Tôhoku Math. Journ.
30 (1978), 307–335.

Γ-FOLIATIONS AND SEMISIMPLE FLAT HOMOGENEOUS SPACES

SEIKI NISHIKAWA AND MASARU TAKEUCHI

(Received January 17, 1977)

Introduction. In this paper we shall study characteristic classes of Γ-foliations. Our object is to prove a strong vanishing theorem for Pontrjagin classes of the normal bundles of certain Γ-foliations.

Let Γ be a pseudogroup acting on a smooth manifold B of dimension q. A Γ-foliation of codimension q on a smooth manifold M is by definition a maximal family $\mathscr{F}$ of submersions

$$f_\alpha : U_\alpha \to B$$

of open sets U_α in M such that the family $\{U_\alpha\}_\alpha$ is an open covering of M and for each $x \in U_\alpha \cap U_\beta$ there exists an element $\gamma^x_{\beta\alpha} \in \Gamma$ with $f_\beta = \gamma^x_{\beta\alpha} \circ f_\alpha$ in some neighborhood of x. The kernels of the differentials $(f_\alpha)_*$ of submersions f_α then constitute a subbundle $\tau(\mathscr{F})$ of the tangent bundle TM of M. The quotient bundle $\nu(\mathscr{F}) = TM/\tau(\mathscr{F})$ is called the normal bundle of $\mathscr{F}$. Let $\mathrm{Pont}^*(\nu(\mathscr{F}))$ denote the subalgebra of $H^*(M; \boldsymbol{R})$ generated by the real Pontrjagin classes of $\nu(\mathscr{F})$. Then the Bott vanishing theorem [3, 4] states that

$$\mathrm{Pont}^k(\nu(\mathscr{F})) = 0 \quad \text{for} \quad k > 2q ,$$

$\mathrm{Pont}^k(\nu(\mathscr{F}))$ denoting the k-dimensional homogeneous part of $\mathrm{Pont}^*(\nu(\mathscr{F}))$. This gives a sharp bound for general Γ-foliations (Thurston [20]).

On the other hand, Pasternack [13] proved a strong vanishing theorem

$$(*) \qquad \mathrm{Pont}^k(\nu(\mathscr{F})) = 0 \quad \text{for} \quad k > q ,$$

for riemannian foliations $\mathscr{F}$, Γ-foliations with Γ consisting of local isometries of a riemannian structure on B. In the previous paper [11] we improved his result by proving a strong vanishing theorem for conformal or projective foliations.

The purpose of this paper is to extend these results. We thereby obtain the following generalization of the strong vanishing theorem.

MAIN THEOREM. *Let L/L_0 be a semisimple flat homogeneous space of dimension q associated with a semisimple graded Lie algebra* $\mathfrak{l} = \mathfrak{g}_{-1} +$

Research supported by the Sonderforschungsbereich "Theoretische Mathematik" (SFB 40) at the University of Bonn.

$\mathfrak{g}_0+\mathfrak{g}_1$ *and* $\mathfrak{k}_0$ *be a maximal compact subalgebra of* $\mathfrak{g}_0$. *Let* Γ *be the pseudogroup of local automorphisms of an* L_0*-structure of second order associated with* L/L_0. *Then for a* Γ*-foliation* $\mathscr{F}$ *of codimension* q, *the strong vanishing theorem* $(*)$ *holds if*

(1) *the Spencer cohomology* $H^{2,1}(\mathfrak{l})=0$ *and*

(2) *the Pontrjagin algebra* $\mathrm{Pont}(\mathfrak{k}_0)\subset I_L(\mathfrak{k}_0)$.

For the terminology and the notation in the Main Theorem, see §§1, 2 and 4.

The method of the proof of the Main Theorem depends heavily on the differential geometry associated with semisimple flat homogeneous spaces, which has been developed extensively by Tanaka [19] and Ochiai [12]. Essentially the idea of the proof is the same as that in [11]. In fact, we define the "prolongation" of the normal frame bundle of $\mathscr{F}$ to construct a "Γ-invariant basic" connection. This is done in §3. The normal Cartan connection plays an important role in the construction.

Examples satisfying the conditions (1) and (2) in the Main Theorem are given in §5. It is known that the condition (1) holds for a fairly general family of Γ-foliations under consideration. We also give a criterion for the condition (2) to be satisfied for every semisimple flat homogeneous space associated with a given semisimple graded Lie algebra $\mathfrak{l}$ in terms of the topology of the symmetric R-space associated with $\mathfrak{l}$ (Theorem 5.3). Structure theorems for automorphisms of real semisimple Lie algebras are of essential use in the argument.

1. Semisimple flat homogeneous spaces. This section is devoted to a brief survey of the basic material on semisimple flat homogeneous spaces. For details, see Kobayashi-Nagano [9] and Ochiai [12].

By a (*transitive*) *graded Lie algebra* we mean a real Lie algebra $\mathfrak{l}=\sum\mathfrak{g}_p$ with a decomposition into a direct sum of subspaces $\mathfrak{g}_p$ ($p\in\boldsymbol{Z}$) satisfying

$$\begin{aligned} &\mathfrak{g}_p=0 && \text{for}\quad p\leqq -2\,,\\ &[\mathfrak{g}_p,\mathfrak{g}_q]\subset\mathfrak{g}_{p+q} && \text{for all}\quad p,q\quad\text{and}\\ &[x,\mathfrak{g}_{-1}]\neq 0 && \text{for each nonzero}\quad x\in\mathfrak{g}_p,\ p\geqq 0\,. \end{aligned}$$

A graded Lie algebra $\mathfrak{l}=\sum\mathfrak{g}_p$ is called *semisimple* if $\mathfrak{l}$ is finite dimensional and semisimple. In the following we are mainly interested in semisimple graded Lie algebras. Let β be the Killing form of $\mathfrak{l}$. It is an immediate consequence of the nondegeneracy of β that $\mathfrak{g}_p=0$ for $p\geqq 2$, that is, $\mathfrak{l}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$, and $\mathfrak{g}_{-1}$ is the dual vector space of $\mathfrak{g}_1$ under the pairing $\mathfrak{g}_{-1}\times\mathfrak{g}_1\ni(x,y)\mapsto\beta(x,y)$. Furthermore there exists a unique element e in $\mathfrak{g}_0$ such that for $k=-1$, 0 and 1

$$\mathfrak{g}_k = \{x \in \mathfrak{l};\ [e, x] = kx\}\ .$$

Let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{l}$ such that $\beta(e, \mathfrak{k}) = 0$. Define $\mathfrak{p}$ by

$$\mathfrak{p} = \{x \in \mathfrak{l};\ \beta(x, \mathfrak{k}) = 0\}\ .$$

Then $\mathfrak{l} = \mathfrak{k} + \mathfrak{p}$, which is a Cartan decomposition of $\mathfrak{l}$. With respect to this Cartan decomposition we define an automorphism τ of $\mathfrak{l}$ by setting $\tau|_{\mathfrak{k}} = 1_{\mathfrak{k}}$ and $\tau|_{\mathfrak{p}} = -1_{\mathfrak{p}}$. A positive definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{l}$ is then given by

$$\langle x, y\rangle = -\beta(x, \tau y) \quad \text{for} \quad x, y \in \mathfrak{l}\ .$$

Semisimple graded Lie algebras have been classified in [9].

The Lie algebra cohomology $H(\mathfrak{l}) = H(\mathfrak{g}_{-1}, \mathrm{ad}_{\mathfrak{l}}|\mathfrak{g}_{-1}, \mathfrak{l})$ of the abelian Lie algebra $\mathfrak{g}_{-1}$ with respect to its adjoint representation on $\mathfrak{l}$ is called the *Spencer cohomology* of a graded Lie algebra $\mathfrak{l} = \sum \mathfrak{g}_p$. More precisely, let

$$C^{p,q} = \mathfrak{g}_{p-1} \otimes \Lambda^q(\mathfrak{g}_{-1})^*$$

be the vector space of all $\mathfrak{g}_{p-1}$-valued q-linear alternating maps on $\mathfrak{g}_{-1}$. Define a coboundary operator $\partial\colon C^{p,q} \to C^{p-1,q+1}$ by

$$(\partial c)(x_1, \cdots, x_{q+1}) = \sum (-1)^{i+1}[x_i, c(x_1, \cdots, \hat{x}_i, \cdots, x_{q+1})]$$

for $c \in C^{p,q}$ and $x_1, \cdots, x_{q+1} \in \mathfrak{g}_{-1}$. Then $\partial^2 = 0$ and the Spencer cohomology $H(\mathfrak{l}) = \sum H^{p,q}(\mathfrak{l})$ is defined by

$$H^{p,q}(\mathfrak{l}) = \partial^{-1}(0) \cap C^{p,q}/\partial C^{p+1,q-1}\ .$$

Let L/L_0 be a connected homogeneous space on which a (not necessarily connected) semisimple Lie group L acts effectively and transitively. L/L_0 is called a *semisimple flat homogeneous space* if the Lie algebra $\mathfrak{l}$ of L has a graded Lie algebra structure $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ such that $\mathfrak{g}_0 + \mathfrak{g}_1$ is the Lie algebra of L_0. We define G_0 as the normalizer of $\mathfrak{g}_0$ in L_0, that is,

$$G_0 = N_{L_0}(\mathfrak{g}_0) = \{x \in L_0;\ \mathrm{Ad}(x)\mathfrak{g}_0 = \mathfrak{g}_0\}\ .$$

Then it is known that the Lie algebra of G_0 coincides with $\mathfrak{g}_0$ and L_0 is a semidirect product $G_0 \cdot G_1$ of G_0 and the vector group $G_1 = \exp \mathfrak{g}_1$. Note that G_0 is also given by

$$G_0 = \{x \in L_0;\ \mathrm{Ad}(x)e = e\}\ .$$

Let $T_o(L/L_0)$ be the tangent space of L/L_0 at the origin $o = L_0$, which is linearly isomorphic to $\mathfrak{g}_{-1}$. We identify $\mathfrak{g}_{-1}$ and hence $T_o(L/L_0)$ with a euclidean vector space $\boldsymbol{R}^q$, $q = \dim \mathfrak{g}_{-1}$, in a natural manner by choosing an orthonormal basis of $\mathfrak{g}_{-1}$ with respect to the inner product $\langle\ ,\ \rangle$ restricted to $\mathfrak{g}_{-1}$. Since L_0 is the isotropy subgroup of L at the origin o, there is a natural representation λ of L_0, called the linear isotropy representation of L_0, on the tangent space $T_o(L/L_0)$. λ is a homomorphism from L_0 into $GL(\mathfrak{g}_{-1}) = GL(q, \boldsymbol{R})$. It follows from the effectiveness of the action of L that the kernel of λ coincides with G_1 so that the restriction $\lambda|G_0$ identifies G_0 with the linear isotropy subgroup in $GL(q, \boldsymbol{R})$. Corresponding to this identification, $\mathfrak{g}_0$ is regarded as a subalgebra of the Lie algebra $\mathfrak{gl}(q, \boldsymbol{R})$ of $GL(q, \boldsymbol{R})$. Let K_0 be the normalizer of $\mathfrak{k}$ in G_0, that is,

$$K_0 = N_{G_0}(\mathfrak{k}) = \{x \in G_0;\ \mathrm{Ad}(x)\mathfrak{k} = \mathfrak{k}\}\ ,$$

$\mathfrak{k}$ being a maximal compact subalgebra of $\mathfrak{l}$ with $\beta(e, \mathfrak{k}) = 0$, and let $\mathfrak{k}_0$ be the Lie algebra of K_0. Then $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$, and we have a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$ by setting $\mathfrak{p}_0 = \mathfrak{g}_0 \cap \mathfrak{p}$. Note that K_0 is regarded as a subgroup of the orthogonal group $O(q)$, for the inner product $\langle\ ,\ \rangle$ is invariant under the adjoint action of the normalizer of $\mathfrak{k}$ in L. It follows from the following lemma that the structure group of any principal G_0-bundle is reducible to the subgroup K_0.

LEMMA 1.1. *K_0 is a maximal compact subgroup of G_0. The map of $K \times \mathfrak{p}_0$ into G_0 defined by*

$$(k, x) \mapsto k \exp x$$

is a diffeomorphism.

PROOF. Let $\mathrm{Aut}\,(\mathfrak{l})$ be the group of automorphisms of $\mathfrak{l}$. Define a closed subgroup $\mathrm{Aut}\,(\mathfrak{l}, e)$ of $\mathrm{Aut}\,(\mathfrak{l})$ by

$$\mathrm{Aut}\,(\mathfrak{l}, e) = \{\alpha \in \mathrm{Aut}\,(\mathfrak{l});\ \alpha e = e\}\ .$$

Note that the Lie algebra of $\mathrm{Aut}\,(\mathfrak{l}, e)$ may be identified with $\mathfrak{g}_0$ provided we identify the Lie algebra of $\mathrm{Aut}\,(\mathfrak{l})$ with $\mathfrak{l}$.

We first prove that the homomorphism $\mathrm{Aut}\,(\mathfrak{l}, e) \to GL(\mathfrak{g}_{-1})$ defined by $\alpha \mapsto \alpha|\mathfrak{g}_{-1}$ is injective. In fact, suppose $\alpha|\mathfrak{g}_{-1} = 1_{\mathfrak{g}_{-1}}$. Then $\alpha|\mathfrak{g}_1 = 1_{\mathfrak{g}_1}$, for $\alpha|\mathfrak{g}_1$ is the contragredient of $\alpha|\mathfrak{g}_{-1}$. Since $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$, $\alpha|\mathfrak{g}_0 = 1_{\mathfrak{g}_0}$ and hence $\alpha = 1_{\mathfrak{l}}$.

It follows from this fact that $\mathrm{Ad}\colon G_0 \to \mathrm{Aut}\,(\mathfrak{l}, e)$ is an injective homomorphism. We identify G_0 with the subgroup $\mathrm{Ad}\,(G_0)$ of $\mathrm{Aut}\,(\mathfrak{l}, e)$.

$\mathrm{Aut}\,(\mathfrak{l}, e)$ is an algebraic subgroup of $GL(\mathfrak{l})$ and is invariant under taking transpose with respect to the inner product $\langle\ ,\ \rangle$. Hence $\mathrm{Aut}\,(\mathfrak{l}, e)$ has a polar decomposition, that is, there is a diffeomorphism

$$\mathrm{Aut}\,(\mathfrak{l}, \mathfrak{k}, e) \times \mathfrak{p}_0 \to \mathrm{Aut}\,(\mathfrak{l}, e)$$

defined by $(k, x) \mapsto k \exp x$, where $\mathrm{Aut}\,(\mathfrak{l}, \mathfrak{k}, e)$ is a maximal compact subgroup of $\mathrm{Aut}\,(\mathfrak{l}, e)$ defined by

$$\mathrm{Aut}\,(\mathfrak{l}, \mathfrak{k}, e) = \{\alpha \in \mathrm{Aut}\,(\mathfrak{l}, e); \alpha\mathfrak{k} = \mathfrak{k}\}\,,$$

whose Lie algebra coincides with $\mathfrak{k}_0$ (cf. Chevalley [6]).

Since $\mathrm{Aut}^0(\mathfrak{l}, \mathfrak{k}, e) \subset K_0 \subset \mathrm{Aut}\,(\mathfrak{l}, \mathfrak{k}, e)$, where $\mathrm{Aut}^0(\mathfrak{l}, \mathfrak{k}, e)$ denotes the identity component of $\mathrm{Aut}\,(\mathfrak{l}, \mathfrak{k}, e)$, K_0 is compact.

For $g \in G_0$, let $g = k \exp x$ ($k \in \mathrm{Aut}\,(\mathfrak{l}, \mathfrak{k}, e)$, $x \in \mathfrak{p}_0$) be the polar decomposition of g. Then it is easy to see that $k = g(\exp x)^{-1}$ is in the normalizer $N_{G_0}(\mathfrak{k})$ of $\mathfrak{k}$ in G_0, that is, in K_0. Hence we obtain the polar decomposition $G_0 = K_0 \exp \mathfrak{p}$, which shows that K_0 is a maximal compact subgroup of G_0. q.e.d.

REMARKS. 1) It is well-known that the maximal compact subalgebras of $\mathfrak{g}_0$ are conjugate with each other under the adjoint action of G_0. Hence each maximal compact subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}_0$ is obtained as

$$\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$$

from a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{l}$ satisfying $\beta(e, \mathfrak{k}) = 0$.

2) It follows from Lemma 1.1 that the maximal compact subgroups of G_0 are conjugate with each other under the inner automorphisms of G_0. Hence each maximal compact subgroup K_0 of G_0 is obtained as

$$K_0 = N_{G_0}(\mathfrak{k})$$

from a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{l}$ satisfying $\beta(e, \mathfrak{k}) = 0$.

3) Let $\mathfrak{k}$ and $\mathfrak{k}'$ be two maximal compact subalgebras of $\mathfrak{l}$ such that $\beta(e, \mathfrak{k}) = \beta(e, \mathfrak{k}') = 0$. Corresponding to $\mathfrak{k}$ and $\mathfrak{k}'$ we get positive definite inner products $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ on $\mathfrak{l}$ respectively in the same way as above. It is then not difficult to show that there exists an element $g_0 \in G_0$ such that

$$\langle x, y\rangle' = \langle \mathrm{Ad}(g_0)x, \mathrm{Ad}(g_0)y\rangle \quad \text{for} \quad x, y \in \mathfrak{l}\,.$$

2. L_0-Structures of 2nd order associated with L/L_0. Let L/L_0 be a semisimple flat homogeneous space as in §1 and G_0 be the linear isotropy subgroup at the origin so that $G_0 \subset GL(q, \boldsymbol{R})$, $q = \dim L/L_0$.

Let B be a smooth manifold of dimension q. Fix a point o of B as the origin. Let $\Gamma(B)$ be the pseudogroup of local diffeomorphisms of B. For each integer $r \geqq 1$, let $P^r(B)$ denote the set of all r-jets $j^r_o(f)$ at o of the local diffeomorphisms $f \in \Gamma(B)$ defined around o. Let $G^r(q)$ be the set

$$\{j^r_o(f) \in P^r(B); f(o) = o\}\,.$$

Then $P^r(B)$ is, in a natural manner, a principal $G^r(q)$-bundle on B with the natural projection π_r defined by $\pi_r(j_o^r(f)) = f(o)$. For details, see Ochiai [12]. Consequently, we have a projective system

$$\cdots \xrightarrow{\pi_4^3} P^3(B) \xrightarrow{\pi_3^2} P^2(B) \xrightarrow{\pi_2^1} P^1(B)$$

by defining the forgetful map $\pi_r^s\colon P^r(B) \to P^s(B)$ $(r > s \geqq 1)$ by $\pi_r^s(j_o^r(f)) = j_o^s(f)$. Each $f \in \Gamma(B)$ is naturally prolonged to a local bundle isomorphism $f^{(r)}$ of $P^r(B)$ by

$$f^{(r)}(j_o^r(g)) = j_o^r(f \circ g) \quad \text{for} \quad j_o^r(g) \in P^r(B) .$$

From now on we are mainly interested in $P^1(B)$ and $P^2(B)$. Note that $G^1(q)$ is isomorphic to $GL(q, \boldsymbol{R})$ and is imbedded canonically into $G^r(q)$, in particular into $G^2(q)$. With respect to this imbedding, π_2^1 is $G^1(q)$-equivariant. By means of the into diffeomorphism Exp defined by $\boldsymbol{R}^q = \mathfrak{g}_{-1} \ni x \mapsto (\exp x)L_0 \in L/L_0$, we have a natural map $\iota\colon L_0 \to G^2(q)$ defined by

$$\iota(a) = j_o^2(\mathrm{Exp}^{-1} \circ a \circ \mathrm{Exp}) .$$

Then it is known (Ochiai [12]) that ι is an injective homomorphism, and we have the following commutative diagrams:

$$\begin{tikzcd} & G^2(q) \arrow[dd, "\pi_2^1"] \\ L_0 \arrow[ur, "\iota"] \arrow[dr, "\lambda"'] & \\ & G^1(q) \end{tikzcd} \qquad \begin{tikzcd} L_0 \arrow[r, "\iota"] & G^2(q) \\ G_0 \arrow[u, hook] \arrow[r, "\lambda"] & G^1(q) \cong GL(q, \boldsymbol{R}) \arrow[u, hook] \end{tikzcd}$$

By this homomorphism ι, we regard L_0 as a subgroup of $G^2(q)$.

Let Q be an L_0-reduction of the principal $G^2(q)$-bundle $P^2(B)$ on B, that is, Q is a principal L_0-subbundle of $P^2(B)$. Q is called an *L_0-structure of 2nd order on B associated with a semisimple flat homogeneous space L/L_0.* For each L_0-structure Q of 2nd order, let Γ denote the pseudogroup of local automorphisms of Q, that is,

$$\Gamma = \{\gamma \in \Gamma(B);\ \gamma^{(2)}Q \subset Q\} .$$

Since L_0 is the semidirect product of G_0 and the vector group $G_1 = \exp \mathfrak{g}_1$, the principal L_0-bundle Q has a G_0-reduction P' on B. Then $P = \pi_2^1(P')$ is a principal G_0-subbundle of $P^1(B)$. It is also given by $P = \pi_2^1(Q)$ by virtue of the above diagrams. P is called the *G_0-structure of 1st order associated with Q.* Note that each element $\gamma \in \Gamma$ is a local automorphism of P as well, that is, $\gamma^{(1)}P \subset P$.

3. Γ-foliations associated with L/L_0. Let L/L_0 be a semisimple flat homogeneous space and Γ be the pseudogroup of local automorphisms of an associated L_0-structure Q of 2nd order on a smooth manifold B of dimension $q = \dim L/L_0$ as in §2.

Let $\mathscr{F}$ be a *Γ-foliation of codimension* q on a smooth manifold M. $\mathscr{F}$ is by definition a maximal family $\mathscr{F} = \{f_\alpha\}_\alpha$ of submersions

$$f_\alpha: U_\alpha \to B$$

of open subsets U_α in M to B such that the family $\{U_\alpha\}_\alpha$ is an open covering of M and for each $x \in U_\alpha \cap U_\beta$ there exists an element $\gamma^x_{\beta\alpha} \in \Gamma$ with

$$f_\beta = \gamma^x_{\beta\alpha} \circ f_\alpha$$

in some neighborhood of x. The fibres of each submersion f_α are then pieced together to define the leaves of the foliation $\mathscr{F}$. The kernels of the differentials $(f_\alpha)_*$ of submersions $f_\alpha \in \mathscr{F}$ constitute a subbundle $\tau(\mathscr{F})$ of the tangent bundle TM of M. $\tau(\mathscr{F})$ is a bundle tangent to the leaves of $\mathscr{F}$. The quotient bundle $\nu(\mathscr{F}) = TM/\tau(\mathscr{F})$ is called the *normal bundle* of $\mathscr{F}$.

With each Γ-foliation $\mathscr{F}$ we associate a $\Gamma(B)$-foliation $\hat{\mathscr{F}}$ which contains $\mathscr{F}$. $\hat{\mathscr{F}}$ is defined in the same way as in the definition of $\mathscr{F}$ by replacing Γ with the pseudogroup $\Gamma(B)$ of local diffeomorphisms of B. Note that $\hat{\mathscr{F}}$ has the same structure of leaves as that of $\mathscr{F}$. We are now in a position to define the "prolongation" of the normal frame bundle of $\mathscr{F}$. Let $P^r(\hat{\mathscr{F}})$ be the set of all r-jets $j^r_x(f_U)$ at x of submersions f_U defined on open subsets U in M such that f_U is constant on the leaves of $\mathscr{F}$ and sends x to the origin o of B, that is,

$$P^r(\hat{\mathscr{F}}) = \{j^r_x(f_U);\, f_U \in \hat{\mathscr{F}},\, x \in U,\, f_U(x) = o\}\,.$$

Then $P^r(\hat{\mathscr{F}})$ is a principal $G^r(q)$-bundle on M with the natural projection π_r defined by $\pi_r(j^r_x(f_U)) = x$, where the group $G^r(q)$ acts on $P^r(\hat{\mathscr{F}})$ from the right by

$$j^r_x(f_U)\cdot j^r_o(h) = j^r_x(h^{-1}\circ f_U)$$

for $j^r_x(f_U) \in P^r(\hat{\mathscr{F}})$ and $j^r_o(h) \in G^r(q)$. In fact, the restriction of $P^r(\hat{\mathscr{F}})$ to U is isomorphic to the pull back by $f_U \in \hat{\mathscr{F}}$ of the bundle $P^r(B)$ on B:

$$f^!_U(P^r(B)) \cong P^r(\hat{\mathscr{F}})\,|\,U\,.$$

The isomorphism is given by

$$(x,\, j^r_o(g)) \mapsto j^r_x(g^{-1}\circ f_U)\,.$$

In particular, $P^1(\hat{\mathscr{F}})$ is the principal $GL(q, \boldsymbol{R})$-bundle associated with the normal bundle $\nu(\mathscr{F})$ of $\mathscr{F}$.

From now on we are mainly interested in $P^1(\hat{\mathscr{F}})$ and $P^2(\hat{\mathscr{F}})$. Note that the prolonged bundle $P^2(\hat{\mathscr{F}})$ of 2nd order has an L_0-reduction $\tilde{Q}$ on M, which is isomorphic locally to the pull back $f_\alpha^! Q$ by $f_\alpha \in \mathscr{F}$ of the L_0-reduction Q of $P^2(B)$ on B. In fact, since each $\gamma \in \Gamma$, or more precisely the prolongation $\gamma^{(2)}$, preserves the L_0-reduction Q of $P^2(B)$, the family of pull back bundles

$$\{f_\alpha^! Q; f_\alpha \in \mathscr{F}\}$$

is glued together to define a principal L_0-bundle on M, which naturally induces an L_0-reduction $\tilde{Q}$ of $P^2(\hat{\mathscr{F}})$.

Let G_0 be the linear isotropy subgroup of L/L_0 as in §1. In the same way as above we get a G_0-reduction $\tilde{P}$ of $P^1(\hat{\mathscr{F}})$, whose restriction $\tilde{P}|U_\alpha$ to U_α is isomorphic to the pull back $f_\alpha^! P$ by $f_\alpha \in \mathscr{F}$ of the G_0-reduction P of $P^1(B)$ in §2. $\tilde{P}$ is a principal G_0-bundle associated with the normal bundle $\nu(\mathscr{F})$ of $\mathscr{F}$.

Let Q^L and $\tilde{Q}^L$ be the group extensions of Q and $\tilde{Q}$ by L respectively, that is, $Q^L = Q \times_{L_0} L$ and $\tilde{Q}^L = \tilde{Q} \times_{L_0} L$. Each element $\gamma \in \Gamma$ naturally induces a local bundle isomorphism $\tilde{\gamma}^{(2)}$ of Q^L, and a local bundle map $\tilde{f}_\alpha^{(2)}$ of $\tilde{Q}^L$ to Q^L is naturally induced by each element $f_\alpha \in \mathscr{F}$.

With these understood, we can state the following lemma which is of essential use.

LEMMA 3.1 (Tanaka-Ochiai). *If the Spencer cohomology $H^{2,1}(\mathfrak{l})$ of the graded Lie algebra $\mathfrak{l}$ of L vanishes, then Q^L has an L-principal connection ω, called the normal Cartan connection of type L/L_0, which is invariant under Γ in the sense that for each $\gamma \in \Gamma$,*

$$\tilde{\gamma}^{(2)*}\omega = \omega .$$

For the proof, see [12, Theorem 11.1]. It should be noted here that ω restricted to the subbundle Q defines an absolute parallelism on Q.

By each submersion $f_\alpha \in \mathscr{F}$, or more precisely by the naturally induced bundle map $\tilde{f}_\alpha^{(2)}: \tilde{Q}^L \to Q^L$, we can pull back ω, the normal Cartan connection of type L/L_0 in Lemma 3.1, to get a family of local forms $\{\tilde{f}_\alpha^{(2)*}\omega\}_\alpha$.

LEMMA 3.2. *The local forms $\{\tilde{f}_\alpha^{(2)*}\omega\}_\alpha$ define a global connection form $\tilde{\omega}$ on $\tilde{Q}^L$.*

PROOF. The local forms $\tilde{f}_\alpha^{(2)*}\omega$ and $\tilde{f}_\beta^{(2)*}\omega$ are identical on $\tilde{Q}^L|U_\alpha \cap U_\beta$. In fact, let $x \in U_\alpha \cap U_\beta$ and W be a neighborhood of x on which $f_\beta = \gamma_{\beta\alpha}^x \circ f_\alpha$ with $\gamma_{\beta\alpha}^x \in \Gamma$. Then

$$\tilde{f}_{\beta}^{(2)*}\omega = \tilde{f}_{\alpha}^{(2)*}\circ\tilde{\gamma}_{\beta\alpha}^{(2)*}\omega = \tilde{f}_{\alpha}^{(2)*}\omega \quad \text{on} \quad \tilde{Q}^{L}|W ,$$

since ω is Γ-invariant. q.e.d.

4. A strong vanishing theorem. Let L/L_0 be a semisimple flat homogeneous space with linear isotropy subgroup $G_0 \subset GL(q, \boldsymbol{R})$ as in §1. Let K_0 be a maximal compact subgroup of G_0 and $\mathfrak{k}_0$ be its Lie algebra (cf. §1).

For later use, some notations are prepared. For a pair of a given Lie group G with Lie algebra $\mathfrak{g}$ and a subgroup H of G with Lie algebra $\mathfrak{h}$, a subalgebra of $\mathfrak{g}$, let $I_G(\mathfrak{h})$ denote the set of the restrictions $\phi|\mathfrak{h}$ to $\mathfrak{h}$ of $\mathrm{Ad}(G)$-invariant polynomials ϕ on $\mathfrak{g}$, that is,

$$I_G(\mathfrak{h}) = \{\phi|\mathfrak{h}\,;\, \phi \text{ is an } \mathrm{Ad}\,(G)\text{-invariant polynomial on } \mathfrak{g}\} .$$

$I_G(\mathfrak{h})$ is a graded commutative algebra in a natural manner and consists of $\mathrm{Ad}(H)$-invariant polynomials on $\mathfrak{h}$ which can be extended to $\mathrm{Ad}\,(G)$-invariant polynomials on $\mathfrak{g}$. For a Lie subgroup G of $GL(q, \boldsymbol{R})$ and its Lie algebra $\mathfrak{g}$, define $\phi_k \in I_G(\mathfrak{g})$ by

$$\phi_k(X) = \text{trace}\, X^{2k} \quad \text{for} \quad X \in \mathfrak{g} \subset \mathfrak{gl}(q, \boldsymbol{R}) .$$

Let $\mathrm{Pont}\,(\mathfrak{g})$ denote the subalgebra of $I_G(\mathfrak{g})$ generated by ϕ_k, $1 \leqq k \leqq [q/2]$. $\mathrm{Pont}(\mathfrak{g})$ is called the *Pontrjagin algebra* of $\mathfrak{g}$. The significance of $\mathrm{Pont}\,(\mathfrak{g})$ is as follows. In general, let ξ be a smooth real vector bundle on M of rank q such that the frame bundle of ξ has a G-reduction P, and $w(P)$ denote the Weil homomorphism of P :

$$w(P)\colon I_G(\mathfrak{g}) \to H^*(M; \boldsymbol{R}) .$$

Then the subalgebra $\mathrm{Pont}^*(\xi)$ of $H^*(M; \boldsymbol{R})$ generated by the real Pontrjagin classes of ξ is given by

$$\mathrm{Pont}^*(\xi) = w(P)(\mathrm{Pont}(\mathfrak{g})) .$$

In the following, we are mainly interested in $I_L(\mathfrak{k}_0)$ and $\mathrm{Pont}(\mathfrak{k}_0)$.

Let Q be an L_0-structure of 2nd order on a smooth manifold B of dimension q, which is associated with L/L_0 as in §2. Let Γ be the pseudogroup of local automorphisms of Q. Consider a Γ-foliation $\mathscr{F}$ of codimension q on a smooth manifold M. Let $\nu(\mathscr{F})$ denote the normal bundle of $\mathscr{F}$ and $\mathrm{Pont}^k(\nu(\mathscr{F}))$ be the k-dimensional homogeneous part of the subalgebra $\mathrm{Pont}^*(\nu(\mathscr{F}))$ of $H^*(M; \boldsymbol{R})$.

With these understood, we can state our strong vanishing theorem.

THEOREM 4.1. *Let L/L_0 be a semisimple flat homogeneous space of dimension q associated with a semisimple graded Lie algebra $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ and $\mathfrak{k}_0$ be a maximal compact subalgebra of $\mathfrak{g}_0$. Let Γ be the* *pseudogroup of local automorphisms of an L_0-structure Q of 2nd order associated with L/L_0 on a smooth manifold B of dimension q. Suppose that*

(1) $H^{2,1}(\mathfrak{l}) = 0$ *and*

(2) $\mathrm{Pont}(\mathfrak{k}_0) \subset I_L(\mathfrak{k}_0)$.

Then for a Γ-foliation $\mathscr{F}$ of codimension q on a smooth manifold M, we have

$$\mathrm{Pont}^k(\nu(\mathscr{F})) = 0 \quad \textit{for} \quad k > q\,.$$

PROOF. Let $Q^L \to B$ be the group extension of Q by L. It then follows from the assumption (1) and Lemma 3.1 that Q^L has the normal Cartan connection ω of type L/L_0. Denote by Ω the curvature form of ω, that is,

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]\,.$$

As in §3, the prolonged bundle $P^2(\hat{\mathscr{F}})$ of 2nd order has an L_0-reduction $\tilde{Q}$ on M which is naturally induced from Q. Let $\tilde{Q}^L \to M$ be the group extension of $\tilde{Q}$ by L. Then by Lemma 3.2, $\tilde{Q}^L$ has an L-principal connection $\tilde{\omega}$ such that for each $f_\alpha \in \mathscr{F}$

$$\tilde{\omega} = \tilde{f}_\alpha^{(2)*}\omega \quad \text{on} \quad \tilde{Q}^L | U_\alpha\,.$$

Denote by $\tilde{\Omega}$ the curvature form of $\tilde{\omega}$. Then by the naturality of the exterior derivative

$$\tilde{\Omega} = \tilde{f}_\alpha^{(2)*}\Omega \quad \text{on} \quad \tilde{Q}^L | U_\alpha\,,$$

from which we get

$$\phi(\tilde{\Omega}) = \tilde{f}_\alpha^{(2)*}\phi(\Omega) \quad \text{on} \quad \tilde{Q}^L | U_\alpha \tag{4.1}$$

for each $\phi \in I_L(\mathfrak{l})$.

Let $\tilde{P}$ be the G_0-reduction on M of the prolonged bundle $P^1(\hat{\mathscr{F}})$ of 1st order associated with Q as in §3. $\tilde{P}$ is a principal G_0-bundle associated with the normal bundle $\nu(\mathscr{F})$ of $\mathscr{F}$. As is seen in §1, $\tilde{P}$ has a K_0-reduction $\tilde{P}_{K_0}$ on M. Corresponding to this reduction, we have, by the naturality of the Weil homomorphism, the following commutative diagram:

$$\begin{tikzcd} I_L(\mathfrak{l}) \arrow[rr, "r"] \arrow[dr, "w(\tilde{Q}^L)"'] & & I_{K_0}(\mathfrak{k}_0) \arrow[dl, "w(\tilde{P}_{K_0})"] \\ & H^*(M; \mathbf{R}) & \end{tikzcd} \tag{4.2}$$

where r is the restriction homomorphism, and $w(\tilde{Q}^L)$ and $w(\tilde{P}_{K_0})$ denote the Weil homomorphisms. By virtue of the diagram (4.2), the assumption (2) implies the strong vanishing

$$\mathrm{Pont}^k(\nu(\mathscr{F})) = 0 \quad \text{for} \quad k > q .$$

In fact, let $\psi_k \in \mathrm{Pont}^k(\nu(\mathscr{F}))$. Then, there exists an element $\Psi \in \mathrm{Pont}(\mathfrak{k}_0)$ such that

$$w(\tilde{P}_{K_0})(\Psi) = \psi_k .$$

By the assumption (2) we have an element $\Psi' \in I_L(\mathfrak{l})$ such that $r(\Psi') = \Psi$. Note that $w(\tilde{Q}^L)(\Psi') = \psi_k$ by virtue of (4.2). Consider the k-forms $\Psi'(\tilde{\Omega})$ and $\Psi'(\Omega)$. These are L-invariant and horizontal forms on $\tilde{Q}^L$ and Q^L respectively. Hence $\Psi'(\tilde{\Omega})$ is pushed down to a k-form $\bar{\Psi}'(\tilde{\Omega})$ on M and $\Psi'(\Omega)$ to a k-form $\bar{\Psi}'(\Omega)$ on B. Then it follows from (4.1) that

$$\bar{\Psi}'(\tilde{\Omega}) = f_\alpha^* \bar{\Psi}'(\Omega) \quad \text{on} \quad U_\alpha . \tag{4.3}$$

Since $\dim B = q$, $\bar{\Psi}'(\Omega)$ vanishes if $k > q$, and hence by (4.3) so does $\bar{\Psi}'(\tilde{\Omega})$. Then we have only to recall that by the definition of the Weil homomorphism

$$w(\tilde{Q}^L)(\Psi') = [\bar{\Psi}'(\tilde{\Omega})] ,$$

where $[\cdot]$ denotes cohomology class in $H^*(M; \boldsymbol{R})$. This completes the proof. q.e.d.

5. The conditions (1) and (2). In this section we study the conditions (1) and (2) in Theorem 4.1 in detail. First, we recall the relevant facts about the structure of automorphisms of real semisimple Lie algebras. For details, see Matsumoto [10], Satake [14] and Takeuchi [17].

For a given Lie algebra $\mathfrak{g}$, let $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Inn}(\mathfrak{g})$ denote the group of automorphisms of $\mathfrak{g}$ and the group of inner automorphisms of $\mathfrak{g}$ respectively. If $A, B, \cdots$ are subsets of $\mathfrak{g}$, then we put

$$\mathrm{Aut}(\mathfrak{g}, A, B, \cdots) = \{\alpha \in \mathrm{Aut}(\mathfrak{g}); \alpha A = A, \alpha B = B, \cdots\} ,$$
$$\mathrm{Inn}(\mathfrak{g}, A, B, \cdots) = \{\alpha \in \mathrm{Inn}(\mathfrak{g}); \alpha A = A, \alpha B = B, \cdots\} .$$

Let $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a semisimple graded Lie algebra and put $\mathfrak{l}_0 = \mathfrak{g}_0 + \mathfrak{g}_1$, a subalgebra of $\mathfrak{l}$. Let L/L_0 be a semisimple flat homogeneous space associated with $\mathfrak{l}$. As in §1, let $\mathfrak{l} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{l}$ and e be a unique distinguished element of $\mathfrak{g}_0$. Set $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$, $G_0 = N_{L_0}(\mathfrak{g}_0)$ and $K_0 = N_{G_0}(\mathfrak{k})$. Take a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ such that $\mathfrak{a}$ contains e. $\mathfrak{a}$ can be extended to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$. Let $\mathfrak{l}^C$ and $\mathfrak{h}^C$ denote the complexifications of $\mathfrak{l}$ and $\mathfrak{h}$ respectively. Let $\sigma: \mathfrak{l}^C \to \mathfrak{l}^C$ be the complex conjugation of $\mathfrak{l}^C$ with respect to $\mathfrak{l}$. By setting $\mathfrak{b} = \mathfrak{h} \cap \mathfrak{k}$, we have a direct sum decomposition $\mathfrak{h} = \mathfrak{b} + \mathfrak{a}$. Then $\mathfrak{h}_{\boldsymbol{R}} = \sqrt{-1}\mathfrak{b} + \mathfrak{a} \subset \mathfrak{h}^C$, where $\sqrt{-1}$ is the imaginary unit, is the real part of

$\mathfrak{h}^C$. σ defines an involutive linear automorphism of $\mathfrak{h}_R$. Let Σ denote the root system of $\mathfrak{l}^C$ with respect to $\mathfrak{h}^C$. We regard Σ as a subset of $\mathfrak{h}_R$, that is $\Sigma \subset \mathfrak{h}_R$, by means of the duality defined by the Killing form β of $\mathfrak{l}^C$. Choose a σ-order $>$ of $\mathfrak{h}_R$ in the sense of Satake [14] such that $\beta(\alpha, e) \geqq 0$ for each positive root α. Let Π be the fundamental system of Σ corresponding to $>$, the σ-fundamental system. We denote by Π_0 the set of roots $\alpha \in \Pi$ satisfying $\beta(\alpha, e) = 0$, and put

$$\mathrm{Aut}_\sigma(\Pi) = \{s \in GL(\mathfrak{h}_R);\ s\Sigma = \Sigma,\ s\Pi = \Pi,\ s\sigma = \sigma s\}\ ,$$
$$\mathrm{Aut}_\sigma(\Pi, \Pi_0) = \{s \in \mathrm{Aut}_\sigma(\Pi);\ s\Pi_0 = \Pi_0\}\ .$$

Considering that $\mathrm{Aut}\,(\mathfrak{l}^C, \mathfrak{l}) = \mathrm{Aut}(\mathfrak{l})$, we have

$$\mathrm{Aut}\,(\mathfrak{l}) = \mathrm{Aut}\,(\mathfrak{l}, \mathfrak{k}, \mathfrak{h}, \Pi)\,\mathrm{Inn}\,(\mathfrak{l}^C, \mathfrak{l})\ .$$

The restriction homomorphism $\mathrm{Aut}\,(\mathfrak{l}, \mathfrak{k}, \mathfrak{h}, \Pi) \to \mathrm{Aut}_\sigma(\Pi)$ then defines a homomorphism γ: $\mathrm{Aut}\,(\mathfrak{l}) \to \mathrm{Aut}_\sigma(\Pi)$, and we have an exact sequence

$$1 \to \mathrm{Inn}\,(\mathfrak{l}^C, \mathfrak{l}) \to \mathrm{Aut}\,(\mathfrak{l}) \xrightarrow{\gamma} \mathrm{Aut}_\sigma(\Pi) \to 1\ .$$

This is, in fact, a split exact sequence, that is, there exists a homomorphism δ: $\mathrm{Aut}_\sigma(\Pi) \to \mathrm{Aut}(\mathfrak{l}, \mathfrak{k}, \mathfrak{h}, \Pi)$ such that $\gamma \circ \delta = \mathrm{id}$. Hence we have a semidirect decomposition

$$\mathrm{Aut}\,(\mathfrak{l}) = \mathrm{Inn}\,(\mathfrak{l}^C, \mathfrak{l}) \cdot \delta(\mathrm{Aut}_\sigma\,(\Pi)) \tag{5.1}$$

from which we obtain a semidirect decomposition

$$\mathrm{Aut}\,(\mathfrak{l}, \mathfrak{l}_0) = \mathrm{Inn}\,(\mathfrak{l}^C, \mathfrak{l}, \mathfrak{l}_0) \cdot \delta(\mathrm{Aut}_\sigma(\Pi, \Pi_0)) \tag{5.2}$$

also. In fact, it is easy to see that

$$\mathrm{Inn}\,(\mathfrak{l}^C, \mathfrak{l}, \mathfrak{l}_0) \cdot \delta(\mathrm{Aut}_\sigma(\Pi, \Pi_0)) \subset \mathrm{Aut}(\mathfrak{l}, \mathfrak{l}_0)\ .$$

Conversely, let $\alpha \in \mathrm{Aut}(\mathfrak{l}, \mathfrak{l}_0)$. Then, according to (5.1), α decomposes into

$$\alpha = g\delta(s)\ , \quad g \in \mathrm{Inn}\,(\mathfrak{l}^C, \mathfrak{l})\ , \quad s \in \mathrm{Aut}_\sigma(\Pi)\ .$$

Since $\alpha\mathfrak{l}_0 = \mathfrak{l}_0$, $g^{-1}\mathfrak{l}_0 = \delta(s)\mathfrak{l}_0$. Then it is known (Matsumoto [10]) that $\mathfrak{l}_0 = \delta(s)\mathfrak{l}_0$ and $\Pi_0 = s\Pi_0$. Therefore $g \in \mathrm{Inn}\,(\mathfrak{l}^C, \mathfrak{l}, \mathfrak{l}_0)$ and $s \in \mathrm{Aut}_\sigma(\Pi, \Pi_0)$. Hence

$$\mathrm{Aut}\,(\mathfrak{l}, \mathfrak{l}_0) \subset \mathrm{Inn}\,(\mathfrak{l}^C, \mathfrak{l}, \mathfrak{l}_0) \cdot \delta(\mathrm{Aut}_\sigma(\Pi, \Pi_0))\ .$$

This completes the proof of (5.2).

LEMMA 5.1. *Let L/L_0 be a semisimple flat homogeneous space associated with a semisimple graded Lie algebra $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. Let* Ad: $L \to \mathrm{Aut}\,(\mathfrak{l})$ *be the adjoint representation of L on $\mathfrak{l}$. Then*

$$\gamma(\mathrm{Ad}\, L) \subset \mathrm{Aut}_\sigma(\Pi, \Pi_0)\ .$$

PROOF. Let $L^* = \mathrm{Ad}\, L$ and $L_0^* = L^* \cap \mathrm{Aut}\,(\mathfrak{l}, \mathfrak{l}_0)$. We consider that

$L^*/L_0^* \subset \mathrm{Aut}(\mathfrak{l})/\mathrm{Aut}(\mathfrak{l}, \mathfrak{l}_0)$. Since the adjoint representation Ad induces a covering

$$N_L(\mathfrak{l}_0)/L_0 \to L/L_0 \xrightarrow{\widetilde{\mathrm{Ad}}} L^*/L_0^* \tag{5.3}$$

in a natural manner, L^*/L_0^* is connected. On the other hand, by virtue of (5.1) and (5.2), γ induces a fibering

$$\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l})/\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l}, \mathfrak{l}_0) \to \mathrm{Aut}(\mathfrak{l})/\mathrm{Aut}(\mathfrak{l}, \mathfrak{l}_0) \xrightarrow{\tilde{\gamma}} \mathrm{Aut}_\sigma(\mathfrak{l})/\mathrm{Aut}_\sigma(\Pi, \Pi_0)$$

in a natural way. We know (Takeuchi [18]) that the fibre $\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l})/\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l}, \mathfrak{l}_0)$ is compact and connected. It follows from these that L^*/L_0^* is diffeomorphic to $\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l})/\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l}, \mathfrak{l}_0)$ and $\tilde{\gamma}(L^*/L_0^*) = \mathrm{Aut}_\sigma(\Pi, \Pi_0)$. The lemma is an immediate consequence of this. q.e.d.

We call the space $R = \mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l})/\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l}, \mathfrak{l}_0)$, which appeared in the proof of Lemma 5.1, the *symmetric R-space* associated with L/L_0 (or often, with $\mathfrak{l}$). Indeed R is a riemannian symmetric space with respect to an $\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l}, \mathfrak{k})$-invariant metric.

Denote the infinitesimal linear isotropy representation $\mathfrak{l}_0 \to \mathfrak{gl}(q, \boldsymbol{R})$, $q = \dim L/L_0$, also by λ. Let $\mathfrak{t}_0$ be a maximal abelian subalgebra of $\mathfrak{k}_0$. An $\boldsymbol{R}$-linear map $\mu: \mathfrak{t}_0 \to \boldsymbol{C}$ is called a *weight* of $\lambda: \mathfrak{k}_0 \to \mathfrak{o}(q)(\subset \mathfrak{gl}(q, \boldsymbol{C}))$ with respect to $\mathfrak{t}_0$ if μ satisfies the condition: let V_μ denote the linear subspace of $\boldsymbol{C}^q$ given by

$$V_\mu = \{v \in \boldsymbol{C}^q;\ \lambda(H)v = \mu(H)v \quad \text{for each} \quad H \in \mathfrak{t}_0\},$$

then $V_\mu \neq 0$. The dimension of V_μ is called the *multiplicity* of μ.

Consider now the multiplicity counted sum

$$\Phi_k = \sum_\mu \mu^{2k}$$

of powers of the weights μ of λ for each $k \in \boldsymbol{Z}$, $k \geqq 1$. Φ_k is a real valued homogeneous polynomial on $\mathfrak{t}_0$ of degree $2k$ and has the property:

$$\Phi_k \in I_{N_{K_0}(\mathfrak{t}_0)}(\mathfrak{t}_0), \quad \phi_k | \mathfrak{t}_0 = \Phi_k .$$

Define a closed subgroup $L^\sharp$ of $\mathrm{Aut}(\mathfrak{l})$ by

$$L^\sharp = \gamma^{-1}(\mathrm{Aut}_\sigma(\Pi, \Pi_0)) = \mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l}) \cdot \delta(\mathrm{Aut}_\sigma(\Pi, \Pi_0)) .$$

Note that the Lie algebra of $L^\sharp$ coincides with $\mathfrak{l}$.

With these understood, we have the following

THEOREM 5.1. *Let $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a semisimple graded Lie algebra.*

(i) *Let L/L_0 be a semisimple flat homogeneous space associated with $\mathfrak{l}$. Then the condition* (2) *in Theorem* 4.1 *is equivalent to the condition*:

$$\Phi_k \in I_L(\mathfrak{t}_0) \quad \text{for each} \quad k \in \boldsymbol{Z},\ 1 \leqq k \leqq [q/2]\ .$$

(ii) *Assume that*

(2)′ $\mathrm{Pont}(\mathfrak{k}_0) \subset I_{L^\#}(\mathfrak{k}_0)$.

Then, for an arbitrary semisimple flat homogeneous space L/L_0 *associated with* $\mathfrak{l}$, *the condition* (2) *holds. The condition* (2)′ *is equivalent to the condition*:

$$\Phi_k \in I_{L^\#}(\mathfrak{t}_0) \quad \text{for each} \quad k \in \boldsymbol{Z},\ 1 \leqq k \leqq [q/2]\ .$$

PROOF. (i) follows directly from the fact that restriction homomorphism $I_{K_0}(\mathfrak{k}_0) \to I_{N_{K_0}(\mathfrak{t}_0)}(\mathfrak{t}_0)$ is an isomorphism. (ii) is an immediate consequence of (i) and Lemma 5.1.

THEOREM 5.2. *Let* L/L_0 *be a semisimple flat homogeneous space associated with a semisimple graded Lie algebra* $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. *If* L/L_0 *is compact, and if the condition* (2) *is satisfied, then the real total Pontrjagin class* $p(R)$ *of the symmetric* R*-space* R *associated with* L/L_0 *is trivial*: $p(R) = 1$.

PROOF. Let $K = N_L(\mathfrak{k})$. Then, the Lie algebra of K is $\mathfrak{k}$ and, as in Lemma 1.1, we get a polar decomposition of L: $L = K \exp \mathfrak{p}$. Let L^0 denote the identity component of L. Then we have the Iwasawa decomposition $L^0 = K^0AN$ of L^0, where K^0 is the identity component of K. In consequence we have a decomposition $L = KAN$ of L. Noticing that $AN \subset L_0$, we finally get

$$L = KL_0\ .$$

Hence $K/N_{L_0}(\mathfrak{k})$ is diffeomorphic to L/L_0.

We next prove that $N_{L_0}(\mathfrak{k}) = K_0$ and hence

(5.4) $$K/K_0 \text{ is diffeomorphic to } L/L_0\ .$$

In fact, it is verified in the same way as in Lemma 1.1 that $N_{L_0}(\mathfrak{k})$ is compact and its Lie algebra is $\mathfrak{k}_0$. Consider the restriction $\lambda' = \lambda \mid N_{L_0}(\mathfrak{k})$ to $N_{L_0}(\mathfrak{k})$ of the linear isotropy representation $\lambda\colon L_0 \to G_0$. Then the image $\lambda(N_{L_0}(\mathfrak{k}))$ of λ' is a compact subgroup of G_0 which contains $K_0 = N_{G_0}(\mathfrak{k})$. Hence $\lambda(N_{L_0}(\mathfrak{k})) = K_0$, for K_0 is a maximal compact subgroup of G_0 by Lemma 1.1. On the other hand, the kernel $G_1 \cap N_{L_0}(\mathfrak{k})$ of λ' is a compact subgroup of the vector group G_1 so that $G_1 \cap N_{L_0}(\mathfrak{k}) = \{1\}$ (cf. §1). Therefore $N_{L_0}(\mathfrak{k}) = K_0$. Since L/L_0 is compact by assumption, K/K_0 is also compact, and hence K is compact. (In fact, K is a maximal compact subgroup of L.)

Now, let B_K and B_{K_0} denote the classifying spaces of principal K-bundles and of principal K_0-bundles respectively. Let

$$K/K_0 \xrightarrow{i} B_{K_0} \xrightarrow{\rho} B_K$$

be the canonical fibering of B_{K_0} over B_K. Let

$$W_K\colon I_K(\mathfrak{k}) \xrightarrow{\cong} H^*(B_K; \boldsymbol{R}) \quad \text{and}$$

$$W_{K_0}\colon I_{K_0}(\mathfrak{k}_0) \xrightarrow{\cong} H^*(B_{K_0}; \boldsymbol{R})$$

be the (universal) Weil homomorphisms, which are isomorphisms, since K and K_0 are both compact. We denote by $\mathrm{Pont}^*(\xi_{K_0})$ the subalgebra of $H^*(B_{K_0}; \boldsymbol{R})$ generated by the real Pontrjagin classes p_k of the real vector bundle ξ_{K_0} of rank q, $q = \dim K/K_0 = \dim L/L_0$, associated with the universal K_0-bundle on B_{K_0}. $\mathrm{Pont}^*(K/K_0)$ denotes the subalgebra of $H^*(K/K_0; \boldsymbol{R})$ generated by the real Pontrjagin classes $p_k(K/K_0)$ of K/K_0.

Consider the following diagram:

$$\begin{array}{ccccccc}
I_L(\mathfrak{l}) \longrightarrow & I_K(\mathfrak{k}) & \longrightarrow & I_{K_0}(\mathfrak{k}_0) & \hookleftarrow & \mathrm{Pont}\,(\mathfrak{k}_0) \\
& \cong\big\downarrow W_K & & \cong\big\downarrow W_{K_0} & & \cong\big\downarrow W_{K_0} \\
& H^*(B_K; \boldsymbol{R}) & \xrightarrow{\rho^*} & H^*(B_{K_0}; \boldsymbol{R}) & \hookleftarrow & \mathrm{Pont}^*(\xi_{K_0}) \\
& & & \big\downarrow i^* & & \big\downarrow i^* \\
& & & H^*(K/K_0; \boldsymbol{R}) & \hookleftarrow & \mathrm{Pont}^*(K/K_0)
\end{array}$$

where the first two arrows in the first row are the restriction homomorphisms. This is a commutative diagram, and

$$p_k(K/K_0) = i^* p_k$$

holds (cf. Borel [1] and Borel-Hirzebruch [2]). Hence we have the following implications:

$$\begin{aligned}
(2)\quad & \mathrm{Pont}(\mathfrak{k}_0) \subset I_L(\mathfrak{t}_0) \\
& \Rightarrow \mathrm{Pont}(\mathfrak{k}_0) \subset I_K(\mathfrak{k}_0) \\
& \Rightarrow \mathrm{Pont}^*(\xi_{K_0}) \subset \rho^* H^*(B_K; \boldsymbol{R}) \\
& \Rightarrow \mathrm{Pont}^*(K/K_0) \subset i^*\rho^* H^*(B_K; \boldsymbol{R}) = (\rho\circ i)^* H^*(B_K; \boldsymbol{R}) \\
& \Rightarrow \mathrm{Pont}^+(K/K_0) = 0\ ,
\end{aligned}$$

where $\mathrm{Pont}^+(K/K_0)$ denotes the sum of positive-dimensional homogeneous parts of $\mathrm{Pont}^*(K/K_0)$. Therefore $p(K/K_0) = 1$, and hence by (5.4), $p(L/L_0) = 1$. Since L/L_0 is compact,

$$N_L(\mathfrak{l}_0)/L_0 \to L/L_0 \xrightarrow{\widetilde{\mathrm{Ad}}} R \tag{5.3}$$

is a finite covering. Consequently, we have

$$p(R) = 1 . \qquad \text{q.e.d.}$$

Before proceeding, let us pay attention to the following observations, which are not difficult to make.

OBSERVATIONS. 1) Let $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a semisimple graded Lie algebra and $\mathfrak{l} = \sum_k \bigoplus \mathfrak{l}^{(k)}$ be the decomposition of $\mathfrak{l}$ into its simple factors. Put $\mathfrak{g}_p^{(k)} = \mathfrak{l}^{(k)} \cap \mathfrak{g}_p$. Then

$$\mathfrak{l}^{(k)} = \mathfrak{g}_{-1}^{(k)} + \mathfrak{g}_0^{(k)} + \mathfrak{g}_1^{(k)}$$

for each k, and each $\mathfrak{l}^{(k)}$ is also a semisimple graded Lie algebra.

2) If each simple factor $\mathfrak{l}^{(k)}$ of $\mathfrak{l}$ satisfies the condition (1), then $\mathfrak{l}$ also satisfies (1).

3) If each simple factor $\mathfrak{l}^{(k)}$ of $\mathfrak{l}$ satisfies the condition (2)′, then $\mathfrak{l}$ also satisfies (2)′.

4) Let L/L_0 be a semisimple flat homogeneous space associated with $\mathfrak{l}$. Let Γ be the pseudogroup of local automorphisms of an L_0-structure Q of 2nd order associated with L/L_0 on a smooth manifold B. If $\mathfrak{l}$ is the scalar restriction to $\boldsymbol{R}$ of a complex Lie algebra, then L/L_0 has an L-invariant complex structure, B is a complex manifold and Γ is a pseudogroup of local holomorphic transformations of B.

5) If Γ is a pseudogroup of local holomorphic transformations of a complex manifold B, then the strong vanishing theorem $(*)$ holds for a Γ-foliation $\mathscr{F}$ (cf. Bott [3] and Bott-Haefliger [4]).

6) If $\mathfrak{l}$ is simple and if $\mathfrak{l}$ is not the scalar restriction to $\boldsymbol{R}$ of a complex Lie algebra, that is, if $\mathfrak{l}^C$ is a complex simple Lie algebra, then the condition (1) holds except the case

$$\mathfrak{l} = \mathfrak{sl}(2, \boldsymbol{R}) , \qquad e = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(cf. Ochiai [12]).

We shall now examine the conditions (1) and (2)′ for semisimple graded Lie algebras $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ such that $\mathfrak{l}^C$ is simple and $p(R) = 1$. In the following, we keep our previous notations. 1_q denotes the identity matrix of degree q, and Tr abbreviates the trace of a matrix unless otherwise mentioned.

First we consider $\mathfrak{l}$ of classical type.

EXAMPLE 1. $\mathfrak{l} = \mathfrak{sl}(q+1, \boldsymbol{R})$ $(q \geqq 1)$;

$$e = \frac{1}{q+1}\left(\begin{array}{c|c} q & \\ \hline & -1_q \end{array}\right).$$

The condition (1) holds if and only if $q \geqq 2$. The graded decomposition $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ is given by

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{c|c} & \\ \hline X & \end{array} \right) ; \quad X \in \boldsymbol{R}^q \right\} ,$$

$$\mathfrak{g}_0 = \left\{ \left(\begin{array}{c|c} \alpha & \\ \hline & B \end{array} \right) ; \quad \begin{array}{l} B \in \mathfrak{gl}(q, \boldsymbol{R}) \\ \alpha = -\operatorname{Tr} B \end{array} \right\} ,$$

$$\mathfrak{g}_1 = \left\{ \left(\begin{array}{c|c} & \xi \\ \hline & \end{array} \right) ; \quad {}^t\xi \in \boldsymbol{R}^q \right\} .$$

The associated symmetric R-space R is diffeomorphic to a real projective space $P_q(\boldsymbol{R})$ of dimension q.

Take $\mathfrak{k} = \mathfrak{o}(q+1)$. Then

$$\mathfrak{k}_0 = \left\{ \left(\begin{array}{c|c} 0 & \\ \hline & B \end{array} \right) ; \quad B \in \mathfrak{o}(q) \right\} .$$

The infinitesimal linear isotropy representation $\lambda\colon \mathfrak{g}_0 \to \mathfrak{gl}(q, \boldsymbol{R})$ is given by

$$\left(\begin{array}{c|c} \alpha & \\ \hline & B \end{array} \right) \mapsto B - \alpha 1_q ,$$

which is an isomorphism. Note that λ induces an isomorphism $\lambda\colon \mathfrak{k}_0 \xrightarrow{\cong} \mathfrak{o}(q)$.

Set

$$H(x_1, \cdots, x_l) = \left(\begin{array}{cccc} \begin{array}{cc} & x_1 \\ -x_1 & \end{array} & & & \\ & \ddots & & \\ & & \begin{array}{cc} & x_l \\ -x_l & \end{array} & \\ & & & 0 \end{array} \right)$$

where $x_i \in \boldsymbol{R}$, $1 \leqq i \leqq l = [q/2]$, and 0 at the (q, q)-component appears only when q is odd. Then take

$$\mathfrak{t}_0 = \left\{ \begin{pmatrix} 0 & \\ & H(x_1, \cdots, x_l) \end{pmatrix} ; \quad \begin{matrix} x_i \in \boldsymbol{R} \\ (1 \leqq i \leqq l) \end{matrix} \right\} .$$

Denoting by x_i the linear form on $\mathfrak{t}_0$ defided by

$$\begin{pmatrix} 0 & \\ & H(x_1, \cdots, x_l) \end{pmatrix} \mapsto x_i$$

we get

$$\Phi_k = 2(-1)^k \sum_i x_i^{2k} .$$

Define a homogeneous polynomial P_k on $\mathfrak{l}$ by

$$P_k(X) = \operatorname{Tr} X^{2k} \quad \text{for} \quad X \in \mathfrak{l} .$$

Then $P_k | \mathfrak{t}_0 = \Phi_k$.

By the explicit computation of $\operatorname{Aut}(\mathfrak{l})$ it is known (Takeuchi [17]) that

$$L^{\#} = \operatorname{Inn}(\mathfrak{sl}(q+1, \boldsymbol{C}), \mathfrak{sl}(q+1, \boldsymbol{R})) .$$

Hence $P_k \in I_{L^{\#}}(\mathfrak{sl}(q+1, \boldsymbol{R}))$, which shows that the condition (2)′ is satisfied for $\mathfrak{sl}(q+1, \boldsymbol{R})$.

An example of associated semisimple flat homogeneous spaces L/L_0 is given as follows:

$$L = PL(q+1, \boldsymbol{R}) = GL(q+1, \boldsymbol{R})/\boldsymbol{R}^* 1_{q+1} ,$$

$$L_0 = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \in GL(q+1, \boldsymbol{R}) \right\} \Big/ \boldsymbol{R}^* 1_{q+1} ,$$

where $\boldsymbol{R}^*$ is the multiplicative group of nonzero reals. Then L/L_0 is diffeomorphic to a real projective space $P_q(\boldsymbol{R})$ of dimension q.

In this case,

$$G_0 = \left\{ \left(\begin{array}{c|c} a & 0 \\ \hline 0 & b \end{array} \right) \in GL(q+1, \boldsymbol{R}) \right\} \Big/ \boldsymbol{R}^* 1_{q+1}$$

and the linear isotropy representation $\lambda: G_0 \to GL(q, \boldsymbol{R})$ is an isomorphism defined by

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \mapsto a^{-1} b .$$

Γ is nothing but the pseudogroup of local projective transformations of a torsionfree linear connection on B.

EXAMPLE 2. $\mathfrak{l} = \mathfrak{o}(S)$
$= \{X \in \mathfrak{gl}(q+2, \boldsymbol{R});\ {}^tXS + SX = 0\}$;

$$e = \left(\begin{array}{c|c|c} 1 & & \\ \hline & 0 & \\ \hline & & -1 \end{array}\right),$$

where $r \geqq s \geqq 0$, $q = r + s \geqq 3$ and

$$S = \left(\begin{array}{c|cc|c} & & & -1 \\ \hline & 1_r & & \\ & & -1_s & \\ \hline -1 & & & \end{array}\right).$$

For this $\mathfrak{l}$ the condition (1) always holds. Set

$$\mathfrak{o}(r, s) = \{X \in \mathfrak{gl}(q, \boldsymbol{R});\ {}^tXS_0 + S_0X = 0\}\ ,$$

$$\mathfrak{co}(r, s) = \{X \in \mathfrak{gl}(q, \boldsymbol{R});\ \exists\alpha \in \boldsymbol{R} \text{ with } {}^tXS_0 + S_0X = \alpha S_0\}\ ,$$

where

$$S_0 = \begin{pmatrix} 1_r & \\ & -1_s \end{pmatrix}.$$

Then the graded decomposition $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ is given by

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{c|cc|c} & & & \\ \hline x' & & & \\ x'' & & & \\ \hline & {}^tx' & -{}^tx'' & \end{array}\right);\quad \begin{array}{l} x' \in \boldsymbol{R}^r \\ x'' \in \boldsymbol{R}^s \end{array} \right\},$$

$$\mathfrak{g}_0 = \left\{ \left(\begin{array}{c|c|c} \alpha & & \\ \hline & B & \\ \hline & & -\alpha \end{array} \right) ; \quad \begin{array}{l} \alpha \in \boldsymbol{R} \\ B \in \mathfrak{o}(r, s) \end{array} \right\} ,$$

$$\mathfrak{g}_1 = \left\{ \left(\begin{array}{c|c|c} & \xi' \quad \xi'' & \\ \hline & & \begin{array}{c} {}^t\xi' \\ -{}^t\xi'' \end{array} \\ \hline & & \end{array} \right) ; \quad \begin{array}{l} {}^t\xi' \in \boldsymbol{R}^r \\ {}^t\xi'' \in \boldsymbol{R}^s \end{array} \right\} .$$

Let $S^r \times S^s$ be the product of two euclidean spheres of dimension r and s respectively and $E_{r,s}$ denote the quotient space $S^r \times S^s/\sim$ of $S^r \times S^s$ by the equivalence relation $\sim$ defined by $(x, y) \sim (-x, -y)$ for $(x, y) \in S^r \times S^s$. Then the associated symmetric R-space R is diffeomorphic to $E_{r,s}$ (Takeuchi [18]).

If we take $\mathfrak{k} = \mathfrak{o}(q + 2) \cap \mathfrak{l}$, then

$$\mathfrak{k}_0 = \left\{ \left(\begin{array}{c|c|c} 0 & & \\ \hline & \begin{array}{cc} B' & \\ & B'' \end{array} & \\ \hline & & 0 \end{array} \right) ; \quad \begin{array}{l} B' \in \mathfrak{o}(r) \\ B'' \in \mathfrak{o}(s) \end{array} \right\} .$$

The infinitesimal linear isotropy representation $\lambda \colon \mathfrak{g}_0 \to \mathfrak{gl}(q, \boldsymbol{R})$ is given by

$$\begin{pmatrix} \alpha & & \\ & B & \\ & & -\alpha \end{pmatrix} \mapsto B - \alpha 1_q \, .$$

λ defines isomorphisms $\lambda \colon \mathfrak{g}_0 \cong \mathfrak{co}(r, s)$ and $\lambda \colon \mathfrak{k}_0 \cong \mathfrak{o}(r) \oplus \mathfrak{o}(s)$ (direct sum). Take

$$\mathfrak{t}_0 = \left\{ \left(\begin{array}{c|c|c} 0 & & \\ \hline & \begin{array}{cc} H(x'_1, \cdots, x'_{l'}) & \\ & H(x''_1, \cdots, x''_{l''}) \end{array} & \\ \hline & & 0 \end{array} \right) ; \quad x'_i, x''_j \in \boldsymbol{R} \right\} ,$$

where $l' = [r/2]$, $l'' = [s/2]$, $1 \leqq i \leqq l'$, $1 \leqq j \leqq l''$. Then, in the same way as in Example 1, we get

$$\Phi_k = 2(-1)^k\Big(\sum_i x_i'^{2k} + \sum_j x_j''^{2k}\Big) .$$

Define a homogeneous polynomial P_k on $\mathfrak{l}$ by

$$P_k(X) = \operatorname{Tr} X^{2k} \quad \text{for} \quad X \in \mathfrak{l} .$$

Then $P_k|\mathfrak{t}_0 = \Phi_k$.

It is known (Takeuchi [17]) that $L^\# = \operatorname{Inn}(\mathfrak{l}^C, \mathfrak{l})$ if q is odd, and $L^\#$ is generated by $\operatorname{Inn}(\mathfrak{l}^C, \mathfrak{l})$ and the involutive automorphism $\tau\colon X \mapsto -{}^tX$ of $\mathfrak{l}$ if q is even. Hence $P_k \in I_{L^\#}(\mathfrak{l})$, which shows that the condition (2)′ is satisfied for $\mathfrak{l} = \mathfrak{o}(S)$.

An example of associated semisimple flat homogeneous spaces L/L_0 is given as follows:

$$\begin{aligned} L &= O(S)/\{\pm 1_{q+2}\} \\ &= \{x \in GL(q+2, \boldsymbol{R});\ {}^txSx = S\}/\{\pm 1_{q+2}\} , \end{aligned}$$

$$L_0 = \left\{ x = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} ;\quad {}^txSx = S \right\}/\{\pm 1_{q+2}\} .$$

Then L/L_0 is diffeomorphic to $E_{r,s}$.

Set

$$\begin{aligned} O(r, s) &= \{x \in GL(q, \boldsymbol{R});\ {}^txS_0x = S_0\} , \\ CO(r, s) &= \{x \in GL(q, \boldsymbol{R});\ \exists a > 0 \quad \text{with} \quad {}^txS_0x = aS_0\} . \end{aligned}$$

Then

$$G_0 = \left\{ \left(\begin{array}{c|c|c} a & & \\ \hline & b & \\ \hline & & a^{-1} \end{array}\right) ;\quad \begin{array}{l} a \in \boldsymbol{R}^* \\ b \in O(r, s) \end{array} \right\}/\{\pm 1_{q+2}\}$$

and the linear isotropy representation $\lambda\colon G_0 \to GL(q, \boldsymbol{R})$ is given by

$$\begin{pmatrix} a & & \\ & b & \\ & & a^{-1} \end{pmatrix} \mapsto a^{-1}b .$$

λ induces an isomorphism $\lambda\colon G_0 \cong CO(r, s)$. Note that Γ is nothing but the pseudogroup of local conformal transformations of a pseudoriemannian metric on B with signature (r, s).

In the same way, we can see that the following Examples 3, 4 and 5 also satisfy the condition (1) and (2)'. Let $\boldsymbol{F}$ denote $\boldsymbol{R}$, $\boldsymbol{C}$ or the real quaternion algebra $\boldsymbol{H}$. The standard units of $\boldsymbol{H}$ are denoted by 1, i, j and k.

EXAMPLE 3.

$$\mathfrak{l} = \begin{cases} \{X \in \mathfrak{gl}(2n, \boldsymbol{F});\ {}^t\bar{X}H + HX = 0\} & \text{if} \quad \boldsymbol{F} = \boldsymbol{R} \quad \text{or} \quad \boldsymbol{H} \\ \{X \in \mathfrak{gl}(2n, \boldsymbol{F});\ \operatorname{Tr} X \in \boldsymbol{R},\ {}^t\bar{X}H + HX = 0\} & \text{if} \quad \boldsymbol{F} = \boldsymbol{C} \end{cases};$$

$$e = \frac{1}{2}\begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix},$$

where $n \geqq 5$ if $\boldsymbol{F} = \boldsymbol{R}$, $n \geqq 3$ if $\boldsymbol{F} = \boldsymbol{C}$, $n \geqq 2$ if $\boldsymbol{F} = \boldsymbol{H}$, $\bar{X}$ denotes the conjugate matix of X, and

$$H = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.$$

$\mathfrak{l}$ is isomorphic to $\mathfrak{o}(n, n)$ if $\boldsymbol{F} = \boldsymbol{R}$, $\mathfrak{su}(n, n)$ if $\boldsymbol{F} = \boldsymbol{C}$ and $\mathfrak{sp}(n, n)$ if $\boldsymbol{F} = \boldsymbol{H}$ in terms of the standard notations.

In this case, the dimension q is:

$$q = \begin{cases} \dfrac{1}{2}n(n-1) & \text{if} \quad \boldsymbol{F} = \boldsymbol{R} \\ n^2 & \text{if} \quad \boldsymbol{F} = \boldsymbol{C} \\ n(2n+1) & \text{if} \quad \boldsymbol{F} = \boldsymbol{H}. \end{cases}$$

The associated symmetric R-sace R is diffeomorphic respectively to:

$$R \approx \begin{cases} SO(n) & \text{if} \quad \boldsymbol{F} = \boldsymbol{R} \\ U(n) & \text{if} \quad \boldsymbol{F} = \boldsymbol{C} \\ \mathrm{Sp}(n) & \text{if} \quad \boldsymbol{F} = \boldsymbol{H}. \end{cases}$$

EXAMPLE 4. $\mathfrak{l} = \mathfrak{sp}(n, \boldsymbol{R})$ $(n \geqq 3)$;

$$e = \frac{1}{2}\begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}.$$

In this case, the dimension $q = n(n+1)/2$ and the associated symmetric R-space $R \approx U(n)/O(n)$.

EXAMPLE 5. $\mathfrak{l} = \{X \in \mathfrak{gl}(2n, \boldsymbol{H});\ {}^t\bar{X}A + AX = 0\}$ $(n \geqq 3)$;

$$e = \frac{i}{2}\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

where $A = j1_{2n}$.

In this case, the dimension $q = n(2n-1)$ and the associated symmetric R-space $R \approx U(2n)/Sp(n)$. $\mathfrak{l}$ is isomorphic to $\mathfrak{so}^*(4n)$ in terms of the standard notations.

Note that these five examples give all the semisimple graded Lie algebras $\mathfrak{l}$ of classical type such that $\mathfrak{l}^C$ is simple and $p(R) = 1$ (cf. Takeuchi [16]).

We shall now consider the exceptional types.

EXAMPLE 6. There are two types of semisimple graded Lie algebras $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ such that $\mathfrak{l}$ is a simple Lie algebra of exceptional type, $\mathfrak{l}^C$ is simple and $p(R) = 1$:

$$\mathfrak{l} = EV\,, \quad \mathfrak{g}_0 = EI \oplus \boldsymbol{R}\,, \quad \text{and}$$
$$\mathfrak{l} = EVII\,, \quad \mathfrak{g}_0 = EIV \oplus \boldsymbol{R}\,.$$

The condition (1) holds for these $\mathfrak{l}$. We show that both $\mathfrak{l}$ satisfy the condition (2)$'$ also.

Let $\mathfrak{l}^C = \mathfrak{g}_{-1}^C + \mathfrak{g}_0^C + \mathfrak{g}_1^C$ be the complexification of $\mathfrak{l}$ and $\lambda\colon \mathfrak{g}_0^C \to \mathfrak{gl}(\mathfrak{g}_{-1}^C)$ denote the complexified infinitesimal linear isotropy representation defined by

$$\lambda(x)y = [x, y] \quad \text{for} \quad x \in \mathfrak{g}_0^C,\ y \in \mathfrak{g}_{-1}^C\,.$$

Denote by $\mathfrak{g}_0'$ the derived algebra of $\mathfrak{g}_0$ and define a homogeneous polynomial $\tilde{\phi}_k$ on the complexification $\mathfrak{g}_0'^C$ of $\mathfrak{g}_0'$ by

$$\tilde{\phi}_k(x) = \mathrm{Tr}\,(\lambda(x)^{2k}) \quad \text{for} \quad x \in \mathfrak{g}_0'^C\,.$$

First we prove the following fact: If there exists a faithful representation $\mathfrak{l}^C \hookrightarrow \mathfrak{gl}(V)$ of $\mathfrak{l}^C$ on a complex vector space V such that for some $c_k \in \boldsymbol{R}^*$

$$(\sharp) \qquad \mathrm{Tr}\,(X^{2k}) = c_k\tilde{\phi}_k(X) \quad \text{for} \quad X \in \mathfrak{g}_0'^C \subset \mathfrak{l}^C \subset \mathfrak{gl}(V)$$

holds, then the condition (2)$'$ is satisfied for $\mathfrak{l}$.

Take e, $\mathfrak{k}$, $\mathfrak{p}$, τ, $\mathfrak{k}_0$ as in §1 and extend τ to a conjugate linear automorphism τ of $\mathfrak{l}^C$. Then $\mathfrak{l}_u = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ is a compact real form of $\mathfrak{l}^C$. Choose an $\mathfrak{l}_u$-invariant hermitian inner product $\langle\ ,\ \rangle$ on V and let X^* denote the adjoint of X with respect to $\langle\ ,\ \rangle$ for $X \in \mathfrak{gl}(V)$. Then $\tau X = -X^*$ holds for $X \in \mathfrak{l}^C$. Denote by σ the complex conjugation of $\mathfrak{l}^C$ with respect to $\mathfrak{l}$. Then $\theta = \sigma\tau$ is an involutive automorphism of $\mathfrak{l}^C$ and the 1-eigenspace of θ coincides with the complexification $\mathfrak{k}^C$ of $\mathfrak{k}$. Note that in our case $\mathfrak{k}$ is $\mathfrak{su}(8)$ or $E_6 \oplus \boldsymbol{R}$ according as $\mathfrak{l} = EV$ or $\mathfrak{l} = EVII$. So $\mathfrak{k}$ has the same rank as that of $\mathfrak{l}$, and thus θ is an inner automorphism of $\mathfrak{l}^C$. Therefore, since $\sigma = \theta\tau$, there exists an element $A \in GL(V)$ such that

$$\sigma X = -AX^*A^{-1} \quad \text{for} \quad X \in \mathfrak{l}^C . \tag{5.4}$$

Define an Inn$(\mathfrak{l}^C)$-invariant polynomial $\tilde{P}_k$ on $\mathfrak{l}^C$ by

$$\tilde{P}_k(X) = \mathrm{Tr}\,(X^{2k}) \quad \text{for} \quad X \in \mathfrak{l}^C \subset \mathfrak{gl}(V) .$$

Then it follows from (5.4) that for $X \in \mathfrak{l}^C$

$$\begin{aligned} \tilde{P}_k(\sigma X) &= \mathrm{Tr}\,((AX^*A^{-1})^{2k}) = \mathrm{Tr}\,(X^{*2k}) = \overline{\mathrm{Tr}\,(X^{2k})} \\ &= \overline{\tilde{P}_k(X)} . \end{aligned}$$

Hence, for $X \in \mathfrak{l}$ we have

$$\tilde{P}_k(X) = \overline{\tilde{P}_k(X)} .$$

Thus, putting $P_k = \tilde{P}_k | \mathfrak{l}$, we get

$$P_k \in I_{\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l})}(\mathfrak{l}) .$$

Since $\mathfrak{k}_0 \subset \mathfrak{g}_0'$ and $\tilde{\phi}_k | \mathfrak{k}_0 = \phi_k$, it follows from (♯) that

$$P_k | \mathfrak{k}_0 = c_k \phi_k .$$

Consequently, Pont$(\mathfrak{k}_0) \subset I_{\mathrm{Inn}(\mathfrak{l}^C, \mathfrak{l})}(\mathfrak{l})$. On the other hand, it is known (Takeuchi [17]) that in our case

$$L^\sharp = \mathrm{Inn}\,(\mathfrak{l}^C, \mathfrak{l}) .$$

Hence the condition (2)′ is satisfied.

Now let us construct a faithful representation $\mathfrak{l}^C \hookrightarrow \mathfrak{gl}(V)$ of $\mathfrak{l}^C$ with the property (♯).

Let $\boldsymbol{K}$ be the Cayley algebra over $\boldsymbol{C}$ and $x \mapsto \bar{x}$ denote the canonical involution of $\boldsymbol{K}$. Identifying $\boldsymbol{C}1$ with $\boldsymbol{C}$, we define a linear form tr and a quadratic form n on $\boldsymbol{K}$ respectively by

$$\mathrm{tr}\,(x) = x + \bar{x} , \qquad n(x) = x\bar{x} \qquad \text{for} \quad x \in \boldsymbol{K} .$$

Let $M_3(\boldsymbol{K})$ denote the total matrix algebra of degree 3 over $\boldsymbol{K}$, and put

$$J = \{u \in M_3(\boldsymbol{K});\ {}^t\bar{u} = u\} .$$

We make J an algebra over $\boldsymbol{C}$ by defining a bilinear product $\circ$ on J by

$$u \circ v = \frac{1}{2}(uv + vu) \quad \text{for} \quad u, v \in J ,$$

and denote it by $\mathfrak{J}$. Then $\mathfrak{J}$ is a complex simple Jordan algebra.

We define a linear form Tr and a cubic form N on J respectively by

$$\begin{aligned} \mathrm{Tr}\,(u) &= \xi_1 + \xi_2 + \xi_3 , \\ N(u) &= \xi_1\xi_2\xi_3 - \sum \xi_i n(x_i) + \mathrm{tr}(x_1x_2x_3) , \end{aligned}$$

for

$$u = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_i \in \boldsymbol{C},\ x_i \in \boldsymbol{K} .$$

Let (u, v, w) denote the tri-linear symmetric form on J obtained from N by linearization and (u, v) denote the nondegenerate symmetric bilinear form on J defined by

$$(u, v) = \mathrm{Tr}\,(u \circ v) \quad \text{for} \quad u, v \in J .$$

The nondegeneracy of $(\ ,\ )$ then defines on J a commutative cross product $u \times v$ by

$$(u \times v, w) = 3(u, v, w) \quad \text{for each} \quad w \in J .$$

Let $R(u)$ denote the translation on the algebra $\mathfrak{J}$, that is, the linear operator on J defined by

$$R(u)v = u \circ v \quad \text{for} \quad u, v \in J .$$

We define a subspace $\mathscr{R}$ of $\mathfrak{gl}(J)$ by

$$\mathscr{R} = \{R(u);\ u \in J,\ \mathrm{Tr}(u) = 0\} .$$

Let $\mathscr{D}$ denote the subalgebra of $\mathfrak{gl}(J)$ consisting of the derivations of the algebra $\mathfrak{J}$. Then $\mathscr{D}$ is a complex simple Lie algebra of type F_4. Note that $\mathscr{D} \cap \mathscr{R} = \{0\}$. We define $\mathscr{E}$ by

$$\mathscr{E} = \mathscr{D} + \mathscr{R} ,$$

which is a subalgebra of $\mathfrak{gl}(J)$. Then $\mathscr{E}$ is a complex simple Lie algebra of type E_6.

Let $u \mapsto u^*$ denote the linear isomorphism of J onto the dual space J^* of J defined by

$$u^*(v) = (u, v) \quad \text{for} \quad v \in J .$$

Take one dimensional complex vector spaces V_1 and V_2, and let f_1 and f_2 denote their bases respectively. Consider the direct sum

$$V = V_1 \oplus J^* \oplus J \oplus V_2$$

of complex vector spaces V_1, J^*, J and V_2. We define a bilinear multiplication $\cdot$ on V as follows:

$$f_i \cdot f_i = f_i \ (i = 1, 2) , \quad f_1 \cdot f_2 = f_2 \cdot f_1 = 0 ;$$

$$f_1 \cdot u = \frac{1}{3}u,\ f_2 \cdot u = \frac{2}{3}u ; \quad f_1 \cdot u^* = \frac{2}{3}u^*,\ f_2 \cdot u^* = \frac{1}{3}u^* ;$$

$$
\begin{aligned}
(5.5) \qquad & u\cdot f_1 = 0,\ u\cdot f_2 = u\ ; \quad u^*\cdot f_1 = u^*,\ u^*\cdot f_2 = 0\ ; \\
& u\cdot v^* = (u, v)f_1,\ u^*\cdot v = (u, v)f_2\ ; \\
& u\cdot v = 2(u \times v)^*,\ u^*\cdot v^* = 2(u \times v)\ ,
\end{aligned}
$$

where $u, v \in J$. Then we get an algebra over C, which is denoted by $\mathscr{V}$. We define a linear form Trace on V by

$$\mathrm{Trace}\,(x) = \alpha + \beta \quad \text{for} \quad x = \alpha f_1 + u^* + v + \beta f_2 \in V\ .$$

Let $L(x)$ denote the left translation of the algebra $\mathscr{V}$, that is, the linear operator on V defined by

$$L(x)y = x\cdot y \quad \text{for} \quad x, y \in V\ .$$

A subspace $\mathscr{L}$ of $\mathfrak{gl}(V)$ is defined by

$$\mathscr{L} = \{L(x);\ x \in V,\ \mathrm{Trace}\,(x) = 0\}\ .$$

The transpose of $E \in \mathfrak{gl}(J)$ is denoted by ${}^tE \in \mathfrak{gl}(J^*)$. We consider that $\mathscr{E} \subset \mathfrak{gl}(V)$ by means of the injective homomorphism $\mathscr{E} \to \mathfrak{gl}(V)$ defined by the correspondence

$$E \mapsto 0 \oplus (-{}^tE) \oplus E \oplus 0\ .$$

Then $\mathscr{E}$ is known to coincide with the subalgebra of $\mathfrak{gl}(V)$ consisting of all derivations of the algebra $\mathscr{V}$. Note that $\mathscr{E} \cap \mathscr{L} = \{0\}$. We define $\mathfrak{G}$ by

$$\mathfrak{G} = \mathscr{E} + \mathscr{L}\ ,$$

which is a subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{G}$ is a complex simple Lie algebra of type E_7. In more detail, we get the following bracket relations:

$$
\begin{aligned}
& [E, L(x)] = L(Ex) \\
(5.6) \qquad & [L(f_1 - f_2),\ L(u)] = \frac{2}{3}L(u) \\
& [L(f_1 - f_2),\ L(u^*)] = -\frac{2}{3}L(u^*)\ ,
\end{aligned}
$$

where $E \in \mathscr{E}$, $x \in V$, $\mathrm{Trace}\,(x) = 0$ and $u \in J$. For the details of those mentioned above, we refer the reader to Brown [5], Ise [7] and Schafer [15].

Now, put

$$e = -\frac{3}{2}L(f_1 - f_2) \in \mathfrak{G}\ .$$

Then it follows from (5.6) that the eigenvalues of $\mathrm{ad}\, e$ are -1, 0 and 1. The eigenspace decomposition $\mathfrak{G} = \mathfrak{G}_{-1} + \mathfrak{G}_0 + \mathfrak{G}_1$ of $\mathrm{ad}\, e$ is given by

$$\begin{aligned}\mathfrak{G}_{-1} &= \{L(u);\ u \in J\}\ ,\\ \mathfrak{G}_0 &= \mathscr{E} \oplus \boldsymbol{C}e\ ,\\ \mathfrak{G}_1 &= \{L(u^*);\ u \in J\}\ .\end{aligned}$$

It is known (Kobayashi-Nagano [9]) that each semisimple graded Lie algebra $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ in Example 6 is then obtained by setting

$$\mathfrak{g}_p = \mathfrak{l} \cap \mathfrak{G}_p\ , \quad p = -1, 0, 1\ ,$$

for an appropriate real form $\mathfrak{l}$ of $\mathfrak{G}$. In particular, remark that $\mathfrak{g}_0'^C = \mathscr{E}$.

We are now in a position to see that the faithful representation $\mathfrak{G} \hookrightarrow \mathfrak{gl}(V)$ satisfies the condition (#).

In fact, it follows from the first equation of (5.6) that the complexified infinitesimal linear isotropy representation $\lambda\colon \mathfrak{g}_0'^C \to \mathfrak{gl}(\mathfrak{g}_{-1}^C)$ is equivalent to the natural representation $\mathscr{E} \hookrightarrow \mathfrak{gl}(J)$. It is then verified from the definition of the imbedding $\mathscr{E} \hookrightarrow \mathfrak{gl}(V)$ that

$$\mathrm{Tr}\,(X^{2k}) = 2\mathrm{Tr}\,(\lambda(X)^{2k}) = 2\tilde{\phi}_k(X)$$

holds for $X \in \mathfrak{g}_0'^C \subset \mathfrak{gl}(V)$. Hence the condition (#) is satisfied.

OBSERVATIONS. 7) From the classification of semisimple graded Lie algebras (Kobayashi-Nagano [9]) and the computation of real Pontrjagin classes of compact symmetric spaces (Takeuchi [16]) the above Examples 1 to 6 are known to give all the semisimple graded Lie algebras $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ such that $\mathfrak{l}$ is simple, $\mathfrak{l}^C$ is simple and $p(R) = 1$.

8) There exists only one semisimple graded Lie algebra $\mathfrak{l}$ such that $\mathfrak{l}$ is simple, $\mathfrak{l}^C$ is not simple and $p(R) = 1$:

$$\mathfrak{l} = \mathfrak{sl}(2, \boldsymbol{C}) \quad \text{regarded as a real Lie algebra ;}$$

$$e = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can see without difficulty that the condition (2)′ is satisfied also for this example. In consequence, we observe that if $\mathfrak{l}$ is simple and $p(R) = 1$, then the condition (2)′ is satisfied for $\mathfrak{l}$.

From the last observation we have the following

THEOREM 5.3. *Let $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a semisimple graded Lie algebra. Then the condition* (2) *is satisfied for every semisimple flat homogeneous space L/L_0 associated with $\mathfrak{l}$ if and only if the real total Pontrjagin class $p(R)$ of the symmetric R-space R associated with $\mathfrak{l}$ is trivial*: $p(R) = 1$.

PROOF. The only if part follows directly from Theorem 5.2. We prove the if part. Let $\mathfrak{l} = \mathfrak{l}^{(1)} \oplus \cdots \oplus \mathfrak{l}^{(s)}$ be the decomposition of $\mathfrak{l}$ into simple factors. If we denote by R_k the symmetric R-space associated with $\mathfrak{l}^{(k)}$ $(1 \leqq k \leqq s)$, then

$$R = R_1 \times \cdots \times R_s$$

and hence

$$H^*(R; \boldsymbol{R}) = H^*(R_1; \boldsymbol{R}) \otimes \cdots \otimes H^*(R_s; \boldsymbol{R}) .$$

In particular, $p(R) = p(R_1) \otimes \cdots \otimes p(R_s)$. Since we assume that $p(R) = 1$, $p(R_k) = 1$ for each k. Then, by Observation 8, the condition $(2)'$ is satisfied for each $\mathfrak{l}^{(k)}$. So, by Observation 3, $\mathfrak{l}$ satisfies the condition $(2)'$. Hence it follows from Theorem 5.1 (ii) that the condition (2) is satisfied for every L/L_0 associated with $\mathfrak{l}$. This completes the proof. q.e.d.

6. Concluding remarks.

1) As a corollary of Theorem 4.1, we obtain the main theorem of Nishikawa-Sato [11], a strong vanishing theorem for projective or conformal foliations. In our terminology of this paper, a Γ-foliation is called projective or conformal according as Γ is the one in Example 1 or the one in Example 2 $(r \geqq s = 0)$.

2) It should be noted that our argument is essentially in the real category as is clear in the light of Theorem 5.3. Compare Observations 4 and 5.

3) The procedure in § 1 of [11] can be naturally extended to yield examples of "locally homogeneous" Γ-foliations associated with semisimple flat homogeneous spaces. As a result, we can get examples of Γ-foliations with nontrivial secondary characteristic classes of foliations (see Bott-Haefliger [4], Kamber-Tondeur [8] and Yamato [21]).

REFERENCES

[1] A. BOREL, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts, Ann. of Math., 57 (1953), 115–207.
[2] A. BOREL AND F. HIRZEBRUCH, Characteristic classes and homogeneous spaces I, Amer. J. Math., 80 (1958), 458–538.
[3] R. BOTT, On a topological obstruction to integrability, Proc. Symp. Pure Math., 16 (1970), 127–131.
[4] R. BOTT AND A. HAEFLIGER, On characteristic classes of Γ-foliations, Bull. Amer. Math. Soc., 78 (1972), 1039–1044.
[5] R. BROWN, A minimal representation for the Lie algebra E_7, Illinois J. Math., 12 (1968), 190–200.
[6] C. CHEVALLEY, Theory of Lie groups I, Princeton Univ. Press, Princeton, 1946.
[7] M. ISE, Bounded symmetric domains of exceptional type, J. Fac. Sci. Univ. Tokyo, Sect. IA, 23 (1976), 75–105.

[8] F. KAMBER AND P. TONDEUR, Non-trivial characteristic invariants of homogeneous foliated bundles, Ann. Ec. Norm. Sup., 8 (1975), 433-486.
[9] S. KOBAYASHI AND T. NAGANO, On filtered Lie algebras and geometric structures I, J. Math. Mech., 13 (1964), 875-908.
[10] H. MATSUMOTO, Quelques remarques sur les groupes de Lie algébriques réels, J. Math. Soc. Japan, 16 (1964), 419-446.
[11] S. NISHIKAWA AND H. SATO, On characteristic classes of riemannian, conformal and projective foliations, J. Math. Soc. Japan, 28 (1976), 223-241.
[12] T. OCHIAI, Geometry associated with semisimple flat homogeneous spaces, Trans. Amer. Math. Soc., 152 (1970), 159-193.
[13] J. PASTERNACK, Foliations and compact Lie group actions, Comm. Math. Helv., 46 (1971), 467-477.
[14] I. SATAKE, On representations and compactifications of symmetric Riemannian spaces, Ann. of Math., 71 (1960), 77-110.
[15] R. SCHAFER, An introduction to nonassociative algebras, Academic Press, New York-London, 1966.
[16] M. TAKEUCHI, On Pontrjagin classes of compact symmetric spaces, J. Fac. Sci. Univ. Tokyo, Sect. IA, 9 (1962), 313-328.
[17] M. TAKEUCHI, On the fundamental group and the group of isometries of a symmetric space, ibid., 10 (1964), 88-123.
[18] M. TAKEUCHI, Cell decompositions and Morse equalities on certain symmetric spaces, ibid., 12 (1965), 81-192.
[19] N. TANAKA, On the equivalence problems associated with a certain class of homogeneous spaces, J. Math. Soc. Japan, 17 (1965), 103-139.
[20] W. THURSTON, Foliations and groups of diffeomorphisms, Bull. Amer. Math. Soc., 80 (1974), 304-307.
[21] K. YAMATO, Examples of foliations with non-trivial exotic characteristic classes, Osaka J. Math., 12 (1975), 407-417.

UNIVERSITY OF TOKYO
AND
OSAKA UNIVERSITY