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COMPLEX MANIFOLDS WITH NONPOSITIVE HOLOMORPHIC SECTIONAL CURVATURE AND HYPERBOLICITY

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1. Introduction. It is known that a hermitian manifold M whose holomorphic sectional curvature is bounded from above by a negative constant is hyperbolic, [5; p. 61]. On the other hand, Brody's theorem states that a compact complex manifold M is hyperbolic if (and only if) there is no nonconstant holomorphic map $f: \boldsymbol{C} \to M$ [1] (A stronger result actually proved will be stated in the next section). Using Brody's result, Green has shown that a closed complex subspace X of a complex torus T is hyperbolic if (and only if) there is no nonconstant affine map $f: \boldsymbol{C} \to T$ such that $f(\boldsymbol{C}) \subset X$, or equivalently, if X contains no subtorus of T, [4].

The purpose of this short note is to show another application of Brody's theorem which is closely related to the result of Green.

THEOREM. *Let M be a compact hermitian manifold with nonpositive holomorphic sectional curvature and X a closed complex subspace. Let ds_M^2 denote the metric of M and $ds_C^2 = dzd\bar{z}$ the usual Euclidean metric on $\boldsymbol{C}$. Then X is hyperbolic if (and only if) there is no totally geodesic, isometric holomorphic immersion $f: \boldsymbol{C} \to M$ such that $f(\boldsymbol{C}) \subset X$.*

COROLLARY. *Let M be as above. Then M is hyperbolic if (and only if) there is no totally geodesic, isometric holomorphic immersion $f: C \to M$.*

It should be pointed out that the theorem is not any more general than its corollary when X is non-singular since the holomorphic sectional curvature of X does not exceed that of M.

2. Proof of Theorem. We restate here Brody's theorem in the way most convenient for us.

BRODY'S THEOREM. *Let M be a compact hermitian manifold with a hermitian metric ds_M^2 and X a closed complex subspace of M. Then X is hyperbolic if (and only if) there is no nonconstant holomorphic*

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map $f: \boldsymbol{C} \rightarrow M$ *such that* $f(\boldsymbol{C}) \subset X$ *and* $f^* ds_M^2 \leqq ds_C^2$.

To start the proof of our theorem, assume that X is not hyperbolic. Then there is a nonconstant holomorphic map $f: \boldsymbol{C} \rightarrow M$ satisfying the conditions above. Write

$$f^* ds_M^2 = \lambda \cdot ds_C^2 \qquad \text{with} \qquad 0 \leqq \lambda \leqq 1 .$$

We shall calculate the Gaussian curvature of the immersed variety $f(\boldsymbol{C}) \subset M$ at a non-singular point, i.e., a point where $\lambda > 0$. In terms of the coordinate z in $\boldsymbol{C}$, the induced metric on $f(\boldsymbol{C})$ is given by $\lambda dz d\bar{z}$. Hence, its Gaussian curvature K is given by

$$K = -\frac{1}{\lambda} \frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}} .$$

In general, if V is a complex submanifold of M, the holomorphic sectional curvature of V does not exceed that of M.[1] In particular, the holomorphic sectional curvature of $f(\boldsymbol{C})$ (at a non-singular point) does not exceed that of M and hence is non-positive. Since $f(\boldsymbol{C})$ is of dimension 1, its holomorphic sectional curvature is nothing but the Gaussian curvature K. Hence,

$$\frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}} \geqq 0 \qquad (\text{wherever } \lambda > 0) .$$

This means that $\log \lambda$ is a subharmonic function on $\boldsymbol{C}$. On the other hand, $\log \lambda \leqq 0$ since $\lambda \leqq 1$. But a bounded subharmonic function on $\boldsymbol{C}$ is constant. Hence, λ is a (positive) constant. Composing f with a homothetic transformation of $\boldsymbol{C}$, we can make $\lambda \equiv 1$. Thus, f is an isometric immersion.

To see that $f(\boldsymbol{C})$ is totally geodesic, we observe that, in general, a complex submanifold V of M is totally geodesic if and only if the holomorphic sectional curvature of V coincides with the restriction to V of the holomorphic sectional curvature of M.[2] If M has non-positive holomorphic sectional curvature and V is flat, then the restriction to V of the holomorphic sectional curvature is zero (since it is non-positive on one hand and is bounded from below by the curvature of V on the other). Hence, $f(\boldsymbol{C})$ is totally geodesic. This completes the proof.

3. Concluding remarks.

(1) The point of the theorem is that, to decide whether X is

[1] This well known result follows from the equations of Gauss-Codazzi which read, in terms of the self-explanatory notation, as follows: $R^M_{i\bar{j}k\bar{l}} - R^V_{i\bar{j}k\bar{l}} = \sum_\alpha h^\alpha_{ik} \bar{h}^\alpha_{jl}$, where (h_{ik}) is the second fundamental form.

[2] See Footnote 1).

hyperbolic or not, it suffices to consider only a very limited class of holomorphic maps $f: \boldsymbol{C} \rightarrow M$ instead of all holomorphic maps.

(2) The theorem extends the result of Green mentioned in §1.

(3) In addition to the result mentioned above, Green proved that if X is a closed hypersurface of a complex torus T containing no complex subtorus of T, then its complement $T - X$ is complete hyperbolic and hyperbolically imbedded. A similar reasoning combined with the proof above yields the following

THEOREM. *Let M be a compact hermitian manifold with non-positive holomorphic sectional curvature and X a closed hypersurface. If there is no totally geodesic, isometric holomorphic immersion $f: \boldsymbol{C} \rightarrow M$ such that $f(\boldsymbol{C}) \subset X$ or $f(\boldsymbol{C}) \subset M - X$, then $M - X$ is complete hyperbolic and hyperbolically imbedded in M.*

The statement above is not as clear-cut as in the case where $M = T$. This is because X cannot be moved within M as freely as in the case of $M = T$ and because we do not know what the closure of $f(\boldsymbol{C})$ in M looks like. It should be pointed out that it is essential that X is a hypersurface. In fact, if X has codimension $\geqq 2$, then the intrinsic pseudo-distance d_{M-X} is the restriction of d_M to $M - X$ according to the theorem of Campbell-Ogawa-Howard-Ochiai [2], [3]. In particular, in the case $M = T$ we have $d_{T-X} = 0$ if X has codimension $\geqq 2$.

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