - e^{-(n+1)p} f(p, q - (n + 1)p) \quad \text{for} \quad p \geqq 0 ;$$
$$[(1 - T)g_n](p, q) = f(p, q) - e^{-(n+1)p} f(p, q + (n + 1)p) \quad \text{for} \quad p \leqq 0 .$$

Hence $(1 - T)g_n \to f$, as $n \to \infty$ in the space $\mathfrak{S}(\boldsymbol{R}^2)$, so that

$$\langle f, \tilde{H} \rangle = \lim_{n\to\infty} \langle (1 - T)g_n, \tilde{H} \rangle = 0 .$$

This means that supp $\tilde{H} \subset \{0\} \times \boldsymbol{R}$, so that

$$\iint e^{isp} f(p) H(q, s) dp ds = 0$$

for every $f \in C_c^\infty(\boldsymbol{R})$ with supp $f \not\ni 0$. Therefore, $H(q, s)$ is a polynomial in s for each fixed q, but being bounded, it has to be independent of s.
q.e.d.

END OF THE PROOF OF THEOREM 5.1. If $a \in C(P; \boldsymbol{R})$ commutes with all $\tilde{F}\pi(x)\tilde{F}^*$, $x \in P$, then the combination of Lemmas 5.2(b) and 5.3 shows that

$$\varphi(a(p)\sigma^\varphi_{-s}(x)) = \varphi(a(p)x), \ (p, s) \in \boldsymbol{R} \times \boldsymbol{R} .$$

Hence we get

$$\varphi(\sigma^\varphi_s(a(p))x) = \varphi(a(p)x), \ s \in \boldsymbol{R}, \ x \in P ,$$

which means that $a(p) = \sigma^\varphi_s(a(p))$, $s \in \boldsymbol{R}$; and hence $a(p) \in P_\varphi$. Thus (**) follows. q.e.d.

PROPOSITION 5.4. *Let φ be a faithful weight on a von Neumann algebra M and ω be the weight on F_∞ as before. If $M'_\varphi \cap M = C_\varphi \subset M_\varphi$, then the center $C_{\varphi\otimes\omega}$ of $P_{\varphi\otimes\omega}$ with $P = M \otimes F_\infty$ is contained in $C_\varphi \otimes U(L^\infty(\boldsymbol{R}))$.*

PROOF. We keep the notations in the proof of Theorem 5.1. We know that $P_{\varphi\otimes\omega}$ is generated by $\pi(M)$ and $\boldsymbol{C} \otimes U(L^\infty(\boldsymbol{R}))$. Since $U(L^\infty(\boldsymbol{R}))$ is maximal abelian in F_∞, $C_{\varphi\otimes\omega}$ is contained in $M \otimes U(L^\infty(\boldsymbol{R}))$. On the other hand, $\pi(x) = x \otimes 1$ for every $x \in M_\varphi$. Hence we have $\pi(M)' \cap P \subset (M'_\varphi \cap M) \otimes F_\infty = C_\varphi \otimes F_\infty$. Thus we get

$$C_{\varphi\otimes\omega} \subset M \otimes U(L^\infty(\boldsymbol{R})) \cap C_\varphi \otimes F_\infty = C_\varphi \otimes U(L^\infty(\boldsymbol{R})) . \qquad \text{q.e.d.}$$

II.6. **Computation of the smooth flow of weights** (2). In Theorem 3.1, we saw that the modular spectrum $S(M)$, or more precisely $S(M) \cap \boldsymbol{R}_+^*$, of a properly infinite factor M with separable predual is precisely the kernel of the smooth flow F^M of weights on M. In Mackey's terminology **[16]**, each ergodic action of a separable locally compact group G on a standard measure space would have a "non-trivial *kernel*", called a *virtual subgroup* of G. Following his theory, the smooth flow F^M of weights on M may be called the *virtual modular spectrum* of M and may be denoted by $S_v(M)$. As a matter of fact, Mackey's theory of virtual groups provides us very useful strategic technique in computing the smooth flow F^M of weights in the case where M is given by the so-called group measure space construction.

Let G be a separable locally compact group acting on a standard measure space $\{\Gamma, \mu\}$. Let C denote the abelian von Neumann algebra $L^\infty(\Gamma, \mu)$. The action of G on Γ gives rise to a continuous action α of G on C as follows:

$$\alpha_g(x) = x(g^{-1}\gamma),\ g \in G,\ x \in C,\ \gamma \in \Gamma\ .$$

Let ρ be a 1-cocycle for the action of G on $\{\Gamma, \mu\}$ with coefficients in another separable locally compact group H, that is, ρ is an H-valued Borel function of $G \times \Gamma$ such that

$$\rho(g_1 g_2, \gamma) = \rho(g_1, g_2\gamma)\rho(g_2, \gamma),\ g_1, g_2 \in G\ , \qquad \gamma \in \Gamma\ .$$

In Mackey's theory, ρ, or more precisely its cohomologus equivalence class, is regarded as a homomorphism of the virtual subgroup $\{G, \Gamma, \mu\}$ of G into H. To this ρ, there correspond a virtual subgroup of G, called the "kernel" of ρ, and a virtual subgroup of H, called the "closure of the range" of ρ, which are defined as follows:

Consider the cartesian product measure space $\{\Gamma \times H, \mu \times \lambda\}$ with λ the left Haar measure in H, and define the actions of G and H on $\Gamma \times H$ by

$$g(\gamma, h) = (g\gamma, \rho(g, \gamma)h),\ g \in G,\ \gamma \in \Gamma,\ h \in H\ ;$$
$$k(\gamma, h) = (\gamma, hk^{-1}),\ \gamma \in \Gamma,\ h, k \in H\ .$$

We note that the actions of G and H commute. Let $D = L^\infty(\Gamma \times H, \mu \times \lambda)$. The action of G on $\Gamma \times H$, or the action $\bar\alpha$ of G on D induced by that of G on $\Gamma \times H$, corresponds to the "kernel" of ρ, and the action β of H on the fixed point subalgebra $D^{\bar\alpha}$ of D under $\bar\alpha$ corresponds to the "closure of the range" of ρ. We denote this "closure of the range" of ρ by $\overline{\rho(G, \Gamma)}$ or $\overline{\rho(G, C)}$.

REMARK 6.1. The closure of the range $\rho(G, \Gamma)$ and the kernel of ρ depends, within equivalence, only on the cohomologus class of ρ. We also observe that $\overline{\rho(G, \Gamma)}$ is independent of the topology in G, and that $\overline{\rho(G_0, \Gamma)} = \overline{\rho(G, \Gamma)}$ for any dense subgroup G_0 of G.

We now apply the above Mackey's procedure to the computation of the smooth flow F^M of weights on M.

We now apply the above Mackey's procedure to the computation of the smooth flow F^M of weights on M.

THEOREM 6.2. *Let M be an infinite factor with separable predual, and φ a faithful weight on M. Suppose that N is a von Neumann subalgebra of M_φ with relative commutant $N' \cap M = C$ contained in N.*

(i) *If a unitary $u \in M$ normalizes N,* i.e., *$uNu^* = N$, then there* *exists a non-singular self-adjoint positive operator* ρ_u *affiliated with* C *such that*

$$\varphi_u = \varphi(\rho_u \cdot) \quad \textit{and} \quad \sigma_t^\varphi(u) = u\rho_u^{it}, \qquad t \in \boldsymbol{R} .$$

(ii) *If the normalizer* $\eta(N)$ *of* N *generates* M, *then for each countable subgroup* G *of* $\eta(N)$ *such that* $M = (N \cup G)''$ *there exists a canonial isomorphism of the closure* $\rho^{-1}(G, C)$ *of the range of* ρ^{-1} *onto the smooth flow* F^M *of weights on* M.

PROOF. (i) We have $(D\varphi_u : D\varphi)_t = u^*\sigma_t^\varphi(u)$, $t \in \boldsymbol{R}$ and $M_{\varphi_u} = u^*M_\varphi u \supset u^*Nu = N$. For each $x \in N$, we get

$$x = \sigma_t^{\varphi_u}(x) = u^*\sigma_t^\varphi(u)\sigma_t^\varphi(x)\sigma_t^\varphi(u^*)u = u^*\sigma_t^\varphi(u)x\sigma_t^\varphi(u^*)u ;$$

hence $u^*\sigma_t^\varphi(u) \in N' \cap M = C$. As we have

$$u^*\sigma_s^\varphi(u)u^*\sigma_t^\varphi(u) = u^*\sigma_s^\varphi(uu^*\sigma_t^\varphi(u)) = u^*\sigma_{s+t}^\varphi(u),\ s,\ t \in \boldsymbol{R} ,$$

there exists a non-singular self-adjoint positive operator ρ_u affiliated with C such that

$$\rho_u^{it} = u^*\sigma_t^\varphi(u) = (D\varphi_u : D\varphi)_t ,$$

so that

$$\varphi_u = \varphi(\rho_u \cdot) .$$

It is straightforward to observe that the map: $u \in G \mapsto \rho_u$ is a 1-cocycle with values in the multiplicative commutative group of non-singular self adjoint positive operators affiliated with C with respect to the action of G on C given by $\mathrm{Ad}(u)$, $u \in G$.

(ii) Representing M on a Hilbert space $\mathfrak{H}$, we consider the tensor product $P = M \otimes F_\infty$ on $L^2(\mathfrak{H}, \boldsymbol{R})$ and the weight $\bar{\omega} = \varphi \otimes \omega$ on P, where ω is the weighe on F_∞ defined in §1. Let U and V be as before. It follows then that the centralizer $P_{\bar{\omega}}$ of $\bar{\omega}$ is generated by $\pi(M)$ and $1 \otimes U_t$, $t \in \boldsymbol{R}$, where

$$\pi(x)\xi(s) = \varphi_{-s}^\varphi(x)\xi(s),\ x \in M,\ s \in \boldsymbol{R},\ \xi \in L^2(\mathfrak{H}; \boldsymbol{R}) .$$

By Proposition 5.4, the center $C_{\bar{\omega}}$ of $P_{\bar{\omega}}$ is contained in $C \otimes U(L^\infty(\boldsymbol{R}))$. Since $\pi(M)$ is generated by $N \otimes \boldsymbol{C}$ and $\pi(G)$, $C_{\bar{\omega}}$ is the fixed point subalgebra of $C \otimes U(L^\infty(\boldsymbol{R}))$ under the automorphism group $\{\mathrm{Ad}(u) : u \in G\}$, that is

$$C_{\bar{\omega}} = [C \otimes U(L^\infty(\boldsymbol{R}))]^G .$$

We now compute the action of G on $C \otimes U(L^\infty(\boldsymbol{R}))$. For each $u \in G$, $x \in C$ and $t \in \boldsymbol{R}$, we have

$$\begin{aligned}[\operatorname{Ad}\pi(u)(x\otimes U_t)\xi](s) &= \pi(u)(x\otimes U_t)\pi(u)^*\xi(s)\\ &= \sigma^{\varphi}_{-s}(u)[(x\otimes U_t)\pi(u)^*\xi](s) = u\rho_u^{-is}x[\pi(u^*)\xi](s+t)\\ &= u\rho_u^{-is}x\rho_u^{i(s+t)}u^*\xi(s+t)\\ &= [(u\rho_u^{it}xu^*\otimes U_t)\xi](s)\ ;\end{aligned}$$

hence

$$\operatorname{Ad}\pi(u)(x\otimes U_t) = u\rho_u^{it}xu^*\otimes U_t,\ x\in C,\ t\in \boldsymbol{R},\ u\in G\ .$$

Let $\{\Gamma, \mu\}$ be a standard measure space with $L^\infty(\Gamma, \mu) = C$. The automorphism $\operatorname{Ad} u$, $u\in G$, of C gives rise to a non-singular transformation α_u of $\{\Gamma, \mu\}$ such that

$$[\operatorname{Ad}(u)x](\gamma) = x(\alpha_u^{-1}\gamma),\ u\in G,\ x\in C,\ \gamma\in\Gamma\ .$$

Identifying $C\otimes U(L^\infty(\boldsymbol{R}))$ with $L^\infty(\Gamma\times\boldsymbol{R}, d\mu\otimes dt)$ under the correspondence: $x\otimes U_t \leftrightarrow \{x(\gamma)e^{itp}\colon (\gamma, p)\in\Gamma\times\boldsymbol{R}\}$, we have

$$[\operatorname{Ad}(u)x](\gamma, p) = x(\alpha_u^{-1}\gamma, p + \log\rho_u(\alpha_u^{-1}\gamma)),\ x\in L^\infty(\Gamma\times\boldsymbol{R}, d\mu\otimes dt)\ .$$

Therefore, G acts on $\Gamma\times\boldsymbol{R}$ by

$$\beta_u(\gamma, p) = (\alpha_u\gamma, p - \log\rho_u(\gamma)),\ u\in G,\ (\gamma, p)\in\Gamma\times\boldsymbol{R}\ .$$

It follows from the proof of Theorem 5.1 that

$$[\theta_t(x)](\gamma, p) = x(\gamma, p - t),\ t\in\boldsymbol{R},\ x\in L^\infty(\Gamma\times\boldsymbol{R})\ .$$

Hence the action of $\boldsymbol{R}$ on $\Gamma\times\boldsymbol{R}$ is given by

$$\theta_t^*(\gamma, p) = (\gamma, p + t)\ .$$

By Corollary 2.5, the smooth flow F^M of weights on M is isomorphic to the action $\{\theta_{-\mathrm{Log}\lambda}\colon \lambda\in\boldsymbol{R}_+^*\}$ of $\boldsymbol{R}_+^*$ on $C_{\tilde\omega}$. Therefore, replacing $\Gamma\times\boldsymbol{R}$ by $\Gamma\times\boldsymbol{R}_+^*$ under the correspondence: $(\gamma, p)\leftrightarrow(\gamma, e^p)$, G and $\boldsymbol{R}_+^*$ act on $\Gamma\times\boldsymbol{R}_+^*$ respectively as follows:

$$\begin{cases}\beta'_u(\gamma, \lambda) = (\alpha_u\gamma, \rho_u^{-1}(\gamma)\lambda),\ u\in G\ ;\\ \bar\theta^*_{\lambda_0}(\gamma, \lambda) = (\gamma, \lambda_0^{-1}\lambda),\ \lambda, \lambda_0\in\boldsymbol{R}_+^*,\ \gamma\in\Gamma\ .\end{cases}$$

Therefore, the smooth flow F^M of weights on M is isomorphic to the action of $\boldsymbol{R}_+^*$ corresponding to the closure $\overline{\rho^{-1}(G, \Gamma)}$ of the range of ρ^{-1}.

q.e.d.

COROLLARY 6.3. *Let N be a semi-finite von Neumann algebra with separable predual, and θ a continuous action on N of a separable locally compact group G such that the restriction of θ to the center C of N is ergodic. Let ρ_g, $g\in G$, be the non-singular self-adjoint positive operator affiliated with C such that $\tau\circ\theta_g = \tau(\rho_g\cdot)$ for a faithful semi-finite normal* *trace τ on N. Let $M = W^*(N, G, \theta)$. If the relative commutant $N' \cap M$ of N in M is contained in N, and hence must be C, then the smooth flow F^M of weights on M is isomorphic to the closure $\overline{(\Delta_G \rho)^{-1}(G, C)}$ of the range of $(\Delta_G \rho)^{-1}$, where Δ_G is the modular function of G.*

PROOF. Let φ be the weight on M dual to trace τ on N in the sense of [8]. Let $\{u(g): g \in G\}$ be the unitary representation of G in M which, together with N, generates M. By a result of [8], we have

$$\begin{cases} \sigma_t^\varphi(x) = x, \ x \in N, \ t \in \boldsymbol{R} \\ \sigma_t^\varphi(u(g)) = \Delta_G(g)^{it} u(g) \rho_g^{it}, \ g \in G\,. \end{cases}$$

Hence the quartet, M, φ, N and $\{u(g): g \in G\}$ satisfies the assumption in Theorem 6.2 with $\rho_{u(g)} = \Delta_G(g)\rho_g$, $g \in G$. q.e.d.

The assumption that $N' \cap M = C$ is satisfied if either (a) G is discrete and θ is free, or (b) the restriction of θ to C is free. Case (a) when N is abelian goes back to the classical work of Murray and von Neumann [19], and when N is a factor it is due to Nakamura and Takeda [21] and Suzuki [27]. Case (b) is relatively new, shown independently by Sauvageot, [26], and the authors, [7]. Since Sauvageot's paper [26] is now available, we will omit the proof for Case (b). However, it is still an open question as to when $N' \cap M = C$ holds with a non-discrete group G.

COROLLARY 6.4. *In the same situation as in Corollary 6.3 with $G = \boldsymbol{Z}$, the smooth flow F^M of weights on M is isomorphic to the ergodic flow built from the ergodic automorphism θ on C under the ceiling function ρ, where $\rho = d\tau \circ \theta^{-1}/d\tau$.*

One should compare this result with Theorem 3.4.

We now turn to the study of an explicit construction of the continuous decomposition of a factor M of type III when M is the crossed product of a semi-finite von Neumann algebra N by a locally compact group G.

Let N be a von Neumann algebra with separable predual, and C the center of N. Let $\{\Gamma, \mu\}$ be a standard measure space with $C = L^\infty(\Gamma, \mu)$. Let

$$N = \int_\Gamma^\oplus N(\gamma) d\mu(\gamma)$$

be the central decomposition of N. Let α be a continuous action of a separable locally compact group G on N. Discarding a null set from Γ, we may assume that G acts on Γ in such a way that

$$\alpha_g(x)(\gamma) = x(g^{-1}\gamma), \ g \in G, \ x \in C, \ \gamma \in \Gamma\,.$$

Furthermore, the action α_g gives rise to a family $\{\alpha_g, \gamma : \gamma \in \Gamma\}$ of isomorphisms from $N(\gamma)$ onto $N(g\gamma)$, $\gamma \in \Gamma$ such that

$$\begin{cases} \alpha_g(x)(\gamma) = \alpha_{g,g^{-1}\gamma}(x(g^{-1}\gamma)),\ x \in N,\ \gamma \in \Gamma\ ; \\ \alpha_{gh,\gamma} = \alpha_{g,h\gamma} \circ \alpha_{h,\gamma},\ g,\ h \in G\ . \end{cases}$$

Suppose that ρ is a 1-cocycle on $G \times \Gamma$ with coefficients in a separable locally compact group H. We then consider the tensor product $N \otimes L^\infty(H)$, and denote it by $\bar{N}$. The central decomposition of $\bar{N}$ is given by

$$\bar{N} = \int_{\Gamma \times H}^{\oplus} N(\gamma, h) d\mu(\gamma) dh\ ,$$

where $N(\gamma, h) = N(\gamma)$, $\gamma \in \Gamma$, $h \in H$. We now define actions $\bar{\theta}$ of G and α of H on $\bar{N}$ respectively as follows:

$$\begin{cases} \bar{\alpha}_g(x)(\gamma, h) = \alpha_{g,g^{-1}\gamma}(x(g^{-1}\gamma, \rho(g^{-1}, \gamma)h)),\ x \in \bar{N},\ g \in G,\ h \in H,\ \gamma \in \Gamma\ ; \\ \beta_k(x)(\gamma, h) = x(\gamma, hk),\ h,\ k \in H\ . \end{cases}$$

Obviously, the actions α and β commute. As a straightforward generalization of Mackey's definition, we say that the action $\bar{\alpha}$ on $\bar{N}$ of G is the kernel of ρ and the action β of H on the fixed point subalgebra $\bar{N}^G$ of $\bar{N}$ under $\bar{\alpha}$ is the closure of the range of ρ.

COROLLARY 6.5. *Let N be a semi-finite von Neumann algebra with separable predual, and α a continuous action of a separable locally compact group G on N such that the restriction of α to the center C of N is ergodic. Let ρ_g, $g \in G$, be a non-singular self-adjoint positive operator affiliated with C such that $\tau \circ \alpha_g = \tau(\rho_g \cdot)$ for a faithful semi-finite normal trace τ on N. Let $M = W^*(N, G, \alpha)$. Let $M = W^*(M_0, \boldsymbol{R}, \theta)$ be a continuous decomposition of M. Then M_0 is isomorphic to the crossed product of the kernel $\{\bar{N}, \bar{\alpha}\}$ of the* 1-*cocycle* $\operatorname{Log}(\Delta_G \rho)^{-1}$, *where Δ_G means of course the modular function of G. The action θ of $\boldsymbol{R}$ on M_0 is isomorphic to the canonical extension $\bar{\beta}$ of the action β of $\boldsymbol{R}$ on $\bar{N}$ to the crossed product $W^*(\bar{N}, G, \bar{\alpha})$.*

The proof does not require much change in the proof of Theorem 6.2; so we leave it to the reader. However, we should observe the following:

REMARK 6.6. In the previous result, we did not require that $N' \cap M = C$. This is because we consider the extended action $\bar{\beta}$ of $\boldsymbol{R}$ on $W^*(\bar{N}, G, \bar{\alpha})$. There are many evidences that the crossed product $W^*(\bar{N}, G, \bar{\alpha})$ of the kernel of ρ in general will be a better substitute of the fixed point subalgebra $\bar{N}^G$ of $\bar{N}$ under $\bar{\alpha}$. We quote Corollary III.2.15 for a reference to this statement.

REMARK 6.7. A slight modification of the above argument shows that if $M = U(G)$ is the von Neumann algebra generated by the left regular representation of a principal virtual group G, then the smooth flow F^M of weights on M is precisely the closure of the range of the module Δ_G of G, where Δ_G is the natural analogue for G of the classical module for a locally compact group—see [26; page 198 (B) and page 203 (i)]. Furthermore the von Neumann algebra N generated by the left regular representation of the kernel of the module Δ_G is precisely the II_∞-von Neumann algebra appearing in the continuous decomposition of M.

We apply now Theorem 6.2 to the tensor product of factors of type III. On the modular spectrum, there is no formula for $S(M_1 \otimes M_2)$ in terms of $S(M_1)$ and $S(M_2)$. However, we do have a formula for computing the virtual modular spectrum $S_v(M_1 \otimes M_2)$ out of $S_v(M_1)$ and $S_v(M_2)$. Once again following Mackey's formalism, we will construct the product of two virtual subgroups of an abelian group G.

Let G be a separable locally compact abelian group acting ergodically on standard measure spaces $\{\Gamma_1, \mu_1\}$ and $\{\Gamma_2, \mu_2\}$. Let α and β be the actions of G on $\Gamma_1 \times \Gamma_2$ given by the following:

$$\alpha_g(\gamma_1, \alpha_2) = (g\gamma_1, \gamma_2),\ \beta_g(\gamma_1, \gamma_2) = (g^{-1}\gamma_1, g\gamma_2)\ , \qquad g \in G\ .$$

The virtual subgroup of G corresponding to the restriction of α to $L^\infty(\Gamma_1 \times \Gamma_2, \mu_1 \times \mu_2)^\beta$, the fixed point subalgebra of $L^\infty(\Gamma_1 \times \Gamma_2, \mu_1 \times \mu_2)$ under β, will be called the *closure of the product* of the virtual subgroups of G corresponding to the actions of G on Γ_1 and Γ_2.

COROLLARY 6.8. *For two factors M_1 and M_2 of type* III *with separable predual, the virtual modular spectrum $S_v(M_1 \otimes M_2)$ is the closure of the product of $S_v(M_1)$ and $S_v(M_2)$.*

PROOF. Let $M_1 = W^*(N_1, \boldsymbol{R}, \theta^1)$ and $M_2 = W^*(N_2, \boldsymbol{R}, \theta^2)$ be continuous decomposition of M_1 and M_2 respectively. We have then $M_1 \otimes M_2 = W^*(N_1 \otimes N_2, \boldsymbol{R}^2, \theta^1 \otimes \theta^2)$. Let $M = M_1 \otimes M_2$, $N = N_1 \otimes N_2$, and C_1 and C_2 be the centers of N_1 and N_2 respectively, and let $C = C_1 \otimes C_2$. We have then, by Theorem 5.1,

$$\begin{aligned} N' \cap M &= (N_1 \otimes N_2)' \cap (M_1 \otimes M_2) = (N_1' \cap M_1) \otimes (N_2' \cap M_2) \\ &= C_1 \otimes C_2 = C\ . \end{aligned}$$

Therefore, we can apply Corollary 6.3 to M, N, $\theta = \theta^1 \otimes \theta^2$ and $\boldsymbol{R}^2$. Let τ_1 and τ_2 be faithful semi-finite normal traces on N_1 and N_2 respectively with $\tau_1 \circ \theta_s^1 = e^{-s}\tau_1$ and $\tau_2 \circ \theta_t^2 = e^{-t}\tau_2$. Put $\tau = \tau_1 \otimes \tau_2$. It follows then that $\tau \circ \theta_{s,t} = e^{-(s+t)}\tau$, $(s, t) \in \boldsymbol{R}^2$; hence $\rho_{s,t} = e^{-(s+t)}$. By Corollary 6.3, we must compute the closure of the range of this ρ with respect to $\{C, \boldsymbol{R}^2, \theta\}$. Let $\{\Gamma_1, \mu_1\}$ and $\{\Gamma_2, \mu_2\}$ be standard measure spaces with $C_1 = L^\infty(\Gamma_1, \mu_1)$ and $C_2 = L^\infty(\Gamma_2, \mu_2)$. Put $\{\Gamma, \mu\} = \{\Gamma, \mu\} = \{\Gamma_1, \mu_1\} \times \{\Gamma_2, \mu_2\}$. Then $C = L^\infty(\Gamma, \mu)$. In order to avoid possible confusion, we denote by $\{\theta_s^{1*}\}$ and $\{\theta_t^{2*}\}$ the flows in Γ_1 and Γ_2 induced by $\{C_1, \theta^1\}$ and $\{C_2, \theta^2\}$. We then have

$$\bar{\theta}^*_{s,t}(\gamma_1, \gamma_2, \lambda) = (\theta_s^{1*}\gamma_1, \theta_t^{2*}\gamma_2, e^{s+t}\lambda),\ s, t \in \boldsymbol{R},\ \lambda \in \boldsymbol{R}_+^* ;$$
$$\alpha^*_{\lambda_0}(\gamma_1, \gamma_2, \lambda) = (\gamma_1, \gamma_2, \lambda_0^{-1}\lambda),\ (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2,\ \lambda_0 \in \boldsymbol{R}_+^* .$$

Put

$$T(\gamma_1, \gamma_2, \lambda) = (\theta^{1*}_{\mathrm{Log}\,\lambda}\gamma_1, \gamma_2, \lambda),\ (\gamma_1, \gamma_2, \lambda) \in \Gamma_1 \times \Gamma_2 \times \boldsymbol{R}_+^* .$$

We have then

$$T^{-1}\bar{\theta}^*_{s,t}T(\gamma_1, \gamma_2, \lambda) = (\theta^{1*}_{-t}\gamma_1, \theta_t^{2*}\gamma_2, e^{s+t}\lambda) ;$$
$$T^{-1}\alpha^*_{\lambda_0}T(\gamma_1, \gamma_2, \lambda) = (\theta^{1*}_{\mathrm{Log}\,\lambda_0}\gamma_1, \gamma_2, \lambda) . \qquad \text{q.e.d.}$$

Therefore, our assertion follows.

CHAPTER III. NON-ABELIAN COHOMOLOGY IN PROPERLY INFINITE VON NEUMANN ALGEBRAS

III.0. **Introduction.** So far we have studied the flow of weights on a factor. As the reader has already noticed, what we have treated there is nothing else but the first cohomology of $\boldsymbol{R}$ in the unitary group of a factor with respect to the modular automorphism group. The techniques developed there can also be applied to the general case, not only to the modular automorphism group. The first cohomology of a locally compact group G in the unitary group $\mathfrak{U}$ of a von Neumann algebra M with respect to an action α of G on M is related to the structure of the crossed product $W^*(M, G, \alpha)$ and its automorphism group. We shall regard a one cocycle in the unitary group as a twisted unitary representation and then follow the well-established multiplicity theory of unitary representations, instead of following the algebraic theory of cohomology. Of course, integrable actions of the group in question will play the role corresponding to that of integrable weights. The result of particular interest is the stability of the single automorphism or of the one parameter automorphism group appearing in the discrete or the continuous decomposition of a factor type III, (see Section 5).

In §1, developing elementary properties of twisted *-representations, we shall lay down our strategic point of view. We shall see in §2 that, as for weights, there exists a unique square integrable twisted unitary representation, called *dominant*, which dominates all other square integrable twisted representations, Theorem 2.12. As a corollary, it will be seen that the fixed point subalgebra of an integrable action is isomorphic to the reduced algebra of the crossed product. Section 3 is devoted to the case of abelian groups. A characterization of a dominant action will be given in terms of the spectrum; and also it will be shown that $\Gamma(\alpha)$, the exterior invariant of α ([3, part II]) is the kernel of the restriction of the dual action $\hat{\alpha}$ to the center of the crossed product $W^*(M, G, \alpha)$, a generalization of [30; Theorem 9.6].

In §4, we shall study the Galois type correspondence between the closed subgroups and the intermediate von Neumann subalgebras for an integrable action of an abelian group. Section 5 is devoted to the study of stability of automorphisms (or one parameter groups of automorphisms) of semi-finite von Neumann algebras.

III.1. **Elementary properties of twisted *-representation.** Let M be a properly infinite von Neumann algebra equipped with a continuous action α of a locally compact group G. We assume the σ-finiteness of M always.

DEFINITION 1.1. *A σ-strong* continuous function $a: s \in G \mapsto a(s) \in M$ is called an α-twisted *-representation of G in M if the following conditions are satisfied:*

$$\begin{cases} a(st) = a(s)\alpha_s(a(t)), \; s, t \in G ; \\ a(s^{-1}) = \alpha_s^{-1}(a(s)^*) . \end{cases}$$

If all $a(s)$ are unitaries, then it is called an α-twisted unitary representation of G in M.

We denote by $Z_\alpha(G, M)$ (resp. $Z_\alpha(G, \mathfrak{U}(M))$) the set of all α-twisted *-representations (resp. unitary representation) of G in M, where $\mathfrak{U}(M)$ denotes the unitary group of M. A straightforward computation gives the following:

LEMMA 1.2. *If $a \in Z_\alpha(G, M)$, then all $a(s)$ are partial isometries such that*

$$a(s)a(s)^* = a(1) \quad \text{and} \quad a(s)^*a(s) = \alpha_s(a(1)), \; s \in G ,$$

where 1 means, of course, the identity of G.

We denote $a(1)$ by e_a. It is also straightforward to observe that by the formula:

$${}_a\alpha_s(x) = a(s)\alpha_s(x)a(s)^* , \quad x \in M_{e_a} , \quad s \in G ,$$

we can define a new action ${}_a\alpha$ of G on the reduced von Neumann algebra $M_{e_a}(=e_aMe_a)$. We denote the fixed point subalgebra of M_{e_a} under this new action ${}_a\alpha$ by M^a. If p is a projection in M^a, then the map: $s \in G \mapsto$

$pa(s) \in M$ is also an α-twisted *-representation of G in M, which will be called the *reduced* α-twisted *-representation by p and denoted by a^p. We call it also a *subrepresentation* of a.

DEFINITION 1.3. We say that a and b in $Z_\alpha(G, M)$ are equivalent and write $a \cong b$ if there exists an element $c \in M$ such that

$$\begin{cases} a(s) = c^* b(s) \alpha_s(c) , & s \in G ; \\ b(s) = c a(s) \alpha_s(c^*) . \end{cases}$$

We write $a \prec b$ if $a \cong b^q$ for some projection q in M^b.

The reader should be aware of the following 2×2-matrix arguments:

LEMMA 1.4. *Let $P = M \otimes F_2$ be the 2×2-matrix algebra over M, and $\bar{\alpha}$ be the action $\alpha \otimes 1$ of G on P. Given $a, b \in Z_\alpha(G, M)$, we define $c \in Z_{\bar{\alpha}}(G, P)$ by*

$$c(s) = a(s) \otimes e_{11} + b(s) \otimes e_{22} , \qquad s \in G ,$$

with a fixed matrix unit $\{e_{ij}\}$ in F_2. Then the following two statements are equivalent:

(i) $\qquad a \prec b \quad (resp.\ a \cong b) :$

(ii) $\quad e_a \otimes e_{11} \leqq e_e \otimes e_{22} \quad (resp.\ e_a \otimes e_{11} \sim e_b \otimes e_{22}) \quad in \quad P^c .$

We leave the proof to the reader.

DEFINITION 1.5. With the same notations as in Lemma 1.4, we call a and b disjoint and write $a \mathrel{\overset{|}{\circ}} b$ if $e_a \otimes e_{11}$ and $e_b \otimes e_{22}$ are centrally orthogonal in P^c. We say that a and b are quasi-equivalent and write $a \sim b$ if $e_a \otimes e_{11}$ and $e_b \otimes e_{22}$ have the same central support, (namely $e_a \otimes e_{11} + e_b \otimes e_{22}$), in P^c.

Given a and b in $Z_\alpha(G, M)$, we set

$$I(a, b) = \{x \in e_a M e_b : x b(s) = a(s) \alpha_s(x),\ s \in G\} .$$

It is not hard to see the following properties of $I(a, b)$:

$$I(b, a) = I(a, b)^* ; \quad I(a, a) = M^a ; \quad I(b, b) = M^b ;$$

$$x = \sum_{i,j=1}^{2} x_{i,j} \otimes e_{i,j} \in P^c \Leftrightarrow \begin{cases} x_{11} \in I(a, a) , & x_{12} \in I(a, b) , \\ x_{21} \in I(b, a) , & x_{22} \in I(b, b) ; \end{cases}$$

$$a \mathrel{\overset{|}{\circ}} b \Leftrightarrow I(a, b) = \{0\} .$$

LEMMA 1.6. (i) *Given a, b and, c in $Z_\alpha(G, M)$, we have*

$$I(a, b) I(b, c) \subset I(a, c) .$$

(ii) *If $x = uh$ is the polar decomposition of $x \in I(a, b)$, then we have*

$$h \in I(b, b) \quad \textit{and} \quad u \in I(a, b)\ .$$

The proof is straightforward, so we leave it to the reader.

DEFINITION 1.7. We say that $a \in Z_\alpha(G, M)$ is of infinite multiplicity if M^a is properly infinite.

LEMMA 1.8. *If a and b in $Z_\alpha(G, M)$ are of infinite multiplicity, then*

$$a \cong b \Leftrightarrow a \sim b\ .$$

PROOF. The implication "$\Rightarrow$" is trivial.

$\Leftarrow$: Suppose $a \sim b$. Let $P = M \otimes F_2$, $\bar{\alpha}$ and $c \in Z_{\bar{\alpha}}(G, M)$ be as in Lemma 1.4. It follows then that $e_a \otimes e_{11}$ and $e_b \otimes e_{22}$ are both properly infinite projections in P^c by assumption; so they are equivalent to their central support in P^c, P being σ-finite. Therefore, we have

$$e_a \otimes e_{11} \sim e_a \otimes e_{11} + e_b \otimes e_{22} \sim e_b \otimes e_{22} \quad \text{in} \quad P^c\ . \qquad \text{q.e.d.}$$

We close this section with the following:

REMARK 1.9. If α is a continuous action of a separable locally compact group G on a von Neumann algebra M with separable predual, then for an M-valued function $a: s \in G \rightarrow a(s) \in M$ to agree almost everywhere with an α-twisted *-representation a' of G in M, it is sufficient that a satisfies the conditions in Definition 1.1 for almost every pair s, t in G, cf [18].

III.2. Tensor product and integrability of twisted *-representations. Let M and N be von Neumann algebras equipped with continuous actions α and β of a locally compact group G respectively. We understand naturally the covariant system $\{M \otimes N, \alpha \otimes \beta\}$ on G. Given $a \in Z_\alpha(G, M)$ and $b \in Z_\beta(G, N)$, we define $a \otimes b \in Z_{\alpha\otimes\beta}(G, M \otimes N)$ by

$$(a \otimes b)(s) = a(s) \otimes b(s)\ , \qquad s \in G\ .$$

It is of our particular interest when $N = \mathfrak{L}(\mathfrak{K})$ and $\beta = 1$. This means that b is an ordinary unitary representation of G of the Hilbert space $\mathfrak{K}$.

THEOREM 2.1. *Let M be a von Neumann algebra equipped with a continuous action α of a locally compact group G. Put $P = M \otimes \mathfrak{L}(L^2(G))$. If λ_r is the right regular representation of G on $L^2(G)$, then $1 \otimes \lambda_r \in Z_{\alpha\otimes 1}(G, P)$ and*

$$W^*(M, G, \alpha) \cong P^{(1\otimes\lambda_r)}\ .$$

PROOF. We may assume that M acts on a Hilbert space $\mathfrak{H}$ in such a way that $\{M, \mathfrak{H}\}$ is standard, so that there exists canonically a unitary representation U of G on $\mathfrak{H}$ such that $\alpha_s(x) = U(s)xU(s)^*$, $x \in M$, $s \in G$. The crossed product $W^*(M, G, \alpha)$ of M by α acts on the Hilbert space $\mathfrak{H} \otimes L^2(G)$. In this situation, the recent result of Digernes, [8], says that the commutant $W^*(M, G, \alpha)'$ of $W^*(M, G, \alpha)$ is generated by $M' \otimes 1$ and $U(s) \otimes \lambda_r(s)$, $s \in G$.

Hence we have

$$\begin{aligned} W^*(M, G, \alpha) &= W^*(M, G, \alpha)'' = \{M' \otimes 1 \cup \{U(s) \otimes \lambda_r(s) : s \in G\}\}' \\ &= M \otimes \mathfrak{L}(L^2(G)) \cap \{U(s) \otimes \lambda_r(s) : s \in G\}' \\ &= P^{(1 \otimes \lambda_r)} . \end{aligned}$$

q.e.d.

Since the left and right regular representations of G are equivalent in $\mathfrak{L}(L^2(G))$ as twisted unitary representation with respect to the trivial action of G on $\mathfrak{L}(L^2(G))$, we have also

$$P^{(1 \otimes \lambda_l)} \cong W^*(M, G, \alpha)$$

with the left regular representation λ_l of G.

The next proposition is classical in homological algebra.

PROPOSITION 2.2. *For any $a \in Z_\alpha(G, \mathfrak{U}(M))$, we have*

$$a \otimes \lambda_r \cong 1 \otimes \lambda_r \quad \textit{in} \quad P = M \otimes \mathfrak{L}(L^2(G)) .$$

PROOF. Suppose that M acts on a Hilbert space $\mathfrak{H}$. Then P acts on $\mathfrak{H} \otimes L^2(G) = L^2(\mathfrak{H}; G)$. We define a unitary b in $M \otimes L^\infty(G) \subset P$ by the following:

$$(b\xi)(s) = a(s^{-1})\xi(s),\ \xi \in L^2(\mathfrak{H}; G),\ s \in G .$$

We compute then

$$[b(1 \otimes \lambda_r(t))\xi](s) = a(s^{-1})\xi(st) ;$$
$$\{[a(t) \otimes \lambda_r(t)]\alpha \otimes 1)_t(b)\xi\}(s) = a(t)\alpha_t(a((st)^{-1}))\xi(st) ,$$

where we use the right invariant Haar measure $d_r s$ in the construction of $L^2(\mathfrak{H}; G)$. We compute further the last term:

$$\begin{aligned} a(t)\alpha_t(a((st)^{-1})) &= a(t)\alpha_t(a(t^{-1}s^{-1})) = a(t)\alpha_t(a(t^{-1})\alpha_t^{-1}(a(s^{-1})) \\ &= a(t)\alpha_t(a(t^{-1}))a(s^{-1}) = a(s^{-1}) . \end{aligned}$$

Hence we get

$$b(1 \otimes \lambda_r(t)) = [a(t) \otimes \lambda_r(t)](\alpha \otimes 1)_t(b) , \qquad t \in G .$$

Therefore, our assertion follows, since b is unitary. q.e.d.

DEFINITION 2.3. Given a σ-finite properly infinite von Neumann algebra M equipped with a continuous action α of a separable locally compact group G, an α-twisted unitary representation a of G in M is said to be dominant if $a \otimes \lambda_r \cong a \otimes 1$ in $M \otimes \mathfrak{L}(L^2(G))$ and a is of infinite multiplicity.

From now on, we assume always that the von Neumann algebras and the groups in question are σ-finite and separable respectively.

COROLLARY 2.4. *Any dominant α-twisted unitary representations are equivalent.*

PROOF. Let a and b be dominant α-twisted unitary representations of G in M. By Theorem 2.2, we have

$$a \otimes 1 \cong a \otimes \lambda_r \cong 1 \otimes \lambda_r \cong b \otimes \lambda_r \simeq b \otimes 1$$

in $M \otimes \mathfrak{L}(L^2(G))$. Therefore, we have only to show that if a and b in $Z_\alpha(G, \mathfrak{U}(M))$ are of infinite multiplicity, then $a \otimes 1 \cong b \otimes 1$ in $M \otimes F_\infty$ implies $a \cong b$ in M with F_∞ a factor of type I_∞. But $a \otimes 1 \cong b \otimes 1$ in $M \otimes F_\infty$ means that $a \sim b$; hence $a \cong b$ by Lemma 1.8. q.e.d.

COROLLARY 2.5. *If $a \in Z_\alpha(G, \mathfrak{U}(M))$ is dominant, then*

$$M^a \cong W^*(M, G, \alpha) .$$

DEFINITION 2.6. A continuous action α of G on M is said to be integrable if the set $\mathfrak{q}_\alpha$ of all x in M such that the integral $\int_G \alpha_s(x^*x)d_ls$ exists in M with respect to the left invariant Haar measure d_ls in G, is σ-weakly dense in M. We say that $a \in Z_\alpha(G, M)$ is square integrable if the action ${}_a\alpha$ of G on M_{e_a} is integrable.

We note here that the integral $\int_G \alpha_s(x^*x)d_ls$ is defined as the limit of the increasing net $\int_K \alpha_s(x^*x)d_ls$ indexed by the net of compact subsets K of G. The very much similar arguments as those in the case of weights show that

a) $\mathfrak{q}_\alpha$ is a left ideal of M;

b) $\mathfrak{p}_\alpha = \mathfrak{q}_\alpha^*\mathfrak{q}_\alpha = \{y^*x : x, y \in \mathfrak{q}_\alpha\}$ is a hereditary *-subalgebra of M generated linearly by the positive part $\mathfrak{p}_\alpha^+ = \mathfrak{p}_\alpha \cap M_+$;

c) $\mathfrak{p}_\alpha^+ = \left\{x \in M_+ : \int_G \alpha_s(x)d_ls \ \text{ exists}\right\}$;

d) The integral

$$E_\alpha(x) = \int_G \alpha_s(x)d_ls$$

makes sense for any $x \in \mathfrak{p}_\alpha$.

The following further properties of E_α are easily verified:

e) $E_\alpha(x)$ lies in the fix point algebra M^α;

f) $E_\alpha(uxv) = uE_\alpha(x)v$, $x \in \mathfrak{p}_\alpha$, $u, v \in M^\alpha$;

g) $E_\alpha(x^*x) \geqq 0$ and $E_\alpha(x^*x) = 0 \Rightarrow x = 0$;

h) $E_\alpha(\sup x_i) = \sup E_\alpha(x_i)$ for any increasing bounded net $\{x_i\}$ in M_+, where $E_\alpha(x) = +\infty$ if $x \in M_+$ is not in $\mathfrak{p}_\alpha^+$, and $\sup y_i = +\infty$ if $\{y_i\}$ is not bounded in M^α.

From property (f), we conclude immediately the following:

LEMMA 2.7. *Any subrepresentation of a square integrable α-twisted $*$-representation of G in M is also square integrable.*

EXAMPLE 2.8. Let $M = \mathfrak{L}(\mathfrak{H})$ and $\alpha = 1$. For a unitary representation $\{U, \mathfrak{H}\}$ of G on $\mathfrak{H}$, U is square integrable as a twisted unitary representation with respect to the trivial action α in the sense of Definition 2.6 if and only if $\{U, \mathfrak{H}\}$ is square integrable in the sense that

$$\int_G |(U(s)\xi|\xi)|^2 d_l s < +\infty$$

for a dense set of ξ in $\mathfrak{H}$.

EXAMPLE 2.9. Let $M = L^\infty(G)$ and α be the translation action of G from the right. It is immediately seen that $\mathfrak{p}_\alpha = L^\infty(G) \cap L^1(G, d_l s)$ and

$$E_\alpha(f) = \int_G f(s) d_l s \ .$$

LEMMA 2.10. *Let M and N be von Neumann algebras equipped with continuous actions α and β of G respectively. If either α or β is integrable, then the tensor product $\alpha \otimes \beta$ on $M \otimes N$ is integrable.* q.e.d.

We leave the proof to the reader.

LEMMA 2.11. *The regular representation of G is square integrable in $Z_1(G, \mathfrak{L}(L^2(G)))$.*

PROOF. Let λ_r be the right regular representation of G on $L^2(G)$. Let $\alpha_s = \mathrm{Ad}\,(\lambda_r(s))$, $s \in G$. It follows that the action α leaves the maximal abelian algebra $L^\infty(G) = \mathfrak{A}$ globally invariant and $\alpha|_{\mathfrak{A}}$ is the right translation action of G on $\mathfrak{A}$. Hence $\mathfrak{p}_\alpha \cap \mathfrak{A} = L^\infty(G) \cap L^1(G, d_l s)$, which contains a net converging σ-strongly to 1. Therefore, $\mathfrak{p}_\alpha$, hence $\mathfrak{q}_\alpha$, is σ-weakly dense in $\mathfrak{L}(L^2(G))$, which means that λ_r is square integrable in

$$Z_1(G, \mathfrak{L}(L^2(G))) \ . \qquad \text{q.e.d.}$$

THEOREM 2.12. *Let M be a σ-finite properly infinite von Neumann algebra equipped with a continuous action α of a separable locally compact group G.*

(i) *There exists a dominant α-twisted unitary representation a of G in M, which is unique up to equivalence.*

(ii) *An α-twisted $*$-representation b of G in M is square integrable if and only if $b \prec a$.*

PROOF. Since M is properly infinite, replacing α by ${}_a\alpha$, we may assume that M^α is properly infinite. Choosing a factor F_∞ of type I_∞ contained in M^α, we may identify $\{M, \alpha\}$ with a covariant system $\{N \otimes F_\infty, \beta \otimes 1\}$ on G. Identifying once again F_∞ with the tensor product $\mathfrak{L}(L^2(G)) \otimes B$ of $\mathfrak{L}(L^2(G))$ and a factor B of type I_∞, we can consider a $(\beta \otimes 1)$-twisted unitary representation $1 \otimes \lambda_r \otimes 1$ of G in $N \otimes \mathfrak{L}(L^2(G)) \otimes B = M$. We have then

$$M^{(1\otimes\lambda_r\otimes 1)} \supseteqq N^\beta \otimes \lambda_r(G)' \otimes B .$$

Hence $1 \otimes \lambda_r \otimes 1$ is of infinite multiplicity. Therefore, $1 \otimes \lambda_r \otimes 1$ is dominant.

For the second assertion, we need the following results:

LEMMA 2.13. *If $b \in Z_\alpha(G, M)$ is square integrable, then*

$$\bigvee \{\operatorname{supp} x^*x : x \in I(b \otimes \lambda_r, b \otimes 1)\} = e_b \otimes 1 \quad \textit{in} \quad M \otimes \mathfrak{L}(L^2(G)) .$$

PROOF. Let e denote the left hand side of the equality. By Lemma 1.6, e belongs to $[M \otimes \mathfrak{L}(L^2(G))]^{(b\otimes 1)}$. For any unitary $u \in [M \otimes \mathfrak{L}^2(G))]^{(b\otimes 1)}$, we have $I(b \otimes \lambda_r, b \otimes 1)u = I(b \otimes \lambda_r, b \otimes 1)$; hence $u^*eu = e$, so that e is a central projection in $[M \otimes \mathfrak{L}(L^2(G))]^{(b\otimes 1)}$. Since $I(b \otimes \lambda_r, b \otimes 1)e = I(b \otimes \lambda_r, b \otimes 1)$, we have only to show

$$I(b \otimes \lambda_r, b \otimes 1)f \neq \{0\}$$

for any non-zero central projection f in $[M \otimes \mathfrak{L}(L^2(G))]^{(b\otimes 1)}$. Since $[M \otimes \mathfrak{L}(L^2(G))]^{(b\otimes 1)} = M^b \otimes \mathfrak{L}(L^2(G))$, f is of the form $p \otimes 1$ with a central projection p in M^b. We consider now M on a Hilbert space $\mathfrak{H}$ and $L^2(G)$ with respect to the right Haar measure d_rs on G. We note, however, that $d_rs^{-1} = d_ls$. Then $M \otimes \mathfrak{L}(L^2(G))$ acts on $L^2(\mathfrak{H}; G)$. Choose an $x \in \mathfrak{p}_{b\alpha}$ with $xp = x \neq 0$ and a continuous function f on G with compact support. Put

$$(y\xi)(s) = {}_b\alpha_s^{-1}(x) \int_G f(t)\xi(t)d_rt,\ \xi \in L^2(\mathfrak{H}; G) .$$

We have then

$$\begin{aligned}
\| y\xi \|^2 &= \int_G \Big\| {}_b\alpha_s^{-1}(x)\Big(\int_G f(t)\xi(t)d_rt\Big)\Big\|^2 d_rs \\
&\leqq \int_G \int_G \| f(t){}_b\alpha_s^{-1}(x)\xi(t) \|^2 d_rt d_rs \\
&= \int_G | f(t) |^2\Big(\int_G \| {}_b\alpha_s^{-1}(x)\xi(t) \|^2 d_rs\Big)d_rt \\
&= \int_G | f(t) |^2\Big(\int_G \| {}_b\alpha_s(x)\xi(t) \|^2 d_ls\Big)d_rt
\end{aligned}$$

$$= \int_G |f(t)|^2 (E_{b^\alpha}(x^*x)\xi(t) \mid \xi(t)) d_r t$$

$$= \int_G |f(t)|^2 \| E_{b^\alpha}(x^*x)^{1/2}\xi(t) \|^2 d_r t$$

$$\leqq \|f\|_\infty^2 \| E_{b^\alpha}(x^*x) \| \, \|\xi\|^2 .$$

Hence y is bounded; so $y \in M \otimes \mathfrak{L}(L^2(G))$. Furthermore, we have, for any $\xi \in L^2(\mathfrak{H}; G)$ and $r, s \in G$,

$$[y(pb(r) \otimes 1)\xi](s) = {}_b\alpha_s^{-1}(x)pb(r) \int_G f(t)\xi(t) d_r t$$

$$= {}_b\alpha_s^{-1}(x)b(r) \int_G f(t)\xi(t) d_r t ;$$

$$\{[b(r) \otimes \lambda_r(s)][\alpha_s \otimes 1](y)\xi\}(s) = b(r)[(\alpha_r \otimes 1)(y)\xi](sr)$$

$$= b(r)\alpha_r[{}_b\alpha_{sr}^{-1}(x)] \int_G f(t)\xi(t) d_r t$$

$$= {}_b\alpha_r \circ {}_b\alpha_{sr}^{-1}(x) b(r) \int_G f(t)\xi(t) d_r t$$

$$= {}_b\alpha_s^{-1}(x)b(r) \int_G f(t)\xi(t) d_r t .$$

Hence y belongs to $I(b \otimes \lambda_r, b \otimes 1)$ and $y(p \otimes 1) = y$. Clearly $y \neq 0$ if $f \neq 0$. q.e.d.

LEMMA 2.14. *For any $b \in Z_\alpha(G, M)$, there exists $\check{b} \in Z_\alpha(G, \mathfrak{U}(M))$ with infinite multiplicity such that $b \prec \check{b}$. If b is square integrable, then we can chose a square integrable $\check{b}$.*

PROOF. Let $e = e_b$ and z be the central support of e in the whole algebra M. Since $\alpha_s(z)$ is the central support of $\alpha_s(e) = b(s)^*b(s)$, $s \in G$, we have $\alpha_s(z) = z$. Therefore, we have $\{M, \alpha\} = \{M_z, \alpha\} \oplus \{M_{1-z}, \alpha\}$ in the obvious sense. It follows from Theorem 2.12 (i) that there exists a dominant $b_2 \in Z_\alpha(M_{1-z}, \mathfrak{U}(M_{1-z}))$. We then restrict our attention to $\{M_z, \alpha\}$. Let $\{e_n\}$ and $\{u_n\}$ be families of orthogonal projections and partial isometries in M respectively such that $\sum_{n=1}^\infty e_n = z$, $u_n^* u_n = e$ and $u_n u_n^* = e_n$, $n = 1, 2, \cdots$, where the existence of such families is guaranteed by the proper infiniteness and the σ-finiteness of M. Put

$$b_1(s) = \sum_{n=1}^\infty u_n b(s)\alpha_s(u_n^*) .$$

It follows that for any $s, t \in G$,

$$b_1(s)\alpha_s(b_1(t)) = \left[\sum_{n=1}^\infty u_n b(s)\alpha_s(u_n^*)\right]\left[\sum_{m=1}^\infty \alpha_s(u_m)\alpha_s(b(t))\alpha_{st}(u_m^*)\right]$$

$$= \sum_{n,m=1}^{\infty} u_n b(s)\alpha_s(u_n^* u_m)\alpha_s(b(t))\alpha_{st}(u_m^*)$$
$$= \sum_{n=1}^{\infty} u_n b(s)\alpha_s(eb(t))\alpha_{st}(u_n^*)$$
$$= \sum_{n=1}^{\infty} u_n b(s)\alpha_s(b(t))\alpha_{st}(u_n^*)$$
$$= b_1(st)\ ;$$
$$b_1(s^{-1}) = \sum_{n=1}^{\infty} u_n b(s^{-1})\alpha_s^{-1}(u_n^*) = \sum_{n=1}^{\infty} u_n \alpha_s^{-1}(b(s)^*)\alpha_s^{-1}(u_n^*)$$
$$= \alpha_s^{-1}\Big(\sum_{n=1}^{\infty} \alpha_s(u_n)b(s)^* u_n^*\Big) = \alpha_s^{-1}(b_1(s)^*)\ ;$$
$$b_1(1) = \sum_{n=1}^{\infty} u_n b(1) u_n^* = \sum_{n=1}^{\infty} u_n e u_n^* = \sum_{n=1}^{\infty} e_n = z\ .$$

Since the map: $s \in G \longrightarrow b_1(s) \in M$ is σ-strongly continuous, b_1 is an α-twisted unitary representation of G in M_z. Put

$$\check{b}(s) = b_1(s) + b_2(s)\ .$$

It follows that $M^{\check{b}} = (M_z)^{b_1} + (M_{1-z})^{b_2}$. By the definition of a dominant representation, $(M_{1-z})^{b_2}$ is properly infinite. We will show that $(M_z)^{b_1}$ is properly infinite. Put $w_{n,m} = u_n u_m^*$, $n, m = 1.2, \cdots$. It follows that

$$w_{n,m}^* w_{n,m} = e_m \quad \text{and} \quad w_{n,m} w_{n,m}^* = e_n$$
$$b_1(s)\alpha_s(w_{n,m})b_1(s)^* = \Big(\sum_{j=1}^{\infty} u_j b(s)\alpha_s(u_j^*)\Big)\alpha_s(w_{n,m})\Big(\sum_{k=1}^{\infty} u_k b(s)\alpha_s(u_k^*)\Big)^*$$
$$= \sum_{j,k=1}^{\infty} (u_j b(s)\alpha_s(u_j^* w_{n,m} u_k)b(s)^* u_k^*)$$
$$= u_n b(s)\alpha_s(u_n^* w_{n,m} u_m)b(s)^* u_m^*$$
$$= u_n b(s)\alpha_s(e)b(s)^* u_m^* = u_n e u_m^* = w_{n,m}\ .$$

Hence $w_{n,m} \in (M_z)^{b_1}$; so b_1 is of infinite multiplicity. By construction, $b \prec b_1$; hence $b \prec \check{b}$.

Suppose now b is square integrable. Since b_2 is square integrable by definition, we need only to show that b_1 is square integrable. Let $\{x_i\}$ be a net in $\mathfrak{p}_{b^\alpha}$ such that $\lim_1 x_i = e$. Let $x_{i,n} = u_n x_i u_n^*$. We have then

$$b_1(s)\alpha_s(x_{i,n})b_1(s)^* = u_n b(s)\alpha_s(x_i)b(s)u_n^*\ ;$$

hence $x_{i,n} \in \mathfrak{p}_{b_1^\alpha}$. Since $\lim_i x_{i,n} = e_n$, the σ-strong closure $\mathfrak{p}_{b_1^\alpha}$ contains all e_n's; hence b_1^α is integrable. Thus, b_1 is square integrable, and so is $\check{b}$.

q.e.d.

PROOF OF THEOREM 2.12. (ii). By Lemma 2.14, we may assume that

b is a square integrable α-twisted unitary representation of G in M with infinite multiplicity. Consider $M \otimes \mathfrak{L}(L^2(G))$, $b \otimes \lambda_r$ and $b \otimes 1$ as well as $P = M \otimes \mathfrak{L}(L^2(G)) \otimes F_2$. Let

$$c(s) = b(s) \otimes \lambda_r(s) \otimes e_{11} + b(s) \otimes 1 \otimes e_{22} .$$

It follows from Lemma 2.13 that the central support of $1 \otimes 1 \otimes e_{11}$ in P^c majorizes $1 \otimes 1 \otimes e_{22}$. Since $M^b \otimes 1 \otimes e_{11}$ is contained in $P^c_{(1\otimes 1\otimes e_{11})}$, $1 \otimes 1 \otimes e_{11}$ is properly infinite in P^c because M^b is. Hence $1 \otimes 1 \otimes e_{11} \succ 1 \otimes 1 \otimes e_{22}$ in P^c; so $b \otimes 1 \prec b \otimes \lambda_r$. By Proposition 2.2, we have

$$b \otimes 1 \prec b \otimes \lambda_r \cong 1 \otimes \lambda_r \cong a \otimes 1$$

if $a \in Z_\alpha(G, \mathfrak{U}(M))$ is dominant. Thus $b \prec a$ because a is of infinite multiplicity. q.e.d.

COROLLARY 2.15. *Let M be a σ-finite von Neumann algebra and G a separable locally compact group. If α is an integrable action of G on M, then the fixed point algebra M^α of M under α is isomorphic to a reduced algebra of the crossed product $W^*(M, G, \alpha)$.*

PROOF. Seeing that $\alpha \otimes 1$ is integrable on $M \otimes F_\infty$ with a factor F_∞ of type I_∞, and that $(M \otimes F_\infty)^{\alpha\otimes 1} = M^\alpha \otimes F_\infty$, we may assume that M^α is properly infinite. Let $b(s) = 1$, $s \in G$, and a be a dominant α-twisted unitary representation of G in M. By Theorem 2.12, $b \prec a$, that is, there exists an isometry u in M such that $u^*u = 1$, $uu^* \in M^a$ and $u^*a(s)\alpha_s(u) = 1$, $s \in G$. Let $e = uu^*$. It follows that $\alpha_r(x) = x$ if and only if ${}_a\alpha_s(uxu^*) = uxu^*$. Hence $M^\alpha \cong M^a_e$. On the other hand, we have $M^a \cong W^*(M, G, \alpha)$ by Corollary 2.5. q.e.d.

COROLLARY 2.16. *Let M be a σ-finite von Neumann algebra and G a finite group. If α is a free action of G on M in the sense that $\alpha_g(x)a = ax$ for every $x \in M$ implies either $g = e$ or $a = 0$, then any pair of α-twisted representations of G in M are equivalent*; i.e., *the equivalence classes in $Z^1_\alpha(G, \mathfrak{U})$ reduces to a singleton.*

PROOF. The discreteness and the free action of G yield, [21], that the relative commutant of M in $W^*(M, G, \alpha)$ is $M^\alpha \cap C$, where C denotes the center of M. This means that if M is properly infinite then every $a \in Z^1_\alpha(G, \mathfrak{U})$ is quasi-equivalent to a dominant one by Theorem 2.12. The finiteness of G implies that M is properly infinite if and only if M^α is also. Hence any $a \in Z^1_\alpha(G, \mathfrak{U})$ is dominant if M is properly infinite.

Suppose M is finite. Considering $M \otimes F_\infty$ and $\alpha \otimes \iota$, we conclude from the above arguments that $M^\alpha \cap C$ is the center of M^α. Hence the uniqueness of the center valued trace in a finite von Neumann algebra implies that the restriction of the center valued trace of M to M^{α} is indeed the center valued trace of M^{α}, which means that for any projections $e, f \in M^{\alpha} e \sim f$ in M if and only if $e \sim f$ in M^{α}. Thus our assertions follows from the well exposed 2×2 matrix arguments. q.e.d.

DEFINITION 2.17. A continuous action α of a locally compact group G on a von Neumann algebra M is said to be stable if for every $a \in Z^1_{\alpha}(G, \mathfrak{U}_M)$ there exists $b \in \mathfrak{U}_M$ such that $a_g = b^* \alpha_g(b)$. A single automorphism α of M is said to be stable if every $u \in \mathfrak{U}_M$ is of the form $u = v^* \alpha(v)$ for some $v \in \mathfrak{U}_M$.

Of course, the stability of an automorphism α of M implies that any automorphism β of the form $\mathrm{Ad}(u) \cdot \alpha$ (and in particular any β with $\|\alpha - \beta\| < 2$, [11]) is conjugate to α under $\mathrm{Int}(M)$. The converse is also true when M is an infinite factor, (cf. Theorem 3.1).

We will discuss further the stability of a single automorphism and a one parameter automorphism group together with its application in Section 5.

III.3. Integrable action of abelian groups, duality and invariant Γ. In this section, we study integrable actions of an abelian group. Let G be a separable locally compact abelian group with dual group $\hat{G}$. We choose Haar measures ds in G and $d\gamma$ in $\hat{G}$ so that the Plancherel formula holds. We denote by $\langle s, \gamma \rangle$ the value of $\gamma \in \hat{G}$ at $s \in G$. An action α of G on M is by definition *dominant* if the trivial α-twisted unitary represention 1 of G in M is dominant.

THEOREM 3.1. *Let M be a properly infinite von Neumann algebra with separable M_*. For a continuous action α of a separable locally compact abelian group G on M with properly infinite M^{α}, the following conditions are equivalent*:

(i) *α is dominant;*

(ii) *For any $\gamma \in \hat{G}$, there exists $u \in \mathfrak{U}(M)$ such that $\alpha_s(u) = \langle s, \gamma \rangle u$, $s \in G$;*

(iii) *There exists a continuous action β of $\hat{G}$ on M^{α} such that*

$$\{W^*(M^{\alpha}, \hat{G}, \beta), \hat{\beta}\} \cong \{M, \alpha\} .$$

PROOF. (i) $\Rightarrow$ (ii): Since M^{α} is properly infinite,

$$\{M, \alpha\} \cong \{M \otimes \mathfrak{L}^2(L(G)), \alpha \otimes 1\} .$$

Denoting the regular representation of G on $L^2(G)$ by λ, we have

$$\{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes 1\} \cong \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes \mathrm{Ad}\, \lambda\} .$$

For each $\gamma \in \hat{G}$, let $\mu(\gamma)$ denote the unitary on $L^2(G)$ given by

$$\mu(\gamma)\xi(s) = \overline{\langle s, \gamma\rangle}\xi(s),\ \xi \in L^2(G),\ s \in G\ .$$

It follows then that

$$\mathrm{Ad}\,(\lambda(s))\mu(\gamma) = \langle s, \gamma\rangle\mu(\gamma)\ .$$

Hence, putting $u(\gamma) = 1 \otimes \mu(\gamma)$, we have

$$\{\alpha_s \otimes \mathrm{Ad}\,(\lambda(s))\}(u(\gamma)) = \langle s, \gamma\rangle u(\gamma)\ .$$

Thus, the isomorphism $\{M, \alpha\} \cong \{M \otimes \mathfrak{L}^2(L(G)), \alpha \otimes \mathrm{Ad}\,\lambda\}$ assures the existence of a unitary $u \in M$ with $\alpha_s(u) = \langle s, \gamma\rangle u$.

(ii) $\Rightarrow$ (i): Suppose that for any $\gamma \in \hat{G}$, there exists a unitary $u \in M$ with $\alpha_s(u) = \langle s, \gamma\rangle u$ for any $s \in G$. Put

$$E = \{(\gamma, u) \in \hat{G} \times \mathfrak{U}(M) : \alpha_s(u) = \langle s, \gamma\rangle u,\ s \in G\}\ .$$

It follows then that E is a closed subset of the polish space $\hat{G} \times \mathfrak{U}(M)$ whose projection to the first coordinate $\hat{G}$ covers the whole dual group $\hat{G}$. Therefore, there exists a $\mathfrak{U}(M)$-valued measurable function $u(\cdot)$ on $\hat{G}$ such that $\alpha_s(u(\gamma)) = \langle s, \gamma\rangle u(\gamma)$. Put

$$u = \int_{\hat{G}}^{\oplus} u(\gamma)d\gamma \in M \otimes L^\infty(\hat{G}) \subset M \otimes \mathfrak{L}(L^2(G))\ .$$

Since $\lambda(s) \in L^\infty(\hat{G})$ such that $\lambda(s)(\gamma) = \langle s, \gamma\rangle$, we have

$$1 \otimes \lambda(s) = \int_{G}^{\oplus} \langle s, \gamma\rangle d_\gamma \in M \otimes L^\infty(\hat{G})\ .$$

Hence we have

$$\begin{aligned} u^*(\alpha_s \otimes 1)(u) &= \int_{\hat{G}}^{\oplus} u(\gamma)^*\alpha_s(u(\gamma))d\gamma = \int_{G}^{\oplus} \langle s, \gamma\rangle d\gamma \\ &= 1 \otimes \lambda(s)\ , \quad s \in G\ . \end{aligned}$$

Therefore we have $1 \otimes 1 \cong 1 \otimes \lambda$ in $Z_{\alpha\otimes 1}(G, \mathfrak{U}(M \otimes \mathfrak{L}(L^2(G)))$. Thus, we get

$$\begin{aligned} \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes \mathrm{Ad}\,\lambda\} &\cong \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes 1\} \\ &\cong \{M, \alpha\}\ , \end{aligned}$$

since M^α is properly infinite.

(iii) $\Rightarrow$ (ii): This follows from the definition of the dual action β.

(i) $\Rightarrow$ (iii): If α is dominant, then we have, by [30; Theorem 4.6],

$$\begin{aligned} \{M, \alpha\} &\cong \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes \mathrm{Ad}\,\lambda\} \cong \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes \mathrm{Ad}\,\lambda^*\} \\ &\cong \{M \otimes \mathfrak{L}(L^2(G)), \hat{\hat{\alpha}}\}\ . \end{aligned}$$

Identifying α with $\hat{\hat{\alpha}}$, the action $\hat{\hat{\alpha}} = \beta$ is the desired action of $\hat{G}$ on M^α. q.e.d.

As in [3; Definition 2.2.1], we define the invariant $\Gamma(\alpha)$ of α as follows:

$$\Gamma(\alpha) = \bigcap \{\operatorname{Sp} \alpha^e : e \text{ runs through all non-zero projections in } M^\alpha\} .$$

We note here that the arguments for [3; Proposition 2.2.2. and Theorem 2.2.4 (c)] do not require the fact that M is a factor. Hence we have

$$\Gamma(\alpha) = \bigcap \{\operatorname{Sp} \alpha^e : e \text{ runs through all non-zero central projections in } M^\alpha\} .$$

THEOREM 3.2. *Let M be a σ-finite von Neumann algebra equipped with a continuous action α of a separable locally compact abelian group G. The invariant $\Gamma(\alpha)$ is the kernel of the restriction of the dual action $\hat{\alpha}$ of $\hat{G}$ on $W^*(M, G, \alpha)$ to the center of $W^*(M, G, \alpha)$. (Hence it is, in particular, a closed subgroup of $\hat{G}$.)*

PROOF. We consider $M \otimes \mathfrak{L}(L^2(G))$, $\alpha \otimes 1$ and $\alpha \otimes \operatorname{Ad} \lambda$ as before. Trivially, we have $\Gamma(\alpha) = \Gamma(\alpha \otimes 1)$; hence $\Gamma(\alpha) = \Gamma(\alpha \otimes \operatorname{Ad} \lambda)$ by [3, 2.2.4]. Hence we may assume that M is properly infinite and α is dominant. It follows from the previous section that there exists a continuous action θ of the dual group $\hat{G}$ on M^α such that

$$\{M, \alpha\} \cong \{W^*(M^\alpha, \hat{G}, \theta), \hat{\theta}\} ;$$
$$\{M^\alpha, \theta\} \cong \{W^*(M, G, \alpha), \hat{\alpha}\}$$

by [30; Theorems 4.5 and 4.6], where $\hat{\alpha}$ and $\hat{\theta}$ mean the dual action of α and θ in the sense of [30; Definition 4.1]. Representing M^α on a Hilbert space $\mathfrak{H}$, we see that M acting on $L^2(\mathfrak{H}; \hat{G})$ is generated by the operators:

$$\pi^\theta(x)\xi(\gamma) = \theta_\gamma(x)\xi(\gamma),\ x\theta M^\alpha,\ \xi \in L^2(\mathfrak{H}; \hat{G}) ;$$
$$u(\gamma_0)\xi(\gamma) = \xi(\gamma + \gamma_0),\ \gamma, \gamma_0 \in \hat{G} .$$

The action α on M is implemented by the unitary representation

$$\{v, L^2(\mathfrak{H}; \hat{G})\}$$

of G defined by

$$v(s)\xi(\gamma) = \overline{\langle s, \gamma\rangle}\xi(\gamma) , \qquad s \in G .$$

Hence have we $\alpha_s(u(\gamma)) = \langle s, \gamma\rangle u(\gamma)$, so that $M(\alpha, \gamma) = M^\alpha u(\gamma)$, $\gamma \in \hat{G}$, where

$$M(\alpha, \gamma) = \{x \in M : \alpha_s(x) = \langle s, \gamma\rangle u(\gamma)\} .$$

If e is a central projection in M^α, then we have

$$eM(\alpha, \gamma)e = e\theta_\gamma(e)M^\alpha u(\gamma),\ \gamma \in \hat{G} ;$$
$$M_e(\alpha^e, \gamma) = e\theta_\gamma(e)M_e^\alpha u(\gamma) .$$

Hence $M_e(\alpha^e, \gamma) \neq \{0\}$ if and only if $e\theta_\gamma(e) \neq 0$. If $\theta_\gamma = \iota$ on the center of M^α, then $e\theta_\gamma(e) \neq 0$ for any non-zero central projection e in M^α; hence

$\gamma \in \Gamma(\alpha)$. A slight modification of the arguments for [30; Lemma 9.5] shows that if $\theta_{\gamma_0} \neq \iota$ on the center of M^α, then there exists a neighborhood V of γ_0 in $\hat{G}$ and a non-zero projection e in the center of M^α such that $e\theta_\gamma(e) = 0$ for every $\gamma \in V$. Hence we have $M_e(\alpha^e, \gamma) = \{0\}$ for every $\gamma \in V$. Since α^e is integrable, our assertion follows from the next lemma.

q.e.d.

LEMMA 3.3. *If α is an integrable action of a locally compact abelian group G on M, then for any open subset V of $\hat{G}$, the spectral subspace $M(\alpha, V) \neq \{0\}$ if and only if $M(\alpha, \gamma) \neq \{0\}$ for some $\gamma \in V$.*

PROOF. Trivially, $M(\alpha, \gamma) \subset M(\alpha, V)$ for any $\gamma \in \hat{G}$. Hence we have only to prove that $M(\alpha, \gamma) = \{0\}$ for every $\gamma \in V$ implies $M(\alpha, V) = \{0\}$. By a simple application of Fubini's theorem, we conclude that $\alpha_f(x) \in \mathfrak{p}_\alpha^+$ for any $f \in L^1(G)$, $f \geqq 0$, and $x \in \mathfrak{p}_\alpha^+$, where $\alpha_f(x) = \int_G f(s)\alpha_s(x)ds$; hence $\alpha_f(\mathfrak{p}_\alpha) \subset \mathfrak{p}_\alpha$ by the linearity for $f \in L^1(G)$. Put

$$\hat{x}(\gamma) = \int_G \overline{\langle s, \gamma\rangle}\alpha_s(x)ds\ , \qquad x \in \mathfrak{p}_\alpha\ .$$

We have then $\hat{x}(\gamma) \in M(\alpha, \gamma)$ for any $x \in \mathfrak{p}_\alpha$. Suppose that $M(\alpha, \gamma) = \{0\}$ for any $\gamma \in V$. Then we have $\hat{x}(\gamma) = 0$ for every $\gamma \in V$. If f is a function in $L^1(G)$ with supp $\hat{f} \subset V$, then we have for any $x \in \mathfrak{p}_\alpha$ and $\gamma \in \hat{G}$

$$\alpha_f(x)\hat{\ }(\gamma) = \hat{f}(\gamma)\hat{x}(\gamma) = 0\ .$$

Hence $\alpha_f(x) = 0$ for every $x \in \mathfrak{p}_\alpha$; so $\alpha_f(M) = \{0\}$ since α_f is σ-weakly continuous and $\mathfrak{p}_\alpha$ is σ-weakly dense in M. Hence $\alpha_f = 0$ whenever supp $\hat{f} \subset V$. Thus $M(\alpha, V) = \{0\}$. q.e.d.

COROLLARY 3.4. *Let α be a continuous action of a separable locally compact abelian group G on a σ-finite von Neumann algebra M. Then the crossed product $W^*(M, G, \alpha)$ is a factor if and only if $\Gamma(\alpha) = \hat{G}$ and α is ergodic on the center of M.*

PROOF. Suppose that $W^*(M, G, \alpha)$ is a factor. By Theorem 3.2, $\Gamma(\alpha) = \hat{G}$. Since $W^*(M, G, \alpha) \cong [M \otimes \mathfrak{L}(L^2(G))]^{\alpha\otimes \mathrm{Ad}\lambda}$, for any central fixed point x under α, $x \otimes 1$ is in $[M \otimes \mathfrak{L}(L^2(G))]^{\alpha\otimes\mathrm{Ad}\lambda}$. Hence $x \otimes 1$ must be a scalar. Hence α is ergodic on the center of M.

Suppose that $\Gamma(\alpha) = \hat{G}$ and α is ergodic on the center of M. Since $\alpha \otimes \mathrm{Ad}\,\lambda$ on $\mathfrak{L}(L^2(G))$ enjoys the same property, we may assume that M is properly infinite and α is dominant. Then there exists an action θ of $\hat{G}$ on M^α such that $\{M, \alpha\} \cong \{W^*(M^\alpha, \hat{G}, \theta), \hat{\theta}\}$. By Theorem 3.2, θ acts trivially on the center C^α of M^α. Therefore, C^α is contained in the center C of M. But α acts ergodically on C, so that $C \cap M^\alpha = \{\lambda 1\}$; Hence $C^\alpha = \{\lambda 1\}$. Thus M^α is a factor. q.e.d.

COROLLARY 3.5. *If α is a continuous action of a separable locally compact abelian group G on a σ-finite von Neumann algebra M with $\Gamma(\alpha) = \hat{G}$, then any square integrable α-twisted unitary representation of G in M with infinite multiplicity is dominant.*

PROOF. Replacing α by a dominant action of G of the form ${}_a\alpha$, we may assume that α is dominant. By Theorem 2.12.ii, every square integrable α-twisted unitary representation of G in M is majorized by a dominant one in the ordering "$\prec$". We have only to prove that α^e on M^e is dominant for any properly infinite projection e of M^α such that $e \sim 1$ in M. Let $\{u(\gamma): \gamma \in \Gamma\}$ be a unitary representation of $\hat{G}$ in M such that $\alpha_s(u(\gamma)) = \langle s, \gamma\rangle u(\gamma)$, so that $\mathrm{Ad}\, u(\gamma)|_{M^\alpha} = \theta_\gamma$ is a continuous action of $\hat{G}$ on M^α with $\{W^*(M^\alpha, \hat{G}, \theta), \hat{\theta}\} \cong \{M, \alpha\}$. By Theorem 3.2, the action of θ on the center C^α of M^α is trivial. Hence e and $\theta_\gamma(e)$ have the same central support in M^α, and are properly infinite in M^α; hence $e \sim \theta_\gamma(e)$. Therefore, there exists a partial isometry v_γ in M^α such that $v_\gamma^* v_\gamma = \theta_\gamma(e)$ and $v_\gamma v_\gamma^* = e$. Let $w_\gamma = v_\gamma u(\gamma) e$. Then we have $w_\gamma^* w_\gamma = e$ and $w_\gamma w_\gamma^* = e$, and also $\alpha_s^e(w_\gamma) = \langle s, \gamma\rangle w_\gamma$. Hence $\{M_e, \alpha^e\}$ satisfies condition (ii) in Theorem 3.1. Thus α^e is dominant. q.e.d.

We close this section with the following:

REMARK 3.6. So far we have mainly dealt with actions and/or weights of infinite multiplicity. The contrast between the following two statements (i) and (ii) might illustrate some of the reasons why the infinite multiplicity has been useful.

(i) If α is a continuous action of a separable locally compact group G on M with infinite multiplicity, then $M(\alpha, V)$ contains a non-zero partial isometry for any open subset V of $\hat{G}$ with $V \cap \Gamma(\alpha) \neq \emptyset$. More strongly, if $\Gamma(\alpha) = \hat{G}$ in addition, then $M(\alpha, V)$ contains a unitary for every non-empty open subset V of $\hat{G}$.

(ii) Let M be an abelian von Neumann algebra and α an ergodic continuous action of $\boldsymbol{R}$. If u is a non-zero partial isometry in $M(\alpha, V)$ for a bounded interval V, then u is unitary and $\alpha_t(u) = e^{ist}u$ for some $s \in V$.

The first assertion can be proven by approximating α with integrable actions. The second statement can be shown by some modification of the Paley-Wiener Theorem for the Fourier transform of distribution with compact support.

III.4. Galois correspondence. In this section, we shall show that given an integrable action α of a locally compact abelian group G on a von Neumann algebra M with M^α a factor, there is a Galois type correspondence between closed subgroups of G and globally α-invariant von Neumann subalgebras of M containing M^α, which generalizes a result in [30; §7].

THEOREM 4.1. *Let M_0 be a factor equipped with a continuous action α of a locally compact abelian group G. Let $M = W^*(M_0, G, \alpha)$. If N is a von Neumann subalgebra of M such that $M_0 \subset N$ and $\hat{\alpha}_p(N) = N$ for every $p \in \hat{G}$, where $\hat{\alpha}$ means the dual action of $\hat{G}$ on M then there is a closed subgroup $\hat{H}$ of $\hat{G}$ such that*

$$N = \{x \in M: \hat{\alpha}_p(x) = x \textit{ for every } p \in \hat{H}\} ;$$
$$\hat{H} = \{p \in \hat{G}: \hat{\alpha}_p(x) = x \textit{ for every } x \in N\} ;$$

therefore N is of the form $N = W^*(M_0, H, \alpha)$ with $H = \hat{H}^\perp$.

We divide the proof into a few steps.

LEMMA 4.2. *Let P be a factor and A an abelian von Neumann algebra. If Q is a factor such that $P \otimes 1 \subset Q \subset P \otimes A$, then $Q = P \otimes 1$.*

PROOF. Representing A as a maximal abelian von Neumann algebra on $\mathfrak{H}$, we have

$$\begin{aligned}(P \otimes 1)' \cap (P \otimes A) &= [P' \otimes \mathfrak{L}(\mathfrak{H})] \cap (P \otimes A)\\ &= 1 \otimes A ;\end{aligned}$$

hence

$$(P \otimes 1)' \cap Q \subset (1 \otimes A) \cap Q = \boldsymbol{C}1 \subset P \otimes 1 .$$

Therefore, there is at most only one normal conditional expectation from Q onto $P \otimes 1$ by [3; Théorème 1.5.5(a)]. Since there are in general many normal conditional expectations from $P \otimes A$ onto $P \otimes 1$, there exists a unique normal conditional expectation, say ε, from Q onto $P \otimes A$. To each normal state ω on A, there corresponds a normal conditional expectation ε_ω of $P \otimes A$ onto $P \otimes 1$ by the formula:

$$\varphi(\varepsilon_\omega(x)) = (\varphi \otimes \omega)(x) , \quad x \in P \otimes A, \varphi \in P_* .$$

By the uniqueness of a conditional expectation, we have, for any $x \in Q$, $\varepsilon(x) = \varepsilon_\omega(x)$, so that

$$(\varphi \otimes \omega)(\varepsilon(x) \otimes 1) = \varphi(\varepsilon_\omega(x)) = (\varphi \otimes \omega)(x) .$$

Therefore, we get $\varepsilon(x) \otimes 1 = x$ for every $x \in Q$; thus $Q = P \otimes 1$. q.e.d.

PROOF OF THEOREM 4.1. We put

$$\hat{H} = \{p \in \hat{G}\colon \alpha_p(x) = x \text{ for every } x \in N\} .$$

By [30; Theorem 7.1], the algebra $M^{\hat{H}}$ of all fixed points in M under $\hat{\alpha}_p$, $p \in \hat{H}$, is $W^*(M_0, H, \alpha)$ with $H = \{g \in G\colon \langle g, p\rangle = 1 \text{ for every } p \in \hat{H}\}$, where the technical assumption in [30; Theorem 7.1] on the existence of a relatively invariant weight on M_0 is not essential because of the commutation theorem for the general crossed product due to T. Digerness [8]. Replacing G by H and M by $W^*(H_0, H, \alpha)$, we may assume that $\hat{H} = \{0\}$, and must show that $N = M$.

We consider the crossed products, $W^*(M, \hat{G}, \hat{\alpha}) = \tilde{M}$, $W^*(N, \hat{G}, \hat{\alpha}) = \tilde{N}$ and $W^*(M_0, \hat{G}, \hat{\alpha}) = \tilde{M}_0$. We have then

$$\tilde{M}_0 = M_0 \otimes L^\infty(G) \subset \tilde{N} \subset \tilde{M} .$$

The action $\hat{\alpha}$ of $\hat{G}$ on N is faithful, and the fixed point algebra $N^{\hat{G}}$ in N under $\hat{\alpha}$ is M_0, hence a factor. Hence $\tilde{N}$ is a factor by Corollary 3.4. By [30; Theorem 4.5], we have

$$\tilde{M} \cong M_0 \otimes \mathfrak{L}(L^2(G)) .$$

Therefore, if we can identify the algebras $\tilde{M}_0$, and $\tilde{M}$ with $M_0 \otimes L^\infty(G)$ and $M_0 \otimes \mathfrak{L}(L^2(G))$, then Lemma 4.2 is applied to the commutants: $M_0' \otimes L^\infty(G) \supset \tilde{N}' \supset M_0' \otimes 1$. Hence $\tilde{N}' = M_0' \otimes 1$, so $\tilde{N} = \tilde{M}$. Since N is the fixed point algebra in $\tilde{N} = \tilde{M}$ under the action $\hat{\hat{\alpha}}$ of G, we have $M = N$. Thus, we must show that $\tilde{M}$ is identified with $M_0 \otimes \mathfrak{L}(L^2(G))$ in such a way that $\tilde{M}_0$ coincides with $M_0 \otimes L^\infty(G)$ under this identification.

Let $\mathfrak{H}$ be the Hilbert space on which M_0 acts. Then M acts on the Hilbert space $L^2(\mathfrak{H}; G)$, and $\tilde{M}$ acts on $L^2(\mathfrak{H}; G \times G)$ and is generated by the following three types of operators:

$$\begin{cases} \bar{x}\xi(s, t) = \alpha_s^{-1}(x)\xi(s, t) , & x \in M_0 ; \\ u(r)\xi(s, t) = \xi(s - r, t - r) , & r \in G ; \\ v(p)\xi(s, t) = \langle t, p\rangle\xi(s, t) , & p \in \hat{G} . \quad (\text{cf. } [30; (4.10)]) . \end{cases}$$

It follows then that $\tilde{M}_0$ is generated by $\{\bar{x}, v(p); x \in M_0, p \in \hat{G}\}$ and identified with $M_0 \otimes L^\infty(G) = L^\infty(M_0; G)$, where the action of $L^\infty(M_0; G)$ is given by the following:

$$x\xi(s, t) = \alpha_s^{-1}(x(t))\xi(s, t)$$

for every $x(\cdot) \in L^\infty(M_0; G)$. We define an automorphism π of $L^\infty(M_0; G)$ by

$$\pi(x)(s) = \alpha_s(x(s)),\ x(\cdot) \in L^\infty(M_0; G) .$$

It follows from the proof of [30; Theorem 4.5] that $\tilde{M}$ is the tensor product of $\pi(M_0 \otimes 1)$ and its relative commutant B in $\tilde{M}$ where B is generated by $u(G)$ and $v(\hat{G})$. Thus we have

$$\tilde{M}_0 = \pi(M_0 \otimes L^\infty(G)) \cong \pi(M_0 \otimes 1) \otimes L^\infty(G) ;$$
$$\tilde{M} = \pi(M_0 \otimes 1) \otimes \mathfrak{B} \supset \tilde{N} \supset \pi(M_0 \otimes 1) \otimes L^\infty(G) = \tilde{M}_0 .$$

q.e.d.

THEOREM 4.3. *Let M be a factor equipped with an integrable action α of a locally compact abelian group G. If $\Gamma(\alpha) = \hat{G}$, then there exists a bijective inclusion reversing correspondence between the closed subgroups H of G and the α-invariant von Neumann subalgebras N of M containing the fixed point algebra M^α in such a way that*

$$N_H = \{x \in M: \alpha_s(x) = x,\ s \in H\} ;$$
$$H_N = \{s \in G: \alpha_s(x) = x,\ x \in N\} .$$

PROOF. We put

$$\bar{M} = M \otimes F_\infty \quad \text{and} \quad \bar{\alpha}_s = \alpha_s \otimes \iota ,\quad s \in G ,$$

with F_∞ a factor of type I_∞. It follows then that $\bar{\alpha}$ is dominant, since the fixed point algebra $\bar{M}^{\bar{\alpha}}$ under $\bar{\alpha}$ is $M^\alpha \otimes F_\infty$. Hence, by Theorem 4.1, the correspondence between H and $\bar{\alpha}$-invariant von Neumann subalgebras $\bar{N}$ of $\bar{M}$ containing $\bar{M}^{\bar{\alpha}}$ given by

$$\bar{N}_H = \{x \in \bar{M}: \bar{\alpha}_s(x) = x,\ s \in H\} ;$$
$$H_{\bar{N}} = \{s \in G: \bar{\alpha}_s(x) = x,\ x \in \bar{N}\}$$

is bijective and inclusion reversing. It is now trivial that $N_{H_N} \supset N$ and $H_{N_H} \supset H$. For a given N, we put $\bar{N} = N \otimes F_\infty$. Trivially we have $H_N = H_{\bar{N}}$. If $x \in N_{H_N}$, then $x \otimes 1 \in \bar{N}_{H_{\bar{N}}}$; so $x \otimes 1 \in \bar{N}$ equivalently $x \in N$. Hence $N = N_{H_N}$. For a given H, we have $\bar{N}_H = N_H \otimes F_\infty (= (N_H)^-)$. Hence we get

$$H = H_{\bar{N}_H} = H_{(N_H \otimes F_\infty)} = H_{N_H} .$$

q.e.d.

EXAMPLE 4.4. Let G be a locally compact abelian group, and $M = \mathfrak{L}(L^2(G))$. Putting

$$\begin{cases} (u(s)\xi)(t) = \xi(t - s) , & \xi \in L^2(G),\ s,\ t \in G ; \\ (v(P)\xi)(t) = \overline{\langle t, p\rangle}\xi(t) , & \xi \in L^2(G),\ p \in \hat{G},\ t \in G , \end{cases}$$

we obtain unitary representations u of G and v of $\hat{G}$ with

$$u(s)v(p)u(s)^*v(p)^* = \langle s, p\rangle 1 ,\quad s \in G,\ p \in \hat{G} .$$

Thus we may define an action α of $G \times \hat{G}$ on M by

$$\alpha_{s,p}(x) = u(s)v(p)xv(p)^*u(s)^* ,\quad s \in G,\ p \in \hat{G},\ x \in M .$$

Since $u(s)$, $s \in G$, and $v(p)$, $p \in \hat{G}$, together generate M, we have

$$M^\alpha = \{\lambda 1 \colon \lambda \in \boldsymbol{C}\} ;$$

hence $\Gamma(\alpha) = (G \times \hat{G})\hat{} = \hat{G} \times G$.

For a pair f, g of functions in $L^2(G)$, we define an operator $x_{f,g} \in M$ by

$$x_{f,g}\xi = (\xi \mid f)g .$$

We have then

$$(u(r)v(p)x_{f,g}v(p)^* u(r)^* \xi \mid \eta) = \iint \overline{\langle s - t, p \rangle f(t)} g(s)\xi(t + r)\overline{\eta(s + r)} ds dt .$$

Therefore, by the Plancherel formula, we get

$$\begin{aligned}\iint (u(r)v(p)x_{f,g}v(p)^* u(r)^* \xi \mid \eta) dp dr &= \iint \bar{f}(s) g(s)\xi(s + r)\overline{\eta(s + r)} ds dr \\ &= (g \mid f)(\xi \mid \eta) ,\end{aligned}$$

so that

$$\int u(r)v(p)x_{f,g}v(p)^* u(r)^* dp dr = (g \mid f)1 .$$

This means that the action α of $G \times \hat{G}$ is integrable. Thus, the α-invariant von Neumann algebras on $L^2(G)$ are labeled by the closed subgroups of $G \times \hat{G}$ by Theorem 4.3. The von Neumann algebras considered in [28] are of the special case where the corresponding subgroups are of the form $H \times \hat{K}$ with H a closed subgroup of G and $\hat{K}$ a closed subgroup of $\hat{G}$.

Since there are many von Neumann algebras not corresponding to any closed subgroup of $G \times \hat{G}$, the invariance of a von Neumann algebra under the action α in Theorem 4.3 is not removable in this general setting. The same is true for Theorem 4.1 because the tensor product with F_∞ a factor of type I_∞ gives counter examples for the Galois correspondence without α-invariance.

The following result strengthens and refines a generalized commutation theorem [28].

PROPOSITION 4.4. *In the setting of Example* 4.4, *let H be a closed subgroup of $G \times \hat{G}$ and $H^\perp = \{(q, t) \in \hat{G} \times G \colon \langle s, q \rangle = \langle t, p \rangle$ for every $(s, p) \in H$. The fixed point algebra M^H under $\alpha_{s,p}$ for every $(s, p) \in H$ is generated by $u(t)v(q)$ with $(q, t) \in H^\perp$.*

PROOF. In general, we have

$$\alpha_{s,p}(u(t)v(q)) = \overline{\langle t, p \rangle} \langle s, q \rangle u(t)v(q) , \quad s, t \in G,\ p, q \in \hat{G} .$$

Hence $u(t)v(q)$ belongs to M^H if and only if $(q, t) \in H^\perp$.

The action of $(G \times \hat{G})/H$ on M^H, denoted by the same notation α, induced by the original action of $G \times \hat{G}$ is integrable; hence M^H is generated by the eigen operators. Let x be an eigen operator in M^H corresponding to $(g, t) \in ((G \times \hat{G})/H)\hat{} = H^\perp$. It forllows then that $(u(t)v(q))^*x$ belongs to the fixed point algebra $M^\alpha = \{\lambda 1\}$. Hence $x = \lambda u(t)v(q)$ for some $\lambda \in \boldsymbol{C}$. Thus M^H is generated by $\{u(t)v(q): (q, t) \in H^\perp\}$. q.e.d.

III.5. Stability of automorphisms. In this section, we shall show that if α is an automorphism (resp. one parameter automorphism group) of a semi-finite von Neumann algebra N scaling a trace down, then every unitary one cocycle is a coboundary. This, in turn, improves the isomorphism criterion for the factors of type III in terms of the conjugacy of discrete as well as continuous decompositions.

THEOREM 5.1. *Let N be a semi-finite von Neumann algebra.*

(i) *If θ is an automorphism of N such that there exists a faithful semi-finite normal trace τ on N such that $\tau \circ \theta \leqq \lambda\tau$ for some $0 < \lambda < 1$, then* (a) *there exists a continuous action α of the torus $\boldsymbol{T}$ on the fixed point algebra N^θ such that*

$$\{W^*(N^\theta, \boldsymbol{T}, \alpha), \hat{\alpha}\} \cong \{N, \theta\} ;$$

(b) *every unitary $u \in N$ is of the form $u = v^*\theta(v)$ for some unitary $v \in N$.*

(ii) *If $\{\theta_t\}$ is a one parameter automorphism group of N such that $\tau \circ \theta_t = e^{-t}\tau$ for some faithful semi-finite normal trace τ on N, then* (a) *there exists a one parameter automorphism group $\{\alpha_s\}$ of the fixed point algebra N^θ such that*

$$\{W^*(N^\theta, \boldsymbol{R}, \alpha), \hat{\alpha}\} \cong \{N, \theta\} ;$$

(b) *every α-twisted unitary representation $\{u_t\}$ of $\boldsymbol{R}$ in N is of the form $u_t = v^*\alpha_t(v)$ for some unitary $v \in N$.*

PROOF. (i) Let θ be an automorphism of N with $\tau \circ \theta \leqq \lambda\tau$. We first claim that for any non-zero projection $p \in N^\theta$ there exists a non-zero projection $q \leqq p$ such that $\{\theta^n(q)\}$ is orthogonal. Let e be a non-zero projection such that $e \leqq p$ and $\tau(e) < +\infty$. Let $f = \bigvee_{n=0}^\infty \theta^n(e)$. We have then

$$\tau(f) \leqq \sum_{n=0}^\infty \tau(\theta^n(e)) \leqq \sum_{n=0}^\infty \lambda^n\tau(e) = \frac{1}{1-\lambda}\tau(e) < +\infty ;$$

$$\theta(f) \leqq f \quad \text{and} \quad \tau(\theta(f)) = \lambda\tau(f) < \tau(f) ;$$

$$q = f - \theta(f) \neq 0 .$$

It is clear that $\{\theta^n(q): n \in \boldsymbol{Z}\}$ is orthogonal. Therefore, the usual exhaustion arguments entail the existence of a projection $q \in N$ such that $\{\theta^n(q): n \in \boldsymbol{Z}\}$ is orthogonal and $\sum_{n \in Z} \theta^n(q) = 1$.

We put, for $0 \leqq s < 1$,

$$u(s) = \sum_{n \in Z} e^{-2\pi i s n} \theta^n(q) .$$

It follows then that $\theta(u(s)) = e^{2\pi i s} u(s)$, $0 \leqq s < 1$. Therefore, $\{u(s): 0 \leqq s < 1\}$ induces a continuous action α of the torus $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ on N^θ by

$$\alpha_s(x) = u(s) x u(s)^* , \quad s \in \boldsymbol{T} ,$$

where we identify the torus $\boldsymbol{T}$ with the half open unit interval $[0, 1)$. Thus, our assertion (a) follows from [15].

For the second assertion, (b), we observe first that if N^θ is properly infinite, then θ is dominant. But we claim that N is properly infinite if and only if N^θ is also. By the usual reduction arguments, it is sufficient to prove the claim that the finiteness of N^θ implies that of N. Suppose N^θ is finite. Let φ be a faithful semi-finite normal trace on N^θ invariant under α, the existence of such a φ being guaranteed by the compactness of $\boldsymbol{T}$. Let $\tilde{\varphi}$ be the weight on N dual to φ. It follows from [30; Proposition 5.16] that $\tilde{\varphi}$ is invariant under θ. Since φ is a faithful semi-finite normal trace on N, $\tilde{\varphi}$ is of the form: $\tilde{\varphi} = \tau(h\cdot)$ for some non-singular positive self-adjoint operator h affiliated with the center C of N. We have then

$$\begin{aligned} \tau(\theta(h)x) &= \tau \circ \theta(h\theta^{-1}(x)) \leqq \lambda\tau(n\theta^{-1}(x)) \\ &= \lambda\tilde{\varphi}(\theta^{-1}(x)) = \lambda\tilde{\varphi}(x) = \lambda\tau(hx) , \quad x \in N_+ . \end{aligned}$$

Hence we get $\theta(h) \leqq \lambda h$. From this, repeating more or less the same arguments as above, we can construct a continuous unitary representation $v(s)$ of $\boldsymbol{T}$ in C such that

$$\theta(v(s)) = e^{2\pi i s} v(s) .$$

Hence the action α' of $\boldsymbol{T}$ on N^θ induced by $\{v(s)\}$ is trivial, and θ is still dual to this new α'. This means that $N \cong N^\theta \otimes l^\infty(\boldsymbol{Z})$ and $\theta \cong 1 \otimes$ (translation on $l^\infty(\boldsymbol{Z})$). Thus N must be finite. In this case, let u be an arbitrary unitary in N, and $u = \{u_n\}$ in the decomposition $N = N^\theta \otimes l^\infty$. Put $v_{n+1} = v_n u_n$ if $n \geqq 1$ and $v_0 = 1$, $v_n = v_{n+1} u_n$ if $n < 0$. We have then $v^*\theta(v) = u$. If N is properly infinite, then every θ with $\tau \circ \theta \leqq \lambda\tau$ is dominant, so that for any $u \in \mathfrak{U}_N$ the new action $\bar{\theta} = \mathrm{Ad}\, u \circ \theta$ is dominant; hence the θ-twisted unitary representation of $\boldsymbol{Z}$ in N generated by u is dominant, which means that $u = v^*\theta(v)$ for some $v \in \mathfrak{U}_N$.

(ii) We apply (i) to $\{\theta_n: n \in \boldsymbol{Z}\}$. Let N_1 denote the fixed point subalgebra of N under $\{\theta_n: n \in \boldsymbol{Z}\}$. It follows then that the restriction $\theta|_{N_1}$ of θ to N_1 is periodic with period one. The action $\{\theta_n: n \in \boldsymbol{Z}\}$ of $\boldsymbol{Z}$ on N is integrable by (i) and $\theta|_{N_1}$ is integrable as an action of the torus $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$. Hence θ itself is integrable, because

$$E(x) = \int_{-\infty}^{\infty} \theta_t(x)dt = \int_0^1 \theta_t(\sum_{n \in \boldsymbol{Z}} \theta_n(x))dt\ , \quad x \in N_+\ .$$

Let ψ be a strictly semi-finite faithful weight on N^θ. It follows then that the weight $\varphi = \psi \circ E$ is a faithful weight on N invariant under θ. By [30; Theorem 5.4], there exists a non-singular self-adjoint operator h affiliated with N such that $\varphi = \tau(h\cdot)$. For any $x \in N_+$, we have

$$\begin{aligned}\tau(\theta_s(h)x) &= \tau \circ \theta_s(h\theta_{-s}(x)) = e^{-s}\tau(h\theta_{-s}(x)) = e^{-s}\varphi(\theta_{-s}(x)) \\ &= e^{-s}\varphi(x) = e^{-s}\tau(hx)\ ;\end{aligned}$$

hence we have $\theta_s(h) = e^{-s}h$. Putting $u(t) = h^{-it}$, $t \in \boldsymbol{R}$, we have

$$\theta_s(u(t)) = e^{ist}u(t)\ .$$

Thus, the one parameter unitary group $\{u(t): t \in \boldsymbol{R}\}$ gives rise to a one parameter automorphism group $\{\alpha_t: t \in \boldsymbol{R}\}$ of N^θ such that $\{N, \theta\} \cong \{W^*(N^\theta, \boldsymbol{R}, \alpha), \hat{\alpha}\}$ by [15]. This proves (a).

To prove the second assertion (b), we first show that N^θ is semi-finite if and only if $\{N, \theta\} \cong \{N^\theta \otimes L^\infty(\boldsymbol{R}), \iota \otimes \text{translation}\}$. Let $P = N \otimes F_\infty$ and $\bar{\theta}_t = \theta_t \otimes \iota$, $t \in R$. It follows then that $\bar{\theta}$ is dominant and $N^\theta \otimes F_\infty = P^{\bar{\theta}}$. If N^θ is semi-finite then so is $P^{\bar{\theta}}$. Hence $W^*(N, \boldsymbol{R}, \theta) \cong P^{\bar{\theta}}$ is semi-finite. Our claim then follows from [30; Section 9], and assertion (b) in this case is standard.

If N^θ is properly infinite, then N^a is also for every $a \in Z^1_\theta(\boldsymbol{R}, \mathfrak{U}_N)$, which means that a is dominant since $\tau \circ {}_a\theta_t = e^{-t}\tau$, $t \in \boldsymbol{R}$. Thus $a \cong 1$.

q.e.d.

COROLLARY 5.2. (i) *Let N_1 and N_2 be properly infinite semi-finite von Neumann algebras equipped with one parameter automorphism groups θ^1 and θ^2 respectively which transform some faithful semi-finite normal traces τ_1 and τ_2 respectively in such a way that*

$$\tau_1 \circ \theta^1_s = e^{-s}\tau_1 \quad \textit{and} \quad \tau_2 \circ \theta^2_s = e^{-s}\tau_2\ , \quad s \in \boldsymbol{R}\ .$$

Then $W^*(N_1, \boldsymbol{R}, \theta^1) \cong W^*(N_2, \boldsymbol{R}, \theta^2)$ if and only if there exists an isomorphism π of N_1 onto N_2 such that $\theta^1_s = \pi^{-1} \circ \theta^2_s \circ \pi$, $s \in \boldsymbol{R}$.

(ii) *If $\{N_1, \theta_1\}$ and $\{N_2, \theta_2\}$ are discrete decompositions of the same factor of type* III_λ, $0 < \lambda < 1$, *then there exists an isomorphism π of N_1 onto N_2 such that $\theta_1 = \pi^{-1} \circ \theta_2 \circ \pi$.*

(iii) *If* $\{N_1, \theta_1\}$ *and* $\{N_2, \theta_2\}$ *are discrete decompositions of the same factor of type* III_0, *then there exist central projections* $e_1 \in N_1$ *and* $e_2 \in N_2$, *and an isomorphism* π *of* N_{1,e_1} *onto* N_{2,e_2} *such that* $\theta_{1,e_1} = \pi^{-1} \circ \theta_{2,e_2} \circ \pi$, *where* θ_{1,e_1} (resp. θ_{2,e_2}) *is an automorphism of* N_{1,e_1} (resp. N_{2,e_2}) *induced by* θ_1 (resp. θ_2) *as described in* [3; Definition 5.4.1.].

PROOF. This is a straightforward consequence of Theorem 5.1 and [30; §8] and [3, Theorems 4.4.1 and 5.4.2]. q.e.d.

COROLLARY 5.3. *An automorphism* α *of a factor* M *of type* II_∞ *is stable if and only if* α *does not preserve the trace* τ *of* M.

PROOF. Suppose α does not preserve the trace τ on M. It follows that $\tau \circ \alpha = \lambda\tau$ for some $\lambda > 0$ by the uniqueness of the trace. Considering α^{-1}, we may assume $\lambda < 1$. Let $\beta = \mathrm{Ad}(u) \circ \alpha$ with u a unitary in M. Then we have $W^*(M, \alpha) \cong W^*(M, \beta)$, and they are of type III_λ. By Theorem 5.1, we have $M^\alpha \otimes \mathfrak{L}(l^2(\boldsymbol{Z})) \cong W^*(M, \alpha)$, so that $M^\alpha \cong W^*(M, \alpha)$. Thus M^α and M^β are both properly infinite, which means that α and β are both dominant. Therefore, there exists a unitary $v \in M$ such that $u = v^*\alpha(v)$, which means that $\beta = \mathrm{Ad}(v)^{-1} \circ \alpha \circ \mathrm{Ad}(v)$.

Suppose conversely α preserves the trace τ. Let e be a projection in M with $\tau(e) < +\infty$. Since $e \sim \alpha(e)$, there exists a unitary $u \in M$ such that $e = u\alpha(e)u^*$, where we note here that the equivalence between finite projections is unitarily implemented. Let $\beta = \mathrm{Ad}(u) \circ \alpha$. It follows then that β preserves a normal positive linear functional $\varphi = \tau(e\cdot)$. Hence $\{\beta^n : n \in \boldsymbol{Z}\}$ is not integrable, so that $\{\beta^n\}$ is not conjugate to any integrable action of $\boldsymbol{Z}$. But there is a unitary $v \in M$ as seen in §2 that $\{(\mathrm{Ad}\, v \circ \beta)^n\}$ is integrable, even dominant. Hence β and $\mathrm{Ad}(v) \cdot \beta$ are not conjugate; therefore either $\beta = \mathrm{Ad}(u) \circ \alpha$ or $\mathrm{Ad}(v) \circ \beta = \mathrm{Ad}(vu) \circ \alpha$ is not conjugate to α. Therefore, α is not stable. q.e.d.

PROOF OF THEOREM II.1.6. Let $\{\bar\omega_1, \bar\omega_2\}$ and $\{\bar\omega_1', \bar\omega_2'\}$ be two quasi-commuting pair of dominant weights on an infinite factor M with separable predual such that $\alpha(\bar\omega_1, \bar\omega_2) = \alpha(\bar\omega_1', \bar\omega_2')$, say α for short. By the uniqueness of a dominant weight, there exists a unitary $u \in M$ such that $\bar\omega_1 = \bar\omega'_{1,u}$. Replacing $\bar\omega_2'$ by $\bar\omega'_{2,u}$, we reduce the situation to the following: given three dominant weights $\bar\omega$, φ, and ψ on M such that $\{\bar\omega, \varphi\}$ and $\{\bar\omega, \psi\}$ are quasi-commuting with $\alpha(\bar\omega, \varphi) = \alpha(\bar\omega, \psi) = \alpha$, we must show that there exists a unitary u in $M_{\bar\omega}$ such that $\psi = \varphi_u$.

Let $M = W^*(N, \boldsymbol{R}, \theta)$ and $\{u(s) : s\theta\boldsymbol{R}\}$ be a continuous decomposition of M and the one parameter unitary group in M associated with this decomposition. We may assume that $\bar\omega$ is the weight on M dual to a trace $\tau \circ \theta_t = e^{-t}\tau$, $t \in \boldsymbol{R}$. For short, put $v_s = (D\varphi\colon D\bar{\omega})_s$, and $w_s = (D\psi\colon D\bar{\omega})_s$, $s \in \boldsymbol{R}$. We have then

$$\sigma_t^{\bar{\omega}}(v_s) = e^{i\alpha st}v_s \quad \text{and} \quad \sigma_t^{\bar{\omega}}(w_s) = e^{i\alpha st}w_s\ ;$$
$$v_{s+t} = e^{i\alpha st}v_s v_t\ , \quad w_{s+t} = e^{i\alpha st}w_s w_t\ .$$

For each $s \in \boldsymbol{R}$, put

$$a_s = e^{-i\alpha s^2/2}v_s u(\alpha s)^* \quad \text{and} \quad b_s = e^{-i\alpha s^2/2}w_s u(\alpha s)^*\ .$$

It is easily seen that $\{a_s\}$ and $\{b_s\}$ are both continuous and parameter families of unitaries in N such that

$$a_{s+t} = a_s\theta_{\alpha s}(a_t) \quad \text{and} \quad b_{s+t} = b_s\theta_{\alpha s}(b_t)\ .$$

By Theorem 5.1, there exists a unitary $u \in N$ such that

$$a_s = ub_s\theta_{\alpha s}(u^*)\ , \quad s \in \boldsymbol{R}\ .$$

Hence we get, for any $s \in \boldsymbol{R}$,

$$\begin{aligned} v_s &= e^{i\alpha s^2/2}a_s u(\alpha s) = e^{i\alpha s^2/2}ub_s\theta_{\alpha s}(u^*)u(\alpha s) \\ &= e^{i\alpha s^2/2}ub_s u(\alpha s)u^* = uw_s u^*\ . \end{aligned}$$

Thus it follows that $\varphi = \psi_u$. q.e.d.

CHAPTER IV. THE FLOW OF WEIGHTS AND THE AUTOMORPHISM GROUP OF A FACTOR OF TYPE III

IV.0. Introduction. The aim of this chapter is to extend the exact sequence of [3, 4.5] to the general case from type III_λ case, $0 < \lambda < 1$, for the automorphism group Aut (M) and/or the outer automorphism group Out $(M) =$ Aut (M)/Int (M) of a factor M of type III in terms of the flow F_M of weights on M and a continuous decomosition $M = W^*(N, \boldsymbol{R}, \theta)$ of M. Since F^M is functorial to each $\alpha \in$ Aut (M) there corresponds a unique automorphism mod (α) of the flow F^M as the restriction of $\bar{\alpha} \in$ Aut $(\mathfrak{P}_M)$ to P_M. Assuming M to be a factor of type II_∞, we will see that mod (α) is precisely the translation of $L^\infty(\boldsymbol{R}_+^*)$ by multiplying $\lambda(\alpha) > 0$ where this positive number $\lambda(\alpha)$ is determined by $\tau \circ \alpha = \lambda(\alpha)\tau$ for the trace τ on M. With this evidence, we call mod the *fundamental homomorphism* of Aut (M) in general. Considering the topologies in Aut (M) and Aut (F^M) as in preliminary, we will show that mod is continuous; hence ker mod contains the closure of Int (M).

We next extend the modular automorphism group $\{\sigma_t^\varphi\}$ from the additive group $\boldsymbol{R}$ to the multiplicative group $Z^1(F^M)$ of unitary one cocycles with respect to the flow F^M of weights. To each $c \in Z^1(F^M)$ and a faithful integrable weight φ on M, we associate an automorphism $\bar{\sigma}_c^\varphi$ of M by $\bar{\sigma}_c^\varphi(x) = p_\varphi^{-1}(c_\lambda p_M(\varphi))x$ for each $x \in M(\sigma^\varphi, \{\lambda\})$. The relative commutant theorem, Theorem II.5.1, then enables us to characterize these automorphisms as those which leave the centralizer elementwise fixed. We then show that for a smooth $c \in Z^1(F^M)$ there exist a map: $\varphi \longrightarrow \bar{\sigma}_c^\varphi$ from the space $\mathfrak{W}_M^0$ of faithful weights to Aut(M) and a map: $(\varphi, \psi) \longrightarrow (D\varphi : \mathfrak{W}\psi)_c$ from $\mathfrak{W}_M^0 \times \mathfrak{W}_M^0$ into the unitary group $\mathfrak{U}$ of M such that

$$\bar{\sigma}_c^\psi(x) = (D\psi : D\varphi)_c \bar{\sigma}_c^\varphi(x)(D\psi : D\varphi)_c^* , \qquad x \in M ,$$

which coincide with σ_t^φ and $(D\psi : D\varphi)_t$ if $c_\lambda = \lambda^{it}$. In this setting, the modular period group $T(M)$ of M is generalized to $B^1(F^M)$ in the sence that $\bar{\sigma}_c^\varphi$ is inner if and only if $c \in B^1(F^M)$, see [30; Theorem 9.4]. Thus we obtain a homomorphism $\bar{\delta}_M$ of $H^1(F^M)$, the first unitary cohomology group of the flow F^M, into Out$(M) =$ Aut(M)/Int(M). Assuming M to be semi-finite, we will see that $(D\varphi : D\,\mathrm{Tr})_c = f(1)^* f(h)$ with $\varphi = \mathrm{Tr}(h\cdot)$ and $c_\lambda = fF_\lambda(f^*)$, $f \in L^\infty(\boldsymbol{R}_+^*)$. From this, we view $\bar{\sigma}_c^\varphi$ and $(D\varphi : D\psi)_c$ as functional calculus of the "generator" of the modular automorphism group $\{\sigma_t^\varphi\}$.

In the last section, fixing a continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$, we obtain an exact sequence:

$$\{1\} \longrightarrow H^1(F^M) \xrightarrow{\bar{\delta}_M} \mathrm{Out}\,(M) \longrightarrow \mathrm{Out}_{\theta,\tau}(N) \longrightarrow \{1\} ,$$

where

$$\mathrm{Out}_{\theta,\tau}(N) = \{\alpha \in \mathrm{Out}\,(N) : \varepsilon_N(\theta_t)\alpha = \alpha\varepsilon_N(\theta_t),\ \tau \circ \alpha = \tau\}$$

and ε_N is the canonical homomorphism of Aut(N) onto Out(N).

IV.1. The fundamental homomorphism. Let M be an infinite factor with separable predual, and F^M the smooth flow of weights on M. Recall that F^M is just the action: $\varphi \longrightarrow \lambda\varphi$ of $\boldsymbol{R}_+^*$ on the classes of integrable weights of infinite multiplicity. Let Aut(F^M) be the group of automorphisms F^M, (i.e., automorphisms of the abelian von Neumann algebra P_M which commute with the action F^M of $\boldsymbol{R}_+^*$). For any $\alpha \in$ Aut(M), the permutation: $\varphi \longrightarrow \varphi \circ \alpha^{-1}$ of classes of integrable weights of infinite multiplicity defines a unique element mod(α) of Aut(F^M) such that

$$\mathrm{mod}\,(\alpha)p_M(\varphi) = p_M(\varphi \circ \alpha^{-1}) , \qquad \alpha \in \mathrm{Aut}\,(M) .$$

DEFINITION 1.1. We call mod the fundamental homomorphism.

This name comes from the following:

PROPOSITION 1.2. *If M is a factor of type* II$_\infty$ *with separable* *predual, then the map:* $\lambda \in \boldsymbol{R}_+^* \to F_\lambda^M \in \operatorname{Aut}(F^M)$ *is an isomorphism and for any* $\alpha \in \operatorname{Aut}(M)$ *and a faithful semi-finite normal trace* τ *we have*

$$\tau \circ \alpha^{-1} = \operatorname{mod}(\alpha)\tau$$

where $\operatorname{mod}(\alpha)$ *is identified to* $\lambda \in \boldsymbol{R}_+^*$ *with* $\operatorname{mod}(\alpha) = F_\lambda^M$.

PROOF. By assumption, F^M is transitive with trivial kernel, so that every automorphism of F^M is of the form F_λ^M, $\lambda \in \boldsymbol{R}_+^*$. Hence for any $\alpha \in \operatorname{Aut}(M)$ there exists $\lambda > 0$ such that $\varphi \circ \alpha^{-1} \sim \lambda\varphi$ for every integrable weight φ of infinite multiplicity. Since M is a factor, we have $\tau \circ \alpha^{-1} = \mu\tau$ for some $\mu > 0$. Let $\varepsilon > 0$. As in the proof of Theorem II.4.7, choose an $h \in M$, $1 - \varepsilon \leqq h \leqq 1 + \varepsilon$, such that $\varphi = \tau(h\cdot)$ is an integrable weight of infinite multiplicity. We have then $\lambda\varphi = \varphi \circ \alpha^{-1} \circ \operatorname{Ad}(u)$ for some unitary $u \in M$, so that for every $x \in M_+$,

$$\begin{aligned} \lambda\tau(hx) &= \lambda\varphi(x) = \tau(h\alpha^{-1}(uxu^*)) = \tau \circ \alpha^{-1}(\alpha(h)uxu^*) \\ &= \mu\tau(\alpha(h)uxu^*) = \mu\tau(u^*\alpha(h)ux)\,. \end{aligned}$$

Thus we get $\lambda h = \mu u^*\alpha(h)u$; hence $(1 - \varepsilon)\lambda \leqq (1 + \varepsilon)\mu$ and $(1 - \varepsilon)\mu \leqq (1 + \varepsilon)\lambda$. Therefore, $\lambda = \mu$, ε being arbitrary. q.e.d.

PROPOSITION 1.3. (i) *If* M *is a factor of type* III_λ, $0 < \lambda < 1$, *with separable predual, then the map:* $\lambda \in \boldsymbol{R}_+^* \to F_\lambda^M \in \operatorname{Aut}(F^M)$ *is a homomorphism of* $\boldsymbol{R}_+^*$ *onto* $\operatorname{Aut}(F^M)$ *with kernel* $S(M) \cap \boldsymbol{R}_+^*$, *and for any* $\alpha \in \operatorname{Aut}(M)$ *and a generalized trace* φ *on* M, [3; 4.3], *we have*

$$\varphi \circ \alpha^{-1} \sim \lambda\varphi \quad \textit{with} \quad \operatorname{mod}(\alpha) = F_\lambda^M\,.$$

(ii) *If* M *is of type* III_1 *instead, then* $\operatorname{mod}(\alpha) = 1$ *for every* $\alpha \in \operatorname{Aut}(M)$.

PROOF. (i) We know that the flow F^M is transitive with kernel $S(M) \cap \boldsymbol{R}_+^*$, so that the first assertion follows. Now let $\alpha \in \operatorname{Aut}(M)$ and φ be as above, and $\lambda_1, \lambda_2 \in \boldsymbol{R}_+^*$ be such that

$$\varphi \circ \alpha^{-1} \sim \lambda_1\varphi \quad \text{and} \quad \psi \circ \alpha^{-1} \sim \lambda_2\psi$$

for any integrable weight ψ of infinite multiplicity on M. As above, for any $\varepsilon > 0$ there exists an $h \in M_\varphi$, $1 - \varepsilon \leqq h \leqq 1 + \varepsilon$, such that $\varphi(h\cdot) = \psi$ is integrable and of infinite multiplicity. For some unitaries $u, v \in M$ we have $\psi \circ \alpha^{-1} = \lambda_2\psi_u$ and $\varphi \circ \alpha^{-1} = \lambda_1\varphi_v$, so that for any $x \in M_+$

$$\begin{aligned} \lambda_2\varphi(huxu^*) &= \lambda_2\psi_u(x) = \psi(\alpha^{-1}(x)) = \varphi(h\alpha^{-1}(x)) \\ &= \varphi(\alpha^{-1}(\alpha(h)x)) = \lambda_1\varphi(v\alpha(h)xv^*)\,; \\ \lambda_2\varphi_u(u^*hux) &= \lambda_1\varphi_v(\alpha(h)x)\,. \end{aligned}$$

Hence we get $(D\varphi_u(u^*hu\cdot): D\varphi_v(\alpha(h)\cdot))_t = \lambda_1^{it}\lambda_2^{-it}$, $t \in \boldsymbol{R}$. Let T_0 be the generator of the modular period group $T(M)$. Then

$$(D\varphi_u(u^*hu\cdot): D\varphi_u)_{T_0}(D\varphi_u: D\varphi_v)_{T_0}(D\varphi_v: D\varphi_v(\alpha(h)\cdot))_{T_0} = \lambda_1^{iT_0}\lambda_2^{-iT_0} .$$

As we have

$$\begin{aligned}(D\varphi_u: D\varphi_v)_{T_0} &= (D\varphi_u: D\varphi)_{T_0}(D\varphi_v: D\varphi)^*_{T_0} \\ &= u^*\sigma^\varphi_{T_0}(u)\sigma^\varphi_{T_0}(v^*)v = 1 ,\end{aligned}$$

we get

$$\lambda_1^{iT_0}\lambda_2^{-iT_0} = (D\varphi_u(u^*hu\cdot): D\varphi_u)_{T_0}(D\varphi_v: D\varphi_v(\alpha(h)\cdot))_{T_0} .$$

The right hand side tends to 1 when $\varepsilon \to 0$, so that $\lambda_1\lambda_2^{-1}$ belongs to $S(M)$.

(ii) We know that the flow F^M is trivial for a factor of type III_1. q.e.d.

PROPOSITION 1.4. (i) *If M is an infinite factor with separable predual, then* Aut(F^M), *equipped with the simple convergence topology with respect to the norm topology in $(P_M)_*$, is a polish topological group.*

(ii) *If M is a factor of type* III_λ, $\lambda \neq 0$, *with separable predual, then the isomorphism of $\boldsymbol{R}^*_+/S(M) \cap \boldsymbol{R}^*_+$ onto* Aut(F^M), *given by Proposition* 1.3, *is a topological isomorphism.*

PROOF. (i) This follows from the fact that Aut(F^M) is a closed subgroup of the automorphism group Aut(P_M) of the separable abelian von Neumann algebra P_M.

(ii) The map: $\lambda \in \boldsymbol{R}^*_+ \to F^M_\lambda \in$ Aut(F^M) is continuous, so the isomorphism of $\boldsymbol{R}^*_+/\boldsymbol{R}^*_+ \cap S(M)$ onto Aut(F^M) is continuous whose domain is compact. Hence it is a homomorphism. q.e.d.

We are now going to show the continuity of the fundamental homomorphism mod. Let M be an infinite factor with separable predual. We represent Aut(M) on the predual M_* by considering the transpose of each automorphism, then consider the pointwise convergence topology in Aut(M) as in the preliminary. What we are going to prove is that mod is a continuous homomorphism of Aut(M) into Aut(F^M).

LEMMA 1.5. *Let M be a von Neumann algebra with separable predual, and $\mathfrak{U}$ the unitary group of M with the uniform structure of the σ-strong* convergence. Let α be a continuous action of a separable locally compact group on M. Then the set $Z^1_\alpha(G, \mathfrak{U})$ of all $\mathfrak{U}$-valued continuous functions on G such that $u_{gh} = u_g\alpha_g(u_h)$, $g, h \in G$, is a Polish space with respect to the uniform convergence topology on compact sets in G.*

PROOF. Let d be a bounded complete metric of $\mathfrak{U}$ giving the uniform structure of the σ-strong* convergence. Let $\{K_n\}$ be an increasing sequence of compact sets in G such that $G = \bigcup_{n=1}^{\infty} \mathring{K}_n$, where $\mathring{K}_n$ means the interior of K_n. Put

$$\delta(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{g \in K_n} d(u_g, v_g)\,, \quad u, v \in Z^1_\alpha(G, \mathfrak{U})\ .$$

It is not hard to see that δ is a complete metric on $Z^1_\alpha(G, \mathfrak{U})$ giving the uniform structure in question. Furthermore, $Z^1_\alpha(G, \mathfrak{U})$ is a closed subset of the separable complete metric space of $C(G, \mathfrak{U})$ of all continuous $\mathfrak{U}$-valued functions on G with the same metric δ. q.e.d.

PROPOSITION 1.6. *In the same situation as above, let* $\mathfrak{U}_0 = \{u \in \mathfrak{U}: \alpha_g(u) = u, g \in G\}$. *Then the map* $d: w \in \mathfrak{U} \to dw \in Z^1_\alpha(G, \mathfrak{U})$ *with* $(dw)_g = w^*\alpha_g(w)$ *induces a Borel isomorphism* $\bar{d}$ *of the quotient Borel space* $\mathfrak{U}_0\backslash\mathfrak{U}$ *onto a Borel subset* B *of* $Z^1_\alpha(G, \mathfrak{U})$.

PROOF. Since $\mathfrak{U}_0$ is a closed subspace, $\mathfrak{U}_0\backslash\mathfrak{U}$ is a Polish space. Now we claim that the map d is continuous. By Akemann's result [1], the σ-strong* topology in a bounded set in M is given by the uniform convergence topology on every weakly compact set in M_*. It follows then that the map: $(\varphi, g) \in L \times G \to \varphi \circ \alpha_g \in M_*$ is continuous on every weakly compact set L in M_*, where we consider the weak topology in M_*; hence the set $\{\varphi \circ \alpha_g: \varphi \in L, g \in K\}$ is weakly compact in M_* for any compact subset K of G and weakly compact subset L of M_*. Hence if $\{w_n\}$ is a sequence in $\mathfrak{U}$ converging to w, then $\{\langle \alpha_g(w_n), \varphi\rangle\}$ converges to $\langle \alpha_g(w), \varphi\rangle$ uniformly for $g \in K$ and $\varphi \in L$ as $n \to \infty$; hence $\alpha_g(w_n)$ tends to $\alpha_g(w)$ uniformly in $\mathfrak{U}$ for $g \in K$. Since $\mathfrak{U}$ is a topological group, $w_n^*\alpha_g(w_n)$ converges to $w^*\alpha_g(w)$ uniformly for $g \in K$. Hence $d(w_n)$ converges to $d(w)$ in $Z^1_\alpha(G, \mathfrak{U})$, which means that d is continuous. Furthermore, $d(w_1) = d(w_2)$, $w_1, w_2 \in \mathfrak{U}$, if and only if $w_1 w_2^* \in \mathfrak{U}_0$. Therefore, d induces a continuous injective map $\bar{d}$ from $\mathfrak{U}_0\backslash\mathfrak{U}$ into $Z^1_\alpha(G, \mathfrak{U})$. Hence it follows from [17] that the induced map $\bar{d}$ is a Borel isomorphism from $\mathfrak{U}_0\backslash\mathfrak{U}$ onto a Borel subset B of $Z^1_\alpha(G, \mathfrak{U})$. q.e.d.

PROPOSITION 1.7. *Let* M *and* $\mathfrak{U}$ *be as before.*

(i) *The space* $\mathfrak{W}_M$ *of all faithful weights* ψ *on* M *is a Polish space with respect to the topology of uniform convergence of the* $(D\psi: D\varphi)_t$ *in* $\mathfrak{U}$ *on compact subsets of* $\boldsymbol{R}$ *with* $\varphi \in \mathfrak{W}_M$ *fixed; and this topology is independent of the choice of* φ.

(ii) *For a faithful weight* φ *on* M, *the set* $\{\psi \in \mathfrak{W}_M: \varphi \sim \psi\} = W_\varphi$ *is a Borel subset of* $\mathfrak{W}_M$, *and there exists a Borel map* $u: \psi \in W_\varphi \to u(\psi) \in \mathfrak{U}$ *such that* $\varphi_{u(\psi)} = \psi$, $\psi \in W_\varphi$.

PROOF. (i) With $\varphi \in \mathfrak{W}_M$ fixed, the topology in $\mathfrak{W}_M$ is identified with that in $Z^1_{\sigma^\varphi}(\boldsymbol{R}, \mathfrak{U})$ under the correspondence: $\psi \leftrightarrow (D\psi: D\varphi) \in Z^1_{\sigma^\varphi}(\boldsymbol{R}, \mathfrak{U})$. Hence the first half of the assertion follows from Lemma 1.5. Let $\{\psi_n\}$ be a sequence in $\mathfrak{W}_M$ converging to ψ. Then $(D\psi_n: D\varphi)_t \to (D\psi: D\varphi)_t$ in $\mathfrak{U}$ uniformly on compact subsets of $\boldsymbol{R}$. For any other faithful weight φ',

$$(D\psi_n: D\varphi')_t = (D\psi_n: D\varphi)_t(D\varphi: D\varphi')_t \to (D\psi: D\varphi)_t(D\varphi: D\varphi')_t = (D\psi: D\varphi')_t$$

in $\mathfrak{U}$ uniformly on compact subsets of $\boldsymbol{R}$. Hence the topology in $\mathfrak{W}_M$ is independent of the choice of φ.

(ii) We apply Proposition 1.6 to $G = \boldsymbol{R}$ and $\alpha = \sigma^\varphi$. It follows then that $\varphi \sim \psi$, $\psi \in \mathfrak{W}_M$, if and only if $(D\psi: D\varphi) \in d(\mathfrak{U})$. Let f be a Borel cross-section from $\mathfrak{U}_0\backslash\mathfrak{U}$ to $\mathfrak{U}$, and put $u(\psi) = f \circ \bar{d}^{-1}(D\psi: D\varphi)$. Then u is a Borel map and $\varphi_{u(\psi)} = \psi$ by construction. q.e.d.

PROPOSITION 1.8. *Let M be as above, and* Aut(M) *be equipped with the simple norm convergence topology in M_*. For any $\varphi \in \mathfrak{W}_M$, the map: $\alpha \in$* Aut(M) *$\to \varphi \circ \alpha^{-1} \in \mathfrak{W}_M$ is continuous in the topology on $\mathfrak{W}_M$ defined above.*

PROOF. Let ψ be a faithful normal state on M. If $\alpha_n \to \alpha_0$ in Aut(M), then $\|\psi \circ \alpha_n^{-1} - \psi \circ \alpha^{-1}\| \to 0$. Hence by [4], $(D\psi \circ \alpha_n^{-1}: D\psi \circ \alpha_0^{-1})_t \to 1$, $n \to \infty$, uniformly on compact subsets of $\boldsymbol{R}$. For any $\varphi \in \mathfrak{W}_M$, we have

$$\begin{aligned}(D\varphi\circ\alpha_n^{-1}: D\varphi\circ\alpha_0^{-1})_t &= (D\varphi\circ\alpha_n^{-1}: D\psi\circ\alpha_n^{-1})_t(D\psi\circ\alpha_n^{-1}: D\psi\circ\alpha_0^{-1})_t(D\psi\circ\alpha_0^{-1}: D\varphi\circ\alpha_0^{-1})_t \\ &= \alpha_n((D\varphi: D\psi)_t)(D\psi \circ \alpha_n^{-1}: D\psi \circ \alpha_0^{-1})_t\alpha_0((D\psi: D\varphi)_t) .\end{aligned}$$

Thus we have only to prove that $\alpha_n(D\varphi: D\psi_t) \to \alpha_0((D\varphi: D\psi)_t)$ in $\mathfrak{U}$ uniformly on compact subsets of $\boldsymbol{R}$. Hence we will show that $\alpha_n(u) \to \alpha(u)$ in $\mathfrak{U}$ uniformly for u in a compact subset of K of $\mathfrak{U}$. For any $u, v \in \mathfrak{U}$, $\alpha, \beta \in$ Aut(M) and $\omega \in M_*$, we have

$$\begin{aligned}|\langle\alpha(u) - \beta(v), \omega\rangle| &\leqq |\langle u, \omega\circ\alpha - \omega\circ\beta\rangle| + |\langle u - v, \omega\circ\beta\rangle| \\ &\leqq \|\omega\circ\alpha - \omega\circ\beta\| + |\langle u - v, \omega\circ\beta\rangle| ,\end{aligned}$$

so that the map: $(\alpha, u) \in$ Aut$(M) \times \mathfrak{U} \to \alpha(u) \in \mathfrak{U}$ is continuous, because the σ-strong* topology and the σ-weak topology in $\mathfrak{U}$ coincide. Hence $A = \{\alpha_n(u): u \in K, n = 0, 1, \cdots\} \subset \mathfrak{U}$ is compact, so that the σ-weak uniform structure and the σ-strong* uniform structure agree in A. For any fixed $\omega \in M_*$, the set $B = \{\omega \circ \alpha_n: n = 0, 1, \cdots\}$ is compact in the norm topology. For any $\varepsilon > 0$, there exist $u_1, u_2, \cdots, u_m$ in K such that $\inf_{1\leqq i\leqq m} |\langle u - u_i, \omega\circ\alpha_n\rangle| < \varepsilon$ for every $u \in K$ and $n = 0, 1, \cdots$, by Akemann's characterization [1] of the σ-strong* topology in M. Let n_0 be large enough so that $|\langle u_i, \omega\circ\alpha_n - \omega\circ\alpha_0\rangle| < \varepsilon$ for every $n \geqq n_0$ and $i = 1, 2, \cdots, m$. We have then, for any $u \in K$ and $n > n_0$,

$$|\langle u, \omega\circ\alpha_n - \omega\circ\alpha_0\rangle| \leqq |\langle u - u_i, \omega\circ\alpha_n - \omega\circ\alpha_0\rangle| + |\langle u_i, \omega\circ\alpha_n - \omega\circ\alpha_0\rangle|$$
$$\leqq 2\varepsilon + \varepsilon = 3\varepsilon\,.$$

Thus $\{\alpha_n(u)\}$ converges to $\alpha_0(u)$ σ-weakly and uniformly for $u \in K$; hence it converges to $\alpha_0(u)$ σ-strongly* uniformly on K. q.e.d.

We are now at the position to state the continuity of γ_M.

THEOREM 1.9. *Let M be an infinite factor with separable predual. Then the fundamental homomorphism* mod *is a continuous homomorphism of* $\mathrm{Aut}(M)$ *into* $\mathrm{Aut}(F^M)$, *where we consider the simple norm convergence topologies in M_* for* $\mathrm{Aut}(M)$ *and in $(P_M)_*$ for* $\mathrm{Aut}(F^M)$ *respectively. Hence* $\mathrm{mod}(\alpha) = \iota$ *for every* $\alpha \in \overline{\mathrm{Int}(M)}$.

PROOF. We know, as in the preliminary, that $\mathrm{Aut}(M)$ is a Polish topological group as well as $\mathrm{Aut}(F^M)$. Hence we just have to prove that γ_M is a Borel map.

By construction, $\mathrm{mod}(\alpha) = \iota$ for every $\alpha \in \mathrm{Int}(M)$. Let $\bar{\omega}$ be a dominant weight on M, and $p_{\bar{\omega}}$ be the isomorphism of the center $C_{\bar{\omega}}$ of $M_{\bar{\omega}}$ onto P_M defined in Theorem I.1.11 and the proof of Theorem II.2.2. We claim that for any $\alpha \in \mathrm{Aut}(M)$ with $\bar{\omega}\circ\alpha^{-1} = \bar{\omega}$

$$(*)\qquad p_{\bar{\omega}}^{-1}\,\mathrm{mod}(\alpha)p_{\bar{\omega}} = \alpha|_{C_{\bar{\omega}}}\,.$$

To see this, let u be an isometry in M with $e = uu^* \in C_{\bar{\omega}}$. Then we have

$$\begin{aligned}\mathrm{mod}(\alpha)(p_M(\bar{\omega}_u)) &= p_M(\bar{\omega}_u\circ\alpha^{-1}) = p_M(\bar{\omega}_{\alpha(u)})\\ &= p_{\bar{\omega}}(\alpha(e)) \qquad \text{by Theorem I.1.11 (ii);}\end{aligned}$$

hence

$$\mathrm{mod}(\alpha)(p_{\bar{\omega}}(e)) = p_{\bar{\omega}}(\alpha(e))\,.$$

Let $u(\cdot)$ be the Borel map from the set $W_{\bar{\omega}}$ of dominant weights on M to the unitary group $\mathfrak{U}$ of M defined in Proposition 1.7(ii) such that $\bar{\omega}_{u(\psi)} = \psi$ for any dominant weight ψ. By Proposition 1.8, the map $h\colon \alpha \in \mathrm{Aut}(M) \to h(\alpha) = \mathrm{Ad}(u(\bar{\omega}\circ\alpha^{-1}))\circ\alpha \in \mathrm{Aut}(M)$ is a Borel map, since the map $\mathrm{Ad}\colon v \in \mathfrak{U} \to \mathrm{Ad}\,v \in \mathrm{Aut}(M)$ is continuous. We then have

$$\mathrm{mod}(\alpha) = \mathrm{mod}(\mathrm{Ad}(u(\bar{\omega}\circ\alpha^{-1})))\,\mathrm{mod}(\alpha),\ \alpha \in \mathrm{Aut}(M)\,;$$
$$\bar{\omega}\circ h(\alpha)^{-1} = (\bar{\omega}\circ\alpha^{-1})_{u(\bar{\omega}\circ\alpha^{-1})} = \bar{\omega}\,;$$

therefore

$$p_{\bar{\omega}}^{-1}\,\mathrm{mod}(\alpha)p_{\bar{\omega}} = h(\alpha)|_{C_{\bar{\omega}}} \qquad \text{by } (*).$$

This shows that mod is a Borel map. q.e.d.

THEOREM 1.10. *Let M be a factor of type* III_λ, $\lambda \neq 1$, *with separa-*

ble predual. Viewing the fundamental homomorphism mod *as a homomorphism of* Out(M) = Aut(M)/Int(M) *into* Aut(F^M) *by the trivial identification, the following three conditions for* $\bar{\alpha} \in$ Out(M) *are equivalent*:

(i) $\mathrm{mod}(\bar{\alpha}) = \iota$;

(ii) *There exists a faithful normal state* φ *on* M *and a representative* α_0 *of* $\bar{\alpha}$ *such that*

$$\varphi \circ \alpha_0 = \varphi \quad \textit{and} \quad \alpha_0|_{C_\varphi} = \iota \,;$$

(iii) *For any* $\varepsilon > 1$ *such that* $]\varepsilon^{-1}, \varepsilon[\cap S(M) = \{1\}$, *there exists a faithful normal state* φ *on* M *and a representative* α_0 *of* $\bar{\alpha}$ *satisfying* (ii) *and*

$$\mathrm{Sp}\,(\Delta_\varphi) \cap]\varepsilon^{-1}, \varepsilon[= \{1\}\,.$$

To prove the theorem, we need the following lemma which is a slight refinement of Lemma I.2.3 and [3; Lemma 5.2.4].

LEMMA 1.11. *If* ψ *is a faithful weight on a factor of type* III$_\lambda$, $\lambda \neq 1$, *then for any* $\varepsilon > 1$ *with* $]\varepsilon^{-1}, \varepsilon[\cap S(M) = \{1\}$ *there exists a positive* $h < C_\psi$ *such that, with* $\varphi = \psi(h\cdot)$ *and* $e = s(h)$,

$$\mathrm{Sp}\,(\Delta_\varphi) \cap]\varepsilon^{-1}, \varepsilon[= \{1\}\,,$$

where Δ_φ *means of course the modular operator corresponding to* $\{M_e, \varphi\}$.

PROOF. This follows from Lemma I.2.3 and the observation that the operator $H \in M_{\psi_1}$ in the proof of Lemma I.2.3 is indeed in C_{ψ_1} because each spectral projection of H is given by the left support projection of $M(\sigma^{\psi_1}, V)$ for each closed subset V of $\boldsymbol{R}$ which belongs to C_{ψ_1}. q.e.d.

PROOF OF THEOREM 1.10. (i) $\Rightarrow$ (iii): Suppose $\gamma_M(\bar{\alpha}) = \iota$ and $\bar{\omega}$ is a dominant weight on M. There exists a representative α_1 of $\bar{\alpha}$ such that $\bar{\omega} \circ \alpha_1 = \bar{\omega}$ and $\alpha_1|_{C_{\bar{\omega}}} = \iota$. Let $h \in C_{\bar{\omega}}$ be a positive operator such that $\varphi = \bar{\omega}(h\cdot)$ satisfies the condition in Lemma 1.11. It follows then that $\varphi \circ \alpha_1 = \varphi$. Since $M_\varphi \supset M_{\bar{\omega}_e}$ with $e = s(h)$, we have $C_\varphi \subset C_{\bar{\omega}_e} = C_{\bar{\omega},e}$ by Theorem II.5.1. Therefore, we have

$$\varphi \circ \alpha_1 = \varphi \quad \text{and} \quad \alpha_1|_{C_\varphi} = \iota\,.$$

Being lacunary, φ is strictly semi-finite, so that the restriction τ of φ to M_φ is a faithful semi-finite normal trace. Since α_1 leaves τ invariant and C_φ elementwise fixed, we have $\alpha_1(p) \sim p$ in M_φ for every projection $p \in M_\varphi$. Let p be a projection in M_φ such that $\varphi(p) < +\infty$. It follows then that $\psi = (1/\varphi(p))\varphi_p$ is a normal state of M. Let u be a unitary in M_φ such that $upu^* = \alpha_1(p)$. Put $\alpha_2 = \mathrm{Ad}\,(u)^{-1} \circ \alpha_1 \in \bar{\alpha}$. We have then

$\psi\circ\alpha_2=\psi$ and that α_2 leaves C_ψ elementwise fixed. Let w be an isometry of M such that $ww^*=p$. Put

$$\alpha_0(x)=w^*\alpha_2(wxw^*)w\,,\qquad x\in M\,;$$
$$\psi_0=\psi_w\,.$$

We have that ψ_0 is a faithful normal state on M, $\psi_0\circ\alpha_0=\psi_0$ and α_0 leaves C_{ψ_0} elementwise fixed. Since $\alpha_0=\mathrm{Ad}\,(w^*\alpha_2(w))\circ\alpha_2$ and $w^*\alpha_2(w)$ is unitary, α_0 belongs to $\bar\alpha$. Thus (iii) follows.

(iii) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (i): Let $\alpha_0\in\mathrm{Aut}\,(M)$ and φ be a faithful normal state on M satisfying the condition in (ii). We consider the tensor products $\bar M=M\otimes F_\infty$, $\bar\omega=\varphi\otimes\omega$ and $\alpha_0\otimes\iota=\tilde\alpha_0$. From the proof of Theorem II.5.1, it follows that the center $C_{\bar\omega}$ of $\bar M_{\bar\omega}$ is a von Neumann subalgebra of $C_\varphi\otimes U(L^\infty(\boldsymbol{R}))$. Since $\alpha_0|_{C_\varphi}=\iota$, $\tilde\alpha_0$ leaves $C_\varphi\otimes U(L^\infty(\boldsymbol{R}))$ elementwise fixed. Hence $C_{\bar\omega}$ is fixed elementwise by $\tilde\alpha_0$. Therefore, we have $\mathrm{mod}\,(\tilde\alpha_0)=\mathrm{mod}\,(\alpha_0)=1$. q.e.d.

IV.2. The extended modular automorphism groups. Throughout this section, let M be an infinite factor with separable predual, P_M, p_M, F^M and so on be as before. Let $Z^1(F^M)$ be the set of all σ-strongly* continuous functions $\{c_\lambda\}$ on $\boldsymbol{R}_+^*$ with values in the unitary group of P_M such that

$$c_{\lambda\mu}=c_\lambda F_\lambda^M(c_\mu)\,,\qquad \lambda,\mu\in\boldsymbol{R}_+^*\,,$$

and $B^1(F^M)$ be the set of all elements in $Z^1(F^M)$ of the form: $\lambda\in\boldsymbol{R}_+^*\to v^*F_\lambda^M(v)$ for some unitary $v\in P_M$. Under the pointwise multiplication, $Z^1(F^M)$ is an abelian group, and $B^1(F^M)$ is a subgroup of $Z^1(F^M)$. Put

$$H^1(F^M)=Z^1(F^M)/B^1(F^M)\,.$$

For each $t\in\boldsymbol{R}$, let $\bar t$ denote the element in $Z^1(F^M)$ defined by

$$\bar t(\lambda)=\lambda^{it}\,,\qquad \lambda\in\boldsymbol{R}_+^*\,.$$

PROPOSITION 2.1. *If φ is an integrable faithful weight on M, then to each $c\in Z^1(F^M)$ there corresponds a unique automorphism $\bar\sigma_c^\varphi$ of M such that*

(i) $\bar\sigma_c^\varphi(x)=p_\varphi^{-1}(c_\lambda p_M(\varphi))x$ *for every* $x\in M(\sigma^\varphi,\{\lambda\})$, $\lambda>0$;

(ii) $\varphi\circ\bar\sigma_c^\varphi=\varphi$ *and* $\bar\sigma_{c_1c_2}^\varphi=\bar\sigma_{c_1}^\varphi\circ\bar\sigma_{c_2}^\varphi$, $c_1,c_2\in Z^1(F^M)$;

(iii) $\bar\sigma_{\bar t}^\varphi=\sigma_t^\varphi$, $t\in\boldsymbol{R}$.

PROOF. (i) The uniqueness of $\bar\sigma_c^\varphi$ follows from Lemma II.2.3. Let $M=W^*(N,\boldsymbol{R},\theta)$ be a continuous decomposition of M, and τ be a faithful semi-finite normal trace on N such that $\tau\circ\theta_s=e^{-s}\tau$, $s\in\boldsymbol{R}$. Let $\{u(s):s\in\boldsymbol{R}\}$

be the one parameter unitary group in M canonically associated with the decomposition $W^*(N, \boldsymbol{R}, \theta) = M$. We know that the dual weight $\bar{\omega} = \tilde{\tau}$ is dominant, and that $\bar{\omega}_{u(s)} = e^{-s}\bar{\omega}$ and $F_\lambda^M \circ p_{\bar{\omega}}(x) = p_{\bar{\omega}} \circ \theta_{-\mathrm{Log}\,\lambda}(x)$ for every x in the center C of N and $\lambda > 0$. For a fixed $c \in Z^1(F^M)$, we put

$$b_s = p_{\bar{\omega}}^{-1}(c_{e^s}) , \qquad s \in \boldsymbol{R} .$$

It follows then that b_s is a unitary in C and

$$b_{s+t} = b_s \theta_s(b_t) , \qquad s, t \in \boldsymbol{R} .$$

Hence there exists a unique automorphism $\bar{\sigma}_c$ of $M = W^*(N, \boldsymbol{R}, \theta)$ such that

$$\bar{\sigma}_c(au(s)) = b_s a u(s) , \qquad a \in N, s \in \boldsymbol{R} .$$

Thus we have shown that $\bar{\sigma}_c^{\bar{\omega}}$ exists for a dominant weight $\bar{\omega}$ on M.

Now, let v be an isometry in M with $vv^* = e \in M_{\bar{\omega}} = N$ such that $\varphi = \bar{\omega}_v$. Observing that e is fixed under $\bar{\sigma}_c^{\bar{\omega}}$, we define an automorphism α of M by

$$\alpha(x) = v^* \bar{\sigma}_c^{\bar{\omega}}(vxv^*)v , \qquad x \in M .$$

Since the map: $x \in M \longrightarrow vxv^* \in M_e$ is an isomorphism of M onto M_e which brings α to $\bar{\sigma}_c^{\bar{\omega}}$ and σ_t^φ to $\sigma_t^{\bar{\omega}}$, $t \in \boldsymbol{R}$, we have

$$\alpha(x) = v^* p_{\bar{\omega}}^{-1}(c_\lambda) v x , \qquad x \in M(\sigma^\varphi, \{\lambda\}) .$$

Thus we must show that

$$v^* p_{\bar{\omega}}^{-1}(a) v = p_\varphi^{-1}(a p_M(\varphi)) , \qquad a \in P_M .$$

To this end, we may assume that $a = P_M(\psi)$ for some integrable ψ, since $p_M(\psi)$'s generate P_M. We have then

$$\begin{aligned} p_\varphi(v^* p_{\bar{\omega}}^{-1}(a) v) &= p_\varphi(v^* c_{\bar{\omega}}(\psi) v) \quad \text{by Theorem I.1.11,} \\ &= p_\varphi(c_{\bar{\omega}_v}(\psi)) \quad \text{by Lemma I.1.6,} \\ &= p_\varphi(c_\varphi(\psi)) = p_M(\psi) p_M(\varphi) \quad \text{by Theorem I.1.11,} \\ &= a p_M(\varphi) . \end{aligned}$$

Thus α satisfies the requirement for $\bar{\sigma}_c^\varphi$.

(ii) We know that $\bar{\omega} \circ \bar{\sigma}_c^{\bar{\omega}} = \bar{\omega}$ by construction. Thus $\bar{\sigma}_c^\varphi$, namely α, preserves φ by definition.

(iii) If $c = \bar{t}$, then $c_\lambda = \lambda^{it}$, so that we get

$$p_\varphi(c_\lambda p_M(\varphi)) = \lambda^{it} , \qquad \lambda > 0 .$$

Hence $\bar{\sigma}_c^\varphi = \sigma_t^\varphi$. q.e.d.

THEOREM 2.2. *Let φ be an integrable weight on M. If $\alpha \in \mathrm{Aut}\,(M)$*

leaves M_φ elementwise fixed, then $\alpha = \bar{\sigma}_c^\varphi$ for some $c \in Z^1(F^M)$.

PROOF. Let $\bar{\omega}$ be dominant, and $M = W^*(N, \boldsymbol{R}, \theta)$ be the associated continuous decomposition of M and $\{u(s)\}$ the one parameter unitary group in M appearing in the decomposition. First we assume that α is an automorphism of M leaving N elementwise fixed. For each $s \in \boldsymbol{R}$, let $b_s = \alpha(u(s))u(s)^*$. By Theorem II.5.1, b_s belongs to the center C of N and

$$b_{s+t} = b_s\theta_s(b_t)\ , \qquad s, t \in \boldsymbol{R}\ .$$

Furthermore, we have

$$\alpha(xu(s)) = b_s xu(s)\ , \qquad x \in N,\ s \in \boldsymbol{R}\ .$$

Hence, putting $c_\lambda = p_{\bar{\omega}}(b_{-\mathrm{Log}\,\lambda})$, $\lambda > 0$, we get $\alpha = \bar{\sigma}_c^{\bar{\omega}}$.

In the general case, there is an isometry u with $uu^* = e \in N$ such that $\varphi = \bar{\omega}_u$. Suppose that $\alpha \in \mathrm{Aut}\,(M)$ leaves M_φ elementwise invariant. Considering the automorphism: $x \in M_e \rightarrow u\alpha(u^*xu)u^* \in M_e$, we may assume that $\alpha \in \mathrm{Aut}\,(M_e)$ leaves N_e elementwise invariant.

For every $x \in N_e$ and $s \in \boldsymbol{R}$, we have

$$\begin{aligned} x\alpha(eu(s)e)eu(s)^*e &= \alpha(xeu(s)e)eu(s)^*e \\ &= \alpha(eu(s)e\theta_{-s}(xe))eu(s)^*e \\ &= \alpha(eu(s)e)\theta_{-s}(xe)eu(s)^*e \\ &= \alpha(eu(s)e)eu(s)^*exe\theta_s(e)\ , \end{aligned}$$

so that $b_s = \alpha(eu(s)e)eu(s)^*e \in Ce\theta_s(e)$. A direct computation shows that

$$b_{s+t}\theta_s(e) = b_s\theta_s(b_t)\ , \qquad s, t \in \boldsymbol{R}\ .$$

Thus there exists, by Proposition A.1, $b' \in Z_\theta^1(\boldsymbol{R}, \mathfrak{U}_C)$ such that $b_s = b'_s e\theta_s(e)$, $s \in \boldsymbol{R}$. Define an $\alpha' \in \mathrm{Aut}\,(M)$ by

$$\alpha'(xu(s)) = b'_s xu(s)\ , \qquad x \in N,\ s \in \boldsymbol{R}\ .$$

We have then

$$\alpha(x) = \alpha'(x) \quad \text{for every} \quad x \in M_e\ .$$

Putting $c_\lambda = p_{\bar{\omega}}(b'_{-\mathrm{Log}\,\lambda})$, we have $\alpha' = \bar{\sigma}_c^{\bar{\omega}}$, so that $\alpha = \bar{\sigma}_c^\varphi$. q.e.d.

EXAMPLE 2.3. Let N be an infinite semi-finite factor with separable predual. We identify $\{P_N, F^N\}$ with $L^\infty(\boldsymbol{R}_+^*, d\lambda)$ acted by the translation of $\boldsymbol{R}_+^*$ as in II.2. We then conclude the following:

(i) For every $c \in Z^1(F^N)$ there exists a unique, up to scalar multiple, unitary $f \in L^\infty(\boldsymbol{R}_+^*, d\lambda)$ such that $c_\lambda = fF_\lambda(f^*)$, $\lambda > 0$.

(ii) With $c = df$ as in (i), and $\varphi = \tau(h_\varphi\cdot)$ as integrable weight, we have

$$\bar{\sigma}_c^{\varphi} = \operatorname{Ad}(f(h_\varphi)) .$$

PROOF. (i) This is known.

(ii) We have first that $\sigma_t^{\varphi} = \operatorname{Ad}(h_\varphi^{it})$, $t \in \boldsymbol{R}$. The integrability of φ implies that the spectrum of h_φ is absolutely continuous with respect to the Lebesgue measure, so that $f(h_\varphi) = u$ makes sence. Let a be a partial isometry in $N(\sigma^\varphi, \{\lambda\})$, $\lambda > 0$. We have then $h_\varphi^{it} a h_\varphi^{-it} = \lambda^{it} a$, $t \in \boldsymbol{R}$, so that $a^* h_\varphi^{it} a = (\lambda h_\varphi)^{it} a^* a$. Therefore, we get

$$\begin{aligned} a^* f(h_\varphi) a &= f(\lambda h_\varphi) a^* a ; \\ f(h_\varphi) a f(h_\varphi)^* &= a f(h_\varphi)^* f(\lambda h_\varphi) a^* a \\ &= f(\lambda^{-1} h_\varphi)^* f(h_\varphi) a = p_\varphi^{-1}(c_\lambda p_M(\varphi)) a \\ &= \bar{\sigma}_c^{\varphi}(a) . \end{aligned}$$

Therefore, $\bar{\sigma}_c^{\varphi}$ and $\operatorname{Ad}(f(h_\varphi))$ agree on the set of partial isometries in $N(\sigma^\varphi, \{\lambda\})$, $\lambda > 0$. But any element of $N(\sigma^\varphi, \{\lambda\})$ is the product of a positive element in $N_\varphi = \{h_\varphi\}' \cap N$ and a partial isometry in $N(\sigma^\varphi, \{\lambda\})$ by polar decomposition. Thus $\bar{\sigma}_c^{\varphi} = \operatorname{Ad}(f(h_\varphi))$. q.e.d.

This example shows what we deal with by considering $\bar{\sigma}_c^{\varphi}$: it may be called a "functional calculus" of the "generator" of the modular automorphism group.

THEOREM 2.4. *Let φ_1 and φ_2 be faithful integrable weights on an infinite factor M with separable predual, and $P = M \otimes F_2$. Put*

$$\varphi\Big(\sum_{i,j=1}^{2} x_{i,j} \otimes e_{i,j}\Big) = \varphi_1(x_{11}) + \varphi_2(x_{22}), \quad x = \sum_{i,j=1}^{2} x_{i,j} \otimes e_{i,j} \in P .$$

We then conclude the following:

(i) *To each $c \in Z^1(F^M)$, there corresponds a unique unitary $u_c = (D\varphi_2 : D\varphi_1)_c$ in M such that*

$$\bar{\sigma}_c^{\varphi}(1 \otimes e_{21}) = u_c \otimes e_{21} ;$$

(ii) *We have*

$$\begin{aligned} \bar{\sigma}_c^{\varphi_2}(x) &= u_c \bar{\sigma}_c^{\varphi_1}(x) u_c^* , \qquad x \in M, \quad c \in Z^1(F^M) ; \\ u_{c_1 c_2} &= u_{c_1} \bar{\sigma}_{c_1}^{\varphi_1}(u_{c_2}) , \qquad c_1, c_2 \in Z^1(F^M) . \end{aligned}$$

PROOF. The integrability of φ follows from that of φ_1 and φ_2. Noticing that $1 \otimes e_{i,i} \in P_\varphi$, $i = 1, 2$, and $\bar{\sigma}_c^{\varphi}(x \otimes e_{ii}) = \bar{\sigma}_c^{\varphi_i}(x) \otimes e_{ii}$, $i = 1, 2$, we follow the arguments for the unitary cocycle Radon-Nikodym theorem, without any alteration. q.e.d.

COROLLARY 2.5. *Let M be an infinite factor with separable predual. Let ε_M denote the canonical homomorphism of* $\operatorname{Aut}(M)$ *onto* $\operatorname{Out}(M) = \operatorname{Aut}(M)/\operatorname{Int}(M)$.

(i) *For every* $c \in Z^1(F^M)$, *the element* $\varepsilon_M(\bar{\sigma}_c^\varphi)$ *of* Out(M) *is independent of the choice of an integrable weight* φ. *Put* $\bar{\delta}_M(c) = \varepsilon_M(\bar{\sigma}_c^\varphi)$.

(ii) $\bar{\delta}_M$ *is an extension of the modular homomorphism* ($\bar{\delta}_M(\bar{t}) = \delta_M(t)$, $t \in \boldsymbol{R}$) *and* $\operatorname{Ker}\bar{\delta}_M = B^1(F^M)$.

(iii) *The range of* $\bar{\delta}_M$ *is a normal subgroup of* Out(M) *with*

$$\alpha\bar{\delta}_M(c)\alpha^{-1} = \bar{\delta}_M(\operatorname{mod}(\alpha)c)\ , \qquad \alpha \in \operatorname{Out}(M)\ .$$

PROOF. (i) Trivial from the previous theorem.

(ii) The first half follows from Proposition 2.1(iii). Let $\bar{\omega}$ be a dominant weight and $c \in Z^1(F^M)$. Assume that $\bar{\sigma}_c^{\bar{\omega}} = \operatorname{Ad}(u)$. Since $\bar{\sigma}_c^{\bar{\omega}}$ leaves $M_{\bar{\omega}}$ pointwise fixed, we have $u \in C_{\bar{\omega}}$ by Theorem II.5.1. It follows then that

$$c_\lambda = p_{\bar{\omega}}(u)^* F_\lambda^M(p_{\bar{\omega}}(u))\ , \qquad \lambda > 0\ .$$

The converse is proven the same way.

(iii) Let $\bar{\omega}$ be dominant as before, and $\alpha \in \operatorname{Aut}(M)$. Multiplying α by an inner automorphism, we assume $\bar{\omega}\circ\alpha = \bar{\omega}$, so that

$$p_{\bar{\omega}}^{-1}\circ\operatorname{mod}(\alpha)\circ p_{\bar{\omega}} = \alpha|_{C_{\bar{\omega}}}\ .$$

If x is an element of $M(\sigma^{\bar{\omega}}, \{\lambda\})$, then $\alpha^{-1}(x) \in M(\sigma^{\bar{\omega}}, \{\lambda\})$, because α and $\sigma^{\bar{\omega}}$ commute; hence

$$\begin{aligned}\alpha\circ\bar{\sigma}_c^{\bar{\omega}}\circ\alpha^{-1}(x) &= \alpha(p_{\bar{\omega}}^{-1}(c_\lambda)\alpha^{-1}(x)) = \alpha(p_{\bar{\omega}}^{-1}(c_\lambda))x \\ &= p_{\bar{\omega}}^{-1}(\operatorname{mod}(\alpha)c_\lambda)x\ .\end{aligned}$$

q.e.d.

THEOREM 2.6. *Let M be an infinite factor with separable predual, and $\mathfrak{W}_M^0$ the space of all faithful weights on M with the metric d defined in* II.4. *If $c \in Z^1(F^M)$ is twice continuously differentiable in norm, then there exist uniquely maps: $\varphi \in \mathfrak{W}_M^0 \to \bar{\sigma}_c^\varphi \in \operatorname{Aut}(M)$ and $(\varphi, \psi) \in \mathfrak{W}_M^0 \times \mathfrak{W}_M^0 \to (D\varphi : D\psi)_c \in \mathfrak{U}(M)$, the unitary group of M, with the following properties*:

(i) *If φ is integrable, then $\bar{\sigma}_c^\varphi$ satisfies condition* (i) *in Proposition* 2.1. *If φ and ψ are both integrable, then $(D\varphi : D\psi)_c$ is given by Theorem* 2.4(i);

(ii) *The both maps are continuous with respect to the norm topologies in* Aut(M) *and* $\mathfrak{U}(M)$;

(iii) *For each $x \in M$, we have*

$$\bar{\sigma}_c^\varphi(x) = (D\varphi : D\psi)_c\bar{\sigma}_c^\psi(x)(D\varphi : D\psi)_c^*\ ;$$

(iv) *For each $\varphi_1, \varphi_2, \varphi_3 \in \mathfrak{W}_M^0$, we have*

$$(D\varphi_1 : D\varphi_3)_c = (D\varphi_1 : D\varphi_2)_c(D\varphi_2 : D\varphi_3)_c\ ;$$
$$(D\varphi_1 : D\varphi_2)_c = (D\varphi_2 : D\varphi_1)_c^*\ ;$$

(v) *For any* $\alpha \in \mathrm{Aut}(M)$ *and* $u \in \mathfrak{U}(M)$, *we have*

$$\bar{\sigma}_c^{\varphi\circ\alpha} = \alpha^{-1} \circ \bar{\sigma}^{\varphi}_{\mathrm{mod}\,(\alpha)c} \circ \alpha \ ;$$
$$(D\varphi \circ \alpha \colon D\psi \circ \alpha)_c = \alpha^{-1}((D\varphi \colon D\psi)_{\mathrm{mod}\,(\alpha)c}) \ ;$$
$$(D\varphi_u \colon D\psi)_c = u^*(D\varphi \colon D\psi)_c \bar{\sigma}_c^{\psi}(u) \ ;$$

(vi) *If* $c_1, c_2 \in Z^1(F^M)$ *are twice differentiable in norm, then*

$$\bar{\sigma}^{\varphi}_{c_1c_2} = \bar{\sigma}^{\varphi}_{c_1} \circ \bar{\sigma}^{\varphi}_{c_2} \ ;$$
$$(D\varphi \colon D\psi)_{c_1c_2} = (D\varphi \colon D\psi)_{c_1} \bar{\sigma}^{\psi}_{c_1}((D\varphi \colon D\psi)_{c_2}) \ .$$

The uniqueness of these maps follows from Proposition 2.1 and the density of integrable weights in $\mathfrak{W}_M^0$.

LEMMA 2.7. *Let* $c \in Z^1(F^M)$ *be as in the theorem. For any* $\varepsilon > 0$ *there exists* $\eta > 0$ *such that for any faithful integrable weight* φ *on* M:

$$x \in M(\sigma^{\varphi}, [e^{-\eta}, e^{\eta}]) \Longrightarrow \|\bar{\sigma}_c^{\varphi}(x) - x\| \leqq \varepsilon \|x\| \ .$$

PROOF. Without loss of generality, we may assume that φ is dominant. Put $b_s = p_{\varphi}^{-1}(c_{e^s})$, $s \in \boldsymbol{R}$. Let $\{u(s)\}$ be the one parameter unitary group in M which, together with M_{φ}, generate M as a continuous decomposition $M = W^*(M_{\varphi}, \boldsymbol{R}, \theta)$. We then have

$$\bar{\sigma}_c^{\varphi}(u(s)) = b_s u(s) \ , \qquad s \in \boldsymbol{R} \ .$$

If f is a function in the Schwartz space $\mathscr{S}(\boldsymbol{R})$, then the M-valued function: $s \in \boldsymbol{R} \longrightarrow \int_{-\infty}^{\infty} e^{-isp} f(p) b_p dp \in M$ is integrable by the twice differentiability of $\{b_p\}$ and we have

$$\bar{\sigma}_c^{\varphi}(\sigma_f^{\varphi}(x)) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-isp} f(p) b_p dp \right) \sigma_s^{\varphi}(x) ds \ , \qquad x \in M \ ,$$

where we recall that the measures dp and ds are the Plancherel measures on $\boldsymbol{R}$. Put

$$a_s = \int_{-\infty}^{\infty} e^{-isp} f(p) b_p dp \ , \qquad s \in \boldsymbol{R} \ .$$

It follows then that

$$\sigma_{\hat{f}}^{\varphi}(x) - \bar{\sigma}_c^{\varphi} \circ \sigma_{\hat{f}}^{\varphi}(x) = \int_{-\infty}^{\infty} (\hat{f}(s) - a_s) \sigma_s^{\varphi}(x) ds \ .$$

Let g be a function in $L^1(\boldsymbol{R})$ such that $\hat{g}(p) = 1$ for p in a neighborhood of 0. If $f(0) = 1$, then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\hat{f}(s) - a_s) g(t) \sigma_t^{\varphi}(x) ds dt = (f(0) - b_0) \sigma_g^{\varphi}(x) = 0 \ .$$

Hence we have

$$\sigma_g^\varphi \circ \sigma_{\hat f}^\varphi(x - \bar\sigma_c^\varphi(x)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\hat f(s) - a_s)g(t)\sigma_{s+t}^\varphi(x)dsdt$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\hat f(s) - a_s)(g(t-s) - g(t))\sigma_t^\varphi(x)dsdt\ ;$$
$$\|\sigma_g^\varphi \circ \sigma_{\hat f}^\varphi(x - \bar\sigma_c^\varphi(x))\| \leqq \|x\| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|\hat f(s) - a_s\|\,|g(t-s) - g(t)|\,dsdt\ .$$

Put $h(s) = \|\hat f(s) - a_s\|$. Then h belongs to $L^1(\boldsymbol{R})$. Hence there exists a sequence $\{g_n\}$ in $L^1(\boldsymbol{R})$ by [25; page 50] such that $\hat g_n(p) = 1$ for $|p| < 1/n$ and

$$\varepsilon_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)\,|g_n(t-s) - g_n(t)|\,dsdt \longrightarrow 0 \quad \text{as} \quad n \longrightarrow 0\ .$$

If $f(p) = 1$ for $|p| < 1/n$, then we have

$$\sigma_{g_n}^\varphi \circ \sigma_{\hat f}^\varphi(x - \bar\sigma_c^\varphi(x)) = x - \bar\sigma_c^\varphi(x)\ , \qquad x \in M(\sigma^\varphi, [e^{-1/n}, e^{1/n}])\ .$$

Thus the conclusion follows. q.e.d.

LEMMA 2.8. *Let $c \in Z^1(F^M)$ be as in Theorem 2.6. For any $\varepsilon > 0$, there exists $\eta > 0$ such that for every faithful integrable weights φ_1 and φ_2 with $d(\varphi_1, \varphi_2) \leqq \eta$ we have*

$$\|(D\varphi_2 : D\varphi_1)_c - 1\| \leqq \varepsilon\ .$$

PROOF. We keep the notations in Theorem 2.4. It follows from II.4 that $d(\varphi_1, \varphi_2) \leqq \eta$ means $1 \otimes e_{21} \in P(\sigma^\varphi, [e^{-\eta}, e^\eta])$. Hence, choosing $\eta > 0$ as in Lemma 2.7, we get

$$\|u_c - 1\| = \|\bar\sigma_c^\varphi(1 \otimes e_{21}) - 1\| \leqq \varepsilon\ .$$

q.e.d.

LEMMA 2.9. *Let $c \in Z^1(F^M)$ be as in Theorem 2.6. Let φ be a faithful weight of infinite multiplicity.*

(a) *$\{\varphi_n\}$ is a sequence of faithful integrable weights such that $\lim_{n\to\infty} d(\varphi, \varphi_n) = 0$, then the sequence $\{\bar\sigma_c^{\varphi_n}\}$ of automorphisms converges to an automorphism, say $\bar\sigma_c^\varphi$, of M.*

(b) *$\bar\sigma_c^\varphi$ does not depend on the choice of a sequence $\{\varphi_n\}$ and satisfies*

$$\varphi \circ \bar\sigma_c^\varphi = \varphi \quad \text{and} \quad \bar\sigma_c^\varphi|_{M_\varphi} = \iota\ .$$

PROOF. Since we have, by the definition of $(D\varphi : D\psi)$,

$$\|(D\varphi_m : D\varphi_1)_c - (D\varphi_n : D\varphi_1)_c\| = \|(D\varphi_m : D\varphi_n)_c - 1\|\ ,$$

it follows from Lemma 2.8 that $\{(D\varphi_n : D\varphi_1)_c\}$ is a Cauchy sequence of unitaries in M. Put

$$(D\varphi: D\varphi_1)_c = u_c = \lim_{n\to\infty} (D\varphi_n: D\varphi_1)_c ,$$

and

$$\bar{\sigma}_c^{\varphi} = \mathrm{Ad}\,(u_c) \circ \bar{\sigma}_c^{\varphi_1} .$$

It follows also from Lemma 2.8 that $(D\varphi: D\varphi_1)_c$ does not depend on the choice of a sequence $\{\varphi_n\}$ but only on φ and φ_1. By construction, we have

$$\lim_{n\to\infty} \|\bar{\sigma}_c^{\varphi} - \bar{\sigma}_c^{\varphi_n}\| = 0 .$$

Let $\{\varphi_n\}$ be a sequence of faithful integrable weights given by $\varphi_n = \varphi(h_n\cdot)$ with $h_n \in M_\varphi$ such that $h_n \leqq h_{n+1}$ and $\lim_{n\to\infty} \|h_n - 1\| = 0$. We have then, for any $x \in M_+$,

$$\begin{aligned} \varphi(x) &= \lim_{n\to\infty} \varphi(x^{1/2}h_n x^{1/2}) = \lim_{n\to\infty} \varphi_n(x) \\ &= \lim_{n\to\infty} \varphi_n \circ \bar{\sigma}_c^{\varphi_n}(x) = \lim_{n\to\infty} \varphi(h_n^{1/2}\bar{\sigma}_c^{\varphi_n}(x)h_n^{1/2}) \\ &\geqq \varphi \circ \bar{\sigma}_c^{\varphi}(x) \end{aligned}$$

by the lower semi-continuity of φ. Replacing c by c^{-1}, we have $\varphi(x) \geqq \varphi \circ \bar{\sigma}_c^{\varphi-1}(x)$. Therefore, we get $\varphi \circ \bar{\sigma}_c^{\varphi} = \varphi$. Let ψ be an integrable faithful weight with $d(\varphi, \psi) < \varepsilon$. Then we have $M_\varphi \subset M(\sigma^\psi, [e^{-2\varepsilon}, e^{2\varepsilon}])$. Therefore, Lemma 2.7 entails the last assertion of (b). q.e.d.

PROOF OF THEOREM 2.6. With possible exception for (vi), all statements for faithful integrable weights follow from Proposition 2.1, Theorem 2.4 and Lemma 2.8. Let $\varphi \in \mathfrak{W}_M^0$ be integrable and $\alpha \in \mathrm{Aut}\,(M)$. It follows then that

$$p_{\varphi\circ\alpha} = \mathrm{mod}\,(\alpha)^{-1} \circ p_\varphi \circ \alpha ;$$

hence for each $x \in M(\sigma^{\varphi\circ\alpha}, \{\lambda\})$ we have

$$\begin{aligned} \bar{\sigma}_c^{\varphi\circ\alpha}(x) &= p_{\varphi\circ\alpha}^{-1}(c_\lambda)x = [\alpha^{-1} \circ p_\varphi^{-1} \circ \mathrm{mod}\,(\alpha)(c_\lambda)]x \\ &= \alpha^{-1}(p_\varphi^{-1}(\mathrm{mod}\,(\alpha)(c_\lambda))\alpha(x)) \\ &= \alpha^{-1} \circ \bar{\sigma}_{\mathrm{mod}\,(\alpha)c}^{\varphi}(\alpha(x)) . \end{aligned}$$

Hence we get the first part of (vi) for integrable weights. The last two formulas for integrable weights follow from this and the usual 2×2-matrix arguments.

Let φ_0 and ψ_0 be arbitrarily fixed faithful integrable weights. For each faithful weight φ of infinite multiplicity, we put

$$(D\varphi: D\varphi_0)_c = \lim_{n\to\infty} (D\varphi_n: D\varphi_0)_c$$

with a sequence $\{\varphi_n\}$ of faithful integrable weights converging to φ in the metric d. We know that this does not depend on the choice of $\{\varphi_n\}$. We define

$$(D\varphi: D\psi)_c = (D\varphi: D\varphi_0)_c(D\psi: D\varphi_0)_c^*$$

for a pair φ, ψ of faithful weights of infinite multiplicity. With sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of integrable weights converging to φ and ψ, we have

$$\begin{aligned}
&\lim_{n\to\infty} (D\varphi_n: D\varphi_0)_c(D\psi_n: D\varphi_0)_c^* \\
&\quad = \lim_{n\to\infty} (D\varphi_n: D\varphi_0)_c(D\varphi_0: D\psi_0)_c(D\psi_0: D\varphi_0)_c(D\psi_n: D\varphi_0)_c^* \\
&\quad = \lim_{n\to\infty} (D\varphi_n: D\psi_0)_c(D\psi_n: D\psi_0)_c^* ;
\end{aligned}$$

hence the above definition of $(D\varphi: D\psi)_c$ makes sense. Given $\varepsilon > 0$, if $d(\varphi, \psi) < \eta$ with $\eta > 0$ in Lemma 2.8, then

$$\begin{aligned}
\|(D\varphi: D\varphi_0)_c - (D\psi: D\varphi_0)_c\| &= \lim_{n\to\infty} \|(D\varphi_n: D\varphi_0)_c - (D\psi_n: D\varphi_0)_c\| \\
&= \lim_{n\to\infty} \|(D\varphi_n: D\psi_n)_c - 1\| \leqq \varepsilon .
\end{aligned}$$

Therefore, if $d(\varphi, \varphi') < \eta$ and $D(\psi, \psi') < \eta$, then we have

$$\begin{aligned}
&\|(D\varphi: D\psi)_c - (D\varphi': D\psi')_c\| \\
&\quad = \|(D\varphi: D\varphi_0)_c(D\psi: D\varphi_0)_c^* - (D\varphi': D\varphi_0)_c(D\psi': D\varphi_0)_c\| \leqq 2\varepsilon .
\end{aligned}$$

Thus, by Lemma 2.9, Theorem 2.4 and continuity, all statements for faithful weights of infinite multiplicity hold.

Let φ be a faithful weight of infinite multiplicity and w be an isometry with $ww^* \in M_\varphi$. We define

$$(D\varphi_w: D\varphi)_c = w^*\bar{\sigma}_c^\varphi(w) .$$

If v is another isometry with $vv^* \in M_\varphi$ such that $\varphi_w = \varphi_v$, then we have

$$v^*\sigma_t^\varphi(v) = (D\varphi_v: D\varphi)_t = (D\varphi_w: D\varphi)_t = w^*\sigma_t^\varphi(w) , \qquad t \in \boldsymbol{R} ,$$

so that we have $vw^* \in M_\varphi$ and $\bar{\sigma}_c^\varphi(vw^*) = vw^*$ by Lemma 2.9. Therefore, $v^*\bar{\sigma}_c^\varphi(v) = w^*\bar{\sigma}_c^\varphi(w)$. Thus $(D\varphi_w: D\varphi)_c$ is well-defined.

If φ and ψ are faithful weights of infinite multiplicity and v and w are isometries of M with $vv^* \in M_\varphi$ and $ww^* \in M_\psi$, then we define

$$(D\varphi_v: D\psi_w)_c = (D\varphi_v: D\varphi)_c(D\varphi: D\psi)_c(D\psi_v: D\psi)_c^* .$$

It is then shown, by the similar arguments as above, that $(D\varphi_v: D\psi_w)_c$ is well-defined. Since any faithful weight is of the form φ_v for some φ of infinite multiplicity, $(D\varphi: D\psi)_c$ is defined for a general pair φ, ψ in $\mathfrak{W}_M^0$. We then define, fixing a faithful weight φ_0 of infinite multiplicity,

$$\bar{\sigma}_c^\varphi(x) = (D\varphi: D\varphi_0)_c\bar{\sigma}_c^{\varphi_0}(x)(D\varphi_0: D\varphi)_c , \qquad x \in M .$$

It follows from the chain rule that $\bar{\sigma}_c^\varphi$ does not depend on the choice of φ_0. A straightforward argument shows that conditions (iv), (v), (vi) and (v) hold.

Thus, the only thing remains to be shown is the continuity of $(D\varphi: D\psi)_c$ in general. We consider $P = M \otimes F_\infty$. It is easily seen that for any $\varphi, \psi \in \mathfrak{W}_M^0$ we have

$$(D(\varphi \otimes \mathrm{Tr}): D(\psi \otimes \mathrm{Tr}))_c = (D\varphi: D\psi)_c \otimes 1 ;$$
$$d(\varphi \otimes \mathrm{Tr}, \psi \otimes \mathrm{Tr}) = d(\varphi, \psi) .$$

Hence the continuity of $(D\varphi: D\psi)_c$ on φ, ψ follows from the continuity of two maps: $(\varphi, \psi) \in \mathfrak{W}_M^0 \times \mathfrak{W}_M^0 \longrightarrow (\varphi \otimes \mathrm{Tr}, \psi \otimes \mathrm{Tr}) \in \mathfrak{W}_P^0 \times \mathfrak{W}_P^0$ and $(\varphi \otimes \mathrm{Tr}, \psi \otimes \mathrm{Tr}) \longrightarrow (D(\varphi \otimes \mathrm{Tr}): D(\psi \otimes \mathrm{Tr}))_c$. The continuity of the map: $\varphi \longrightarrow \bar{\sigma}_c^\varphi$ is automatic after this. q.e.d.

EXAMPLE 2.10. Let N be an infinite semi-finite factor with separable predual. As in Example 2.3, let $c = df \in Z^1(F^M)$ and $\varphi = \tau(h_\varphi \cdot)$ a faithful weight on N. Then we have

$$(D\varphi: D\tau)_c = f(1)^* f(h_\varphi) .$$

We leave the proof to the reader.

COROLLARY 2.11. *Let M be an infinite factor with separable predual. Let $c \in Z^1(F^M)$ be as in Theorem 2.6. Let φ be a faithful weight on M and put*

$$c(h) = (D(\varphi(h \cdot)): D\varphi)_c$$

for each non-singular self-adjoint positive operator h affiliated with M_φ. We conclude the following:

(i) *$c(h)$ falls in the center of $\{h\}' \cap M_\varphi$;*

(ii) *$c_1c_2(h) = c_1(h)c_2(h)$ for every twice differentiable $c_1, c_2 \in Z^1(F^M)$.*

PROOF. (i) Let $P = M \otimes F_2$ and

$$\psi(x) = \varphi(x_{11}) + \varphi(hx_{22}), \quad x = \sum_{i,j=1}^{2} x_{ij} \otimes e_{i,j} \in P .$$

Let $u = 1 \otimes e_{21}$. We have then

$$c(h) \otimes e_{21} = \bar{\sigma}_c^\psi(u) .$$

Since $\sigma_t^\psi(u) = h^{it} \otimes e_{21}$, we have $\sigma_t^\psi(u)u^* \in P_\psi$, so that

$$\sigma_t^\psi(u^*)u = \bar{\sigma}_c^\psi(\sigma_t^\psi(u^*)u) = \sigma_t^\psi(\bar{\sigma}_c^\psi(u^*))\bar{\sigma}_c^\psi(u) ;$$

hence

$$c(h) \otimes e_{11} = u^*\bar{\sigma}_c^\psi(u) = \sigma_t^\psi(u^*\bar{\sigma}_c^\psi(u)) \in P_\psi ,$$

which means that $c(h) \in M_\varphi$.

If $x \in \{h\}' \cap M_\varphi \subset M_\varphi \cap M_{\varphi(h\cdot)}$, then we have

$$x = \bar{\sigma}_c^{\varphi(h\cdot)}(x) = c(h)\bar{\sigma}_c^{\varphi}(x)c(h)^* = c(h)xc(h)^* ,$$

so that $c(h) \in (\{h\}' \cap M_\varphi)' \cap M_\varphi =$ the center of $\{h\}' \cap M_\varphi$.

(ii) This follows from (i) and Theorem 2.6 (vii). q.e.d.

We now apply Theorem 2.6 to a factor given by the group measure space construction, and then compute the extended modular automorphism. Let M be an infinite factor with separable predual and φ a faithful weight. Suppose that there exists a von Neumann subalgebra N of M_φ with relative commutant $N' \cap M = C$ contained in N and a continuous unitary representation $u(\cdot)$ of a separable locally compact group G in M such that

$$u(g)Nu(g)^* = N , \qquad g \in G ;$$
$$M = \{N \cup u(G)\}'' .$$

By Theorem II.6.2, there exists a non-singular self-adjoint operator ρ_g affiliated with C such that

$$\sigma_t^\varphi(u(g)) = u(g)\rho_g^{it} , \qquad t \in \boldsymbol{R},\ g \in G .$$

It is also easy to see, using $N' \cap M = C \subset N$, that if $\alpha \in \operatorname{Aut}(M)$ leaves N elementwise fixed, then there exists a one-cocycle $\{a_g\} \in Z_\beta^1(G, \mathfrak{U}_C)$ such that

$$\alpha(u(g)) = a_g u(g) , \qquad g \in G ,$$

where the action β of G on N, hence on C, is given by

$$\beta_g(x) = u(g)xu(g)^* , \qquad x \in N,\ g \in G .$$

Let $\{\Gamma, \mu\}$ be a standard measure space with $C = L^\infty(\Gamma, \mu)$, on which G acts in such a way that

$$\beta_g(x)(\gamma) = x(g^{-1}\gamma) , \qquad x \in C,\ g \in G,\ \gamma \in \Gamma .$$

We consider the action of G on $\Gamma \times \boldsymbol{R}$ defined by:

$$T_g(\gamma, s) = (g\gamma, s - \log \rho_g(\gamma)) , \qquad \gamma \in \Gamma,\ s \in \boldsymbol{R},\ g \in G .$$

Let $k_g(\gamma) = -\log \rho_g(\gamma)$, $g \in G$, $\gamma \in \Gamma$. By Theorem II.6.2, the center $C_{\bar{\omega}}$ of the dominant weight $\bar{\omega} = \varphi \otimes \omega$ on $M \otimes F_\infty$ is identified with $L^\infty(\Gamma \times \boldsymbol{R}, \mu \otimes m)^G$, where m means, of course, the Plancherel measure on $\boldsymbol{R}$.

COROLLARY 2.12. *In the above situation, if $c \in Z^1(F^M)$ is as in Theorem 2.6, then the cocycle $a \in Z_\beta^1(G, \mathfrak{U}_C)$ corresponding to the extended* *modular automorphism* $\alpha = \bar{\sigma}_c^{\varphi}$ *is given by the formula:*

$$a_g(\gamma) = b_{k_g(g^{-1}\gamma)}(\gamma, 0)\ ,$$

where $b_s = p_{\bar{\omega}}^{-1}(c_{e^{-s}})$, $s \in \boldsymbol{R}$.

PROOF. For $n = 1, 2, \cdots$, put

$$\Phi_n(t) = \frac{1}{n}\tan^{-1} t\ , \qquad t \in \boldsymbol{R}$$

$$\Psi_n(s) = \tan ns\ , \qquad -\frac{\pi}{2n} < s < \frac{\pi}{2n}\ .$$

We define an isometry w_n of $L^2(\boldsymbol{R})$ onto $L^2(-\pi/2n, \pi/2n)$ by

$$(w_n\xi)(s) = \sqrt{\Psi_n'(s)}\xi\circ\Psi_n(s)\ , \quad -\frac{\pi}{2n} < s < \frac{\pi}{2n}\ , \quad \xi \in L^2(\boldsymbol{R})\ .$$

Clearly we have

$$(w_n^*\xi)(t) = \sqrt{\Phi_n'(t)}\xi\circ\Phi_n(t)\ , \quad t \in \boldsymbol{R},\ \xi \in L^2(-\pi/2n, \pi/2n)\ .$$

Let ω be the weight on $F_\infty = \mathfrak{L}(L^2(\boldsymbol{R}))$ such that

$$(D\omega: D\,\mathrm{Tr})_t = V_t\ ,$$

where $\{U_s\}$ and $\{V_t\}$ mean the one parameter unitary groups defined in Chapter II. We have then

$$\begin{aligned}\{(D\omega_{w_n}: D\,\mathrm{Tr})_t\xi\}(s) &= (w_n^* V_t w_n\xi)(s)\\ &= e^{it\Phi_n(s)}\xi(s)\ .\end{aligned}$$

Hence we get $d(\omega_{w_n}, \mathrm{Tr}) = \pi/2n$, so that ω_{w_n} converges to Tr uniformly. Therefore $\varphi\otimes\omega_{w_n}$ converges to $\varphi\otimes\mathrm{Tr}$ uniformly; thus we get

$$\begin{aligned}(D\varphi\otimes\mathrm{Tr}: D\bar{\omega})_c &= \lim_{n\to\infty}(D\varphi\otimes\omega_{w_n}: D\bar{\omega})_c\\ &= \lim_{n\to\infty}(1\otimes w_n)^*\bar{\sigma}_c^{\bar{\omega}}(1\otimes w_n)\ .\end{aligned}$$

Let $u_c = (D\varphi\otimes\mathrm{Tr}: D\bar{\omega})_c$ and $u_{n,c} = (D\varphi\otimes\omega_{w_n}: D\bar{\omega})_c$. It follows from the proof of Lemma 2.7 that

$$u_{n,c} \in \{b_t,\ (1\otimes w_n^*)\sigma_s^{\bar{\omega}}(1\otimes w_n): s, t \in \boldsymbol{R}\}'' \subset C\otimes L^\infty(\boldsymbol{R})\ ,$$

and that

$$u_{n,c}(\gamma, s) = b_{\Phi_n(s)-s}(\gamma, \Phi_n(s))\ , \qquad \gamma \in \Gamma,\ s \in \boldsymbol{R}\ .$$

Therefore, we get $u_c \in L^\infty(\Gamma\times\boldsymbol{R})$ and

$$u_c(\gamma, s) = b_{-s}(\gamma, 0)\ ,$$

where we use the fact that the differentiability of b in norm, together with the cocycle property, implies the continuity of $b_s(\gamma, t)$ in t.

We next have

$$\sigma_t^{\bar{\omega}}(u(g) \otimes 1)(u(g)^* \otimes 1) = \sigma_t^{\varphi}(u(g))u(g)^* \otimes 1 = \beta_g(\rho_g^{it}) \otimes 1 \in C \otimes C ,$$

so that $\sigma_c^{\bar{\omega}}(u(g) \otimes 1)(u(g)^* \otimes 1) = d_g$ belongs to $C \otimes L^\infty(\boldsymbol{R}) = L^\infty(\Gamma \times \boldsymbol{R})$ and we get

$$d_g(\gamma, s) = b_{k_g(g^{-1}\gamma)}(\gamma, s) .$$

Since we have

$$\begin{aligned} a_g \otimes 1 = \bar{\sigma}_c^{\varphi \otimes \mathrm{Tr}}(u(g) \otimes 1)(u(g)^* \otimes 1) &= u_c \sigma_c^{\bar{\omega}}(u(g) \otimes 1)(u(g)^* \otimes 1)(\beta_g \otimes \iota)(u_c^*) \\ &= u_c d_g(\beta_g \otimes \iota)(u_c^*) , \end{aligned}$$

we have

$$\begin{aligned} a_g(\gamma) = u_c(\gamma, s) d_g(\gamma, s)\overline{u_c(g^{-1}\gamma, s)} &= b_{-s}(\gamma, 0) b_{k_g(g^{-1}\gamma)}(\gamma, s)\overline{b_{-s}(g^{-1}\gamma, 0)} \\ &= b_{k_g(g^{-1}\gamma)}(\gamma, 0) . \end{aligned}$$

q.e.d.

IV.3. The exact sequence for the group of all automorphisms. Given a factor M of type III with separable predual, we have constructed various mathematical objects: the flow F^M of weights, the fundamental homomorphism γ_M of Out(M) into Aut(F^M), the extension $\bar{\delta}_M$ of the modular homomorphism and a continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$. Putting these things together, we compute Out(M) = Aut(M)/Int(M), and generalize the exact sequence in [3; Chapter IV].

THEOREM 3.1. *Let M be a factor of type* III *with separable predual. If $M = W^*(N, \boldsymbol{R}, \theta)$ is a continuous decomposition of M, then there exists a homomorphism $\bar{\gamma}$ of* Out(M) *onto* Out$_{\theta,\tau}(N)$ *which makes the following sequence exact*:

$$\{1\} \longrightarrow H^1(F^M) \xrightarrow{\bar{\delta}_M} \mathrm{Out}\,(M) \xrightarrow{\bar{\gamma}} \mathrm{Out}_{\theta,\tau}\,(N) \longrightarrow \{1\} ,$$

where

$$\mathrm{Out}_{\theta,\tau}\,(N) = \{\varepsilon_N(\alpha) \colon \alpha \in \mathrm{Aut}\,(N),\ \alpha\theta_s = \theta_s\alpha,\ s \in \boldsymbol{R},\ \tau \circ \alpha = \tau\} .$$

PROOF. Let $\bar{\omega}$ be the dominant weight of M dual to the trace τ on N with $\tau \circ \theta_s = e^{-s}\tau$. By Theorem 2.2, if $\alpha \in$ Aut(M) leaves N pointwise fixed, then $\alpha = \bar{\sigma}_c^{\bar{\omega}}$ for some $c \in Z^1(F^M)$. By Corollary 2.5. (ii), α is inner if and only if $c \in B^1(F^M)$. Hence the map $\bar{\delta}_M \colon c \in Z^1(F^M) \longrightarrow \varepsilon_M(\bar{\sigma}_c^{\bar{\omega}}) \in$ Out(M) gives rise to an isomorphism of $H^1(F^M)$ into Out(M) which will be denoted by $\bar{\delta}_M$ again.

Let α be an arbitrary automorphism of M. Then $\bar{\omega} \circ \alpha$ is again dominant. By the uniqueness of a dominant weight, there exists a unitary $u \in M$ such that $\bar{\omega} \circ \alpha \circ \mathrm{Ad}(u) = \bar{\omega}$. Hence, putting

$$\mathrm{Aut}_{\bar{\omega}}(M) = \{\alpha \in \mathrm{Aut}(M) : \bar{\omega} \circ \alpha = \bar{\omega}\} ,$$

we have $\mathrm{Out}(M) = \varepsilon_M(\mathrm{Aut}_{\bar{\omega}}(M))$. Let $\alpha \in \mathrm{Aut}_{\bar{\omega}}(M)$. If $\alpha = \bar{\sigma}_c^{\bar{\omega}}$ for some $c \in Z^1(F^M)$, then $\alpha|_N = \iota$ by construction. If $\alpha|_N = \mathrm{Ad}(u)$ for some $u \in \mathfrak{U}(N)$, then we have $\alpha \circ \mathrm{Ad}(u)^{-1}|_N = \iota$, so that $\alpha \circ \mathrm{Ad}(u)^{-1} = \bar{\sigma}_c^{\bar{\omega}}$ for some $c \in Z^1(F^M)$ by Theorem 2.2. Hence the kernel of the homomorphism $\gamma: \alpha \in \mathrm{Aut}_{\bar{\omega}}(M) \rightarrow \varepsilon_N(\alpha|_N) \in \mathrm{Out}(N)$ is precisely the image of $Z^1(F^M)$ under $\bar{\sigma}^{\bar{\omega}}$. Since we have

$$\mathrm{Aut}_{\bar{\omega}}(M) \cap \mathrm{Int}(M) = \{\mathrm{Ad}(u) : u \in \mathfrak{U}(N)\} ,$$

γ gives rise to a unique homomorphism $\bar{\gamma}$ of $\mathrm{Out}(M)$ into $\mathrm{Out}(N)$ such that $\bar{\gamma} \circ \varepsilon_M = \gamma$.

We examine the range of γ. Put

$$\mathrm{Aut}_{\theta,\tau}(N) = \{\alpha \in \mathrm{Aut}(N) : \alpha\theta_s = \theta_s\alpha,\ s \in \boldsymbol{R},\ \tau \circ \alpha = \tau\} .$$

Let $\{u(s)\}$ be the one parameter unitary group in M which appears in the crossed product decomposition $M = W^*(N, \boldsymbol{R}, \theta)$. Let $\alpha \in \mathrm{Aut}_{\bar{\omega}}(M)$ and $\beta = \alpha|_N$. Since α and $\{\sigma_t^{\bar{\omega}}\}$ commute, we have $\sigma_t^{\bar{\omega}}(\alpha(u(s))) = e^{ist}\alpha(u(s))$, so that $a_s = \alpha(u(s))u(s)^* \in \mathfrak{U}(N)$. It is straightforward to see that

$$a_{s+t} = a_s\theta_s(a_t) , \qquad s, t \in \boldsymbol{R} ;$$

hence $a \in Z_\theta^1(\boldsymbol{R}, \mathfrak{U}(N))$. By Theorem III.5.1, we have $a = b^*\theta_s(b)$ for some $b \in \mathfrak{U}(N)$. Thus we get $\alpha(u(s)) = b^*\theta_s(b)u(s) = b^*u(s)b$, so that $\alpha \circ \mathrm{Ad}(b)$ leaves $u(s)$ fixed for every $s \in \boldsymbol{R}$, which means that $\beta \circ \mathrm{Ad}(b) = \alpha \circ \mathrm{Ad}(b)|_N$ and $\{\theta_s\}$ commute. Since $\bar{\omega} \circ \alpha = \bar{\omega}$, $\alpha|_N$ leaves τ invariant by the equalities $\bar{\omega} = \tau \circ E_{\bar{\omega}}$ and $E_{\bar{\omega}} \circ \alpha = E_{\bar{\omega}}$, so that $\beta \circ \mathrm{Ad}(b)$ leaves τ invariant. Thus we conclude the inclusion:

$$\bar{\gamma}(\mathrm{Out}(M)) \subset \varepsilon_N(\mathrm{Aut}_{\theta,\tau}(N)) = \mathrm{Out}_{\theta,\tau}(N) .$$

Suppose $\beta \in \mathrm{Aut}_{\theta,\tau}(N)$. A standard argument shows that β is extended uniquely to an $\alpha \in \mathrm{Aut}(M)$ such that $\alpha(xu(s)) = \beta(x)u(s)$, $x \in N$, $s \in \boldsymbol{R}$. Trivially, we have $\alpha|_N = \beta$. Thus we have

$$\bar{\gamma}(\mathrm{Out}(M)) \supset \varepsilon_N(\mathrm{Aut}_{\theta,\tau}(N)) .$$

q.e.d.

THEOREM 3.2. *In the same situation as in Theorem* 3.1,

$$\mathrm{Out}_{\theta,\tau}(N) = \{\bar{\alpha} \in \mathrm{Out}(N) : \varepsilon_N(\theta_s)\bar{\alpha} = \bar{\alpha}\varepsilon_N(\theta_s),\ s \in \boldsymbol{R},\ \tau \circ \bar{\alpha} = \tau\} .$$

PROOF. Let C denote the center of N. The unitary group $\mathfrak{U}(N)$ of N is a polish group with respect to the σ-strong* topology and $\mathfrak{U}(C)$ is a closed subgroup of $\mathfrak{U}(N)$. We consider the pointwise convergence topology in $\mathrm{Aut}(N)$ with respect to the norm topology in N_*. The map $\mathrm{Ad}: u \in \mathfrak{U}(N) \to \mathrm{Ad}(u) \in \mathrm{Aut}(N)$ is a continuous homomorphism with kernel $\mathfrak{U}(C)$. Hence the naturally induced map $\overline{\mathrm{Ad}}: \bar{u} \in \mathfrak{U}(N)/\mathfrak{U}(C) \to \mathrm{Ad}(u) \in \mathrm{Aut}(N)$ is a continuous isomorphism from the polish group onto $\mathrm{Int}(N)$. Hence $\mathrm{Int}(N)$ is a Borel subset of $\mathrm{Aut}(N)$ and the inverse map $\overline{\mathrm{Ad}}^{-1}$ is a Borel map from $\mathrm{Int}(N)$ onto $\mathfrak{U}(N)/\mathfrak{U}(C)$. Let T be a Borel transversal of $\mathfrak{U}(N)/\mathfrak{U}(C)$ in $\mathfrak{U}(N)$, and let $\pi = T \circ \overline{\mathrm{Ad}}^{-1}$. Then π is a Borel map from $\mathrm{Int}(N)$ into $\mathfrak{U}(N)$ such that $\mathrm{Ad}(\pi(\alpha)) = \alpha$ for every $\alpha \in \mathrm{Int}(N)$.

Suppose $\alpha \in \mathrm{Aut}(M)$ commute with θ_s, $s \in \boldsymbol{R}$, modulo $\mathrm{Int}(M)$, that is, $\varepsilon_N(\alpha)\varepsilon(\theta_s) = \varepsilon(\theta_s)\varepsilon_N(\alpha)$. Put $\beta_s = \alpha \circ \theta_s \circ \alpha^{-1} \circ \theta_s^{-1} \in \mathrm{Int}(N)$ and $b_s = \pi(\beta_s) \in \mathfrak{U}(N)$, $s \in \boldsymbol{R}$. We have then

$$\mathrm{Ad}(b_s) \circ \theta_s = \alpha \circ \theta_s \circ \alpha^{-1}, \qquad s \in \boldsymbol{R}.$$

By the one parameter group property of $\{\alpha \circ \theta_s \circ \alpha^{-1}\}$, we have

$$\mathrm{Ad}(b_s\theta_s(b_t)) = \mathrm{Ad}(b_{s+t}), \qquad s, t \in \boldsymbol{R}.$$

Put

$$c(s, t) = b_s^* b_{s+t} \theta_s(b_t^*) \in \mathfrak{U}(C), \qquad s, t \in \boldsymbol{R}.$$

By a direct computation, we get

$$c(r, s)c(r + s, t) = \theta_r(c(s, t))c(r, s + t), \qquad r, s, t \in \boldsymbol{R}.$$

Hence c is a Borel unitary 2-cocycle of the flow $\{C, \theta\}$. By the triviality $H_\theta^2(\boldsymbol{R}, \mathfrak{U}(C)) = \{0\}$ of the second cohomology group of a flow, see Appendix, we can find a $\mathfrak{U}(c)$-valued Borel function $\{d_s\}$ such that

$$c(s, t) = d_s^* d_{s+t} \theta_s(d_t^*), \qquad \text{for almost } s, t \in \boldsymbol{R}.$$

Let $a_s = d_s b_s$, $s \in \boldsymbol{R}$. We then obtain a $\mathfrak{U}(N)$-valued Borel function $\{a_s\}$ such that for almost every s, t in $\boldsymbol{R}$,

$$\begin{cases} a_{s+t} = a_s\theta_s(a_t), & s, t \in \boldsymbol{R}; \\ \mathrm{Ad}(a_s) \circ \theta_s = \alpha \circ \theta_s \circ \alpha^{-1}. \end{cases}$$

By Remark III.1.9, there exists $a \in Z_\theta^1(\boldsymbol{R}, \mathfrak{U}(N))$ such that $a_s' = a_s$ for almost every $s \in \boldsymbol{R}$.

By the triviality of $H_\theta^1(\boldsymbol{R}, \mathfrak{U}(N))$, Theorem III.5.1, we have an element $u \in \mathfrak{U}(N)$ such that $a_s = u^*\theta_s(u)$, $s \in \boldsymbol{R}$. Thus we get $\mathrm{Ad}(u^*) \circ \theta_s \circ \mathrm{Ad}(u) = \alpha \circ \theta_s \circ \alpha^{-1}$ for almost every $s \in \boldsymbol{R}$. Namely, $\mathrm{Ad}(u) \circ \alpha$ and $\{\theta_s\}$ commute in $\mathrm{Aut}(M)$ by continuity. q.e.d.

REMARK 3.3. The exact sequence in Theorem 3.1 does not split in general.

APPENDIX

PROPOSITION A.1. *Let G and H be separable locally compact groups and $\{\Gamma, \mu\}$ a standard measure space on which G acts ergodically. Let E be a Borel subset of Γ with $\mu(E) > 0$. Put $A = \{(g, \gamma) \in G \times E: g\gamma \in E\}$. If b is an H-valued Borel function on A such that for every $g_1, g_2 \in G$ with $\mu(E \cap g_2^{-1}E \cap g_2^{-1}g_1^{-1}E) > 0$*

$$b(g_1g_2, \gamma) = b(g_1, g_2\gamma)b(g_2, \gamma)$$

for almost every $\gamma \in E \cap g_2^{-1}E \cap g_2^{-1}g_1^{-1}E$, then there exists an H-valued Borel function c on $G \times \Gamma$ such that

$$c(g, \gamma) = b(g, \gamma) , \qquad (g, \gamma) \in A ;$$

for every $g_1, g_2 \in G$

$$c(g_1g_2, \gamma) = c(g_1, g_2\gamma)c(g_2, \gamma)$$

for almost every $\gamma \in \Gamma$.

PROOF. Let G_0 be a dense countable subgroup of G. Let $\Gamma_0 = \bigcup_{g \in G_0} gE$. By ergodicity, we have $\mu(\Gamma - \Gamma_0) = 0$. Hence we may assume $\Gamma = \Gamma_0$. Then, there exists a family $\{E_g: g \in G_0\}$ of Borel subsets of E such that

$$\Gamma = \bigcup_{g \in G_0} gE_g , \quad gE_g \cap hE_h = \emptyset , \quad g \neq h .$$

Define a G-valued Borel function $a(\cdot)$ on Γ by

$$a(\gamma) = g \quad \text{if} \quad \gamma \in gE_g ,$$

and put $\omega(\gamma) = a(\gamma)^{-1}\gamma \in E$, and $\rho(g, \gamma) = a(g\gamma)^{-1}ga(\gamma)$. We have then

$$\gamma = a(\gamma)\omega(\gamma) , \quad \omega(g\gamma) = \rho(g, \gamma)\omega(\gamma) ;$$
$$\rho(g_1g_2, \gamma) = \rho(g_1, g_2\gamma)\rho(g_2, \gamma) .$$

Furthermore, for each fixed $g \in G$, $\rho(g, \cdot)$ takes only countably many values: indeed $\rho(g, \gamma) \in G_0gG_0$ for every $\gamma \in \Gamma$. Define

$$c(g, \gamma) = b(\rho(g, \gamma), \omega(\gamma)) , \qquad g \in G, \gamma \in \Gamma .$$

Since we can choose $E_1 = E$ where 1 means the unit of G, we have $c(g, \gamma) = b(g, \gamma)$ for $(g, \gamma) \in A$. Furthermore, we have

$$\begin{aligned} c(g_1g_2, \gamma) &= b(\rho(g_1g_2, \gamma), \omega(\gamma)) = b(\rho(g_1, g_2\gamma)\rho(g_2, \gamma), \omega(\gamma)) \\ &= b(\rho(g_1, g_2\gamma), \rho(g_2, \gamma)\omega(\gamma))b(\rho(g_2, \gamma), \omega(\gamma)) \\ &= b(\rho(g_1, g_2\gamma), \omega(g_2\gamma))b(\rho(g_2, \gamma), \omega(\gamma)) \\ &= c(g_1, g_2\gamma)c(g_2, \gamma) \end{aligned}$$

for almost every $\gamma \in \Gamma$, where we use, in order to exclude a null set of

γ, the fact that $\rho(g_1, g_2\gamma)$ and $\rho(g_2, \gamma)$, $\gamma \in \Gamma$, are at most countable.

q.e.d.

The authors learned that the following result had been proven by L. Brown sometime earlier. We present, however, a proof for the sake of convenience of the reader, since Brown's work is not yet available in print.

PROPOSITION A.2. *Let A be an abelian von Neumann algebra with separable predual, and $\{\alpha_t : t \in \boldsymbol{R}\}$ be an ergodic one parameter automorphism group of A. Then for every $n \geqq 2$, we have $H^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A) = \{1\}$.*

PROOF. By virtue of the representation theorem for flows, due to Ambrose, Kakutani, Krengel and Kubo [12], [16], we may assume that the flow $\{A, \alpha\}$ is built under a ceiling function from a single ergodic automorphism. Let $\{\Gamma, \mu)$ be a standard measure space equipped with an ergodic transformation T. Let f be a positive Borel function on Γ. Consider the abelian von Neumann algebra $B = L^\infty(\Gamma \times \boldsymbol{R}, \mu \otimes m)$, where m means the Lebesgue measure on $\boldsymbol{R}$. We define a one parameter automorphism group $\{\beta_t\}$ and an automorphism θ of B as follows:

$$\begin{aligned} &\beta_t(x)(\gamma, s) = x(\gamma, s - t)\ , \quad x \in B\ , \quad (\gamma, s) \in \Gamma \times \boldsymbol{R}\ , \quad t \in \boldsymbol{R}\ , \\ &\theta(x)(\gamma, s) = x(T^{-1}\gamma, s + f(\gamma))\ . \end{aligned}$$

The representation theorem says that $\{A, \alpha\} \cong \{B^\theta, \beta\}$ for a suitable choice of Γ, μ, T, and f.

An n-cochain $c \in C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A)$ is by definition a unitary of $L^\infty(\boldsymbol{R}^n) \otimes A$ considered as a $\mathfrak{U}_A$-valued function on $\boldsymbol{R}^n$. In particular, $C^0_\alpha(\boldsymbol{R}, \mathfrak{U}_A) = \mathfrak{U}_A$. For each $n \geqq 0$, and $c \in C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A)$, the coboundary dc is given by the formula:

$$\begin{aligned} dc(s_1, \cdots, s_{n+1}) &= \alpha_{s_1}(c(s_2, \cdots, s_{n+1}))c(s_1 + s_2, s_3, \cdots, s_{n+1})^{-1} \cdots c(s_1, \cdots, s_n)^{(-1)^{n+1}} \\ &= \alpha_{s_1}(c(s_2, \cdots, s_{n+1})) \prod_{j=1}^{n} c(s_1, \cdots, s_j + s_{j+1}, \cdots, s_{n+1})^{(-1)^j} \\ &\quad \times c(s_1, s_2, \cdots, s_n)^{(-1)^{n+1}}\ . \end{aligned}$$

Thus we obtain a cochain complex:

$$(1) \qquad \mathfrak{U}_A = C^0_\alpha(\boldsymbol{R}, \mathfrak{U}_A) \xrightarrow{d} C^1_\alpha(\boldsymbol{R}, \mathfrak{U}_A) \cdots \xrightarrow{d} C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A) \xrightarrow{d} \cdots\ .$$

We have then by definition $H^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A) = \{\text{the kernel of } d \text{ in } C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A)\}/\{\text{the range of } d\}$.

Let $\mathfrak{U}^n$ be the unitary group of $L^\infty(\boldsymbol{R}^{n+1}) \otimes B = L^\infty(\boldsymbol{R}^{n+1} \times \Gamma)$. For each $c \in \mathfrak{U}^n$, we define the coboundary dc by the formula:

$$dc(t_0, t_1, \cdots, t_{n+1}, \gamma) = \prod_{j=0}^{n+1} c(t_0, t_1, \cdots, \hat{t}_j, \cdots, t_{n+1}\gamma)^{(-1)^j},$$

where $\hat{t}_j$ indicates that the term t_j is missing. We then have a long exact sequence:

$$\mathfrak{U}^0 \xrightarrow{d} \mathfrak{U}^1 \xrightarrow{d} \mathfrak{U}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{U}^n \xrightarrow{d} \cdots . \tag{2}$$

For each $n \geqq 0$, we define an automorphism of $L^\infty(\boldsymbol{R}^{n+1}) \otimes B$, denoted by θ again for the obvious reason, by the following:

$$\theta(x)(t_0, t_1, \cdots, t_n, \gamma) = x(t_0 + f(\gamma), t_1 + f(\gamma), \cdots, t_n + f(\gamma), T^{-1}\gamma).$$

Let π be a map of $L^\infty(\boldsymbol{R}^n) \otimes A$ into $L^\infty(\boldsymbol{R}^{n+1}) \otimes B$ defined by the following:

$$\pi(x)(t_0, t_1, \cdots, t_n, \gamma) = x(t_1 - t_0, t_2 - t_1, \cdots, t_n - t_{n-1}, \gamma, t_0),$$

where we identify A with B^θ. It follows then that π is an isomorphism of $L^\infty(\boldsymbol{R}^n) \otimes A$ onto $(L^\infty(\boldsymbol{R}^{n+1}) \otimes B)^\theta$ which makes the following diagram commute:

$$\begin{array}{ccccccccc}
C_\alpha^0(\boldsymbol{R}, \mathfrak{U}_A) & \xrightarrow{d} & C_\alpha^1(\boldsymbol{R}, \mathfrak{U}_A) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C_\alpha^n(\boldsymbol{R}, \mathfrak{U}_A) & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \pi} & & \\
\mathfrak{U}^0 & \xrightarrow{d} & \mathfrak{U}^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathfrak{U}^n & \xrightarrow{d} & \cdots .
\end{array}$$

Moreover, we have $\pi(C_\alpha^n(\boldsymbol{R}, \mathfrak{U}_A)) = (\mathfrak{U}^n)^\theta =$ the fixed point subgroup of $\mathfrak{U}^n$ under θ. Therefore, cochain complex (1) is isomorphic to the following cochain complex:

$$(\mathfrak{U}^0)^\theta \xrightarrow{d} (\mathfrak{U}^1)^\theta \xrightarrow{d} \cdots \xrightarrow{d} (\mathfrak{U}^n)^\theta \xrightarrow{d} \cdots . \tag{3}$$

Now, let $C = L^\infty(\Gamma, \mu)$ and $\theta(x)(\gamma) = x(T^{-1}\gamma)$ for each $x \in C$. Putting $\varepsilon(x) = 1 \otimes x \in L^\infty(\boldsymbol{R}) \otimes C$ for each $x \in C$, we obtain an injective resolution of the $\boldsymbol{Z}$-module $\mathfrak{U}_C$:

$$\{1\} \longrightarrow \mathfrak{U}_C \xrightarrow{\varepsilon} \mathfrak{U}^0 \xrightarrow{d} \mathfrak{U}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{U}^n \xrightarrow{d} \cdots \tag{4}$$

where $\boldsymbol{Z}$ acts on each group, of course, via θ and the injectivity follows from the divisibility of the unitary group of a von Neumann algebra. Hence the cohomology groups of cochain complex (3), hence (1), are isomorphic to the cohomology groups $H_\theta^n(\boldsymbol{Z}, \mathfrak{U}_C)$, $n \geqq 1$, (cf. [10; page 105]). This means then that

$$H_\alpha^n(\boldsymbol{R}, \mathfrak{U}_A) \cong H_\theta^n(\boldsymbol{Z}, \mathfrak{U}_C), \qquad u \geqq 1.$$

It is known, however, that

$$H_\theta^n(\boldsymbol{Z}, \mathfrak{U}_C) = \{1\}, \qquad n \geqq 2.$$

q.e.d.

The above result, or more precisely the proof, is known in homological algebra as Shapiro's lemma.

References

[1] C. Akemann, The dual space of an operator algebra, Trans. Amer. Math. Soc., 126 (1967), pp. 286-302.

[2] N. Bourbaki, Topologic Générale, Chapter 9, 2nd. Ed., Paris (1958).

[3] A. Connes, Une classification des facteurs de type III, Ann. Sci. École Norm. Sup., 4 eme Ser., 6 (1973), pp. 133-252.

[4] A. Connes, États presque périodiques sur une algèbra de von Neumann, C. R. Acad. Sci., Paris, Sér. A 274 (1972), pp. 1402-1405.

[5] A. Connes, Caracterisation des algebres de von Neumann comme espaces vectoriels ordonnes, to appear.

[6] A. Connes, Sur le thèoreme de Radon-Nikodym pour les normaux fideles semi-finis, to appear.

[7] A. Connes and M. Takesaki, Flots des poids sur les facteurs de type III, C. R. Acad. Sci., Paris, Sér. A 278 (1974), pp. 945-948.

[8] T. Digerness, Poids dual sur un produit croisé, C. R. Sci., Paris, Sér. A 278 (1974), pp. 937-940.

[9] T. Digerness, Duality for weights on covariant systems and its applications, Thesis, UCLA (1975).

[10] R. Godement, Theorie des faisceaux, Herman, Paris, (1964).

[11] R. Kadison and J. Ringrose, Derivations and automorphisms of operator algebras, Comm. Math. Phys., 4 (1967), pp. 32-63.

[12] U. Krengel, Darstellungssätze für Strömungen und Halb-strömungen II, Math. Ann. 182 (1969), pp. 1-39.

[13] W. Krieger, On ergodic flows and the isomorphism of factors, to appear.

[14] I. Kubo, Quasi-flows, Nagoya Math. J., 35 (1669), 1-30.

[15] M. Landstad, Duality theory of covariant systems, to appear.

[16] G. Mackey, Ergodic theory and virtual groups, Math. Ann., 166 (1966), pp. 187-207.

[17] G. Mackey, Borel structures in groups and their duals, Trans. Amer. Math. Soc., 85 (1957), pp. 265-311.

[18] C. C. Moore, Group extensions and group cohomology, Group representations in Mathematics and physics, Lecture Notes in Physics, Springer, 6 (1970), pp. 17-35.

[19] F. J. Murray and V. von Neumann, On rings of operators, Ann. of Math., 37 (1936), pp. 116-229.

[20] F. J. Murray and J. von Neumann, On rings of operators IV, Ann. of Math., 44 (1943), pp. 716-808.

[21] M. Nakamura and Z. Takeda, On some elementary properties of the crossed products of von Neumann algebras, Proc. Japan Acad., 34 (1958), pp. 489-494.

[22] M. Nakamura and Z. Takada, A Galois theory for finite factors, Proc. Japan Acad., 36 (1960), 258-260.

[23] M. Nakamura and Z. Takada, On the fundamental theorem of the Galois theory for finite factors, Proc. Japan Acad., 36 (1960), pp. 313-318.

[24] G. K. Pederson and M. Takesaki, The Radon-Nikodym theorem for von Neumann algebras, Acta Math., 130 (1973), pp. 53-87.

[25] W. Rudin, Fourier analysis on groups, Intersciences, (1960).

[26] J. L. Sauvageot, Sur le type du produit croisé d'une algèbre de von Neumann par un

groupe localement compact d'automorphisms, C. R. Acad. Sci. Paris, Sér. A, 278, (1974), pp. 941-944.

[27] N. Suzuki, Cross products of rings of operators, Tôhoku Math. J., 11 (1959), pp. 113-124.

[28] M. Takesaki, A generalized commutation relation for the regular representation, Bull. Soc. Math., France, 97 (1969), pp. 289-297.

[29] M. Takesaki, Tomita's theory of modular Hilbert algebras and its applications, Lecture Notes in Math., 128, Springer, (1970).

[30] M. Takesaki, Duality for crossed products and the structure of von Neumann algebras of type III, Acta Math., 131 (1974), pp. 249-310.

[31] M. Takesaki and N. Tatsuuma, Duality and subgroups, Ann. of Math., 93 (1971), pp. 344-364.

[32] G. Zeller-Meier, Produits croisés d'une C^*-algebra par un groupe d'automorphismes, J. Math. Pures Appl., 47 (1968), pp. 101-239.

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