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HOLOMORPHIC FAMILIES OF RIEMANN SURFACES AND TEICHMÜLLER SPACES II

Applications to the uniformization of algebraic surfaces and the compactification of two dimensional Stein manifolds

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Introduction. Let $\bar{\mathscr{S}}$ be a two dimensional complex manifold and let C be a non-singular one dimensional analytic subset of $\bar{\mathscr{S}}$ or an empty set. Denote by D the unit disc $|t| < 1$ and by D^* the punctured unit disc $0 < |t| < 1$ in the complex t-plane. We assume that a proper holomorphic mapping $\bar{\pi}: \bar{\mathscr{S}} \to D^*$ satisfies the following two conditions;

1) $\bar{\pi}$ is of maximal rank at every point of $\bar{\mathscr{S}}$, and

2) by setting $\mathscr{S} = \bar{\mathscr{S}} - C$ and $\pi = \bar{\pi}|\mathscr{S}$, the fibre $S_t = \pi^{-1}(t)$ of $\mathscr{S}$ over each $t \in D^*$ is an irreducible analytic subset of $\mathscr{S}$ and is of fixed finite type (g, n) with $2g - 2 + n > 0$ as a Riemann surface, where g is the genus of S_t and n is the number of punctures of S_t. We call such a triple $(\mathscr{S}, \pi, D^*)$ a holomorphic family of Riemann surfaces of type (g, n) over D^*. We also say that $\mathscr{S}$ has a holomorphic fibration $(\mathscr{S}, \pi, D^*)$ of type (g, n).

Our main problem is to construct a completion of $(\mathscr{S}, \pi, D^*)$ canonically in such a way that the central fibre is a Riemann surface (possibly with nodes) of the same type (g, n) modulo a finite group of automorphisms.

As a continuation of the preceeding paper [6], we treat the completion of $(\mathscr{S}, \pi, D^*)$ in the first half of this paper. For a holomorphic family $(\mathscr{S}, \pi, D^*)$ of Riemann surfaces of type (g, n) with $2g - 2 + n > 0$, we regard the fibre S_t over $t \in D^*$ as a point $\Phi(t)$ in a Teichmüller space. It should be noted that, in general, Φ is a multi-valued analytic mapping. In §1 and §2, we recall terminologies and notations in [6]. In §3, we study the behavior of Φ as t tends to zero. In §4, using the result of §3, we canonically construct a completion $(\hat{\mathscr{S}}, \hat{\pi}, D)$ of $(\mathscr{S}, \pi, D^*)$ and, in §5, we prove an extension theorem for a holomorphic mapping F of $\mathscr{S}$ into $\hat{\mathscr{S}}$ with $\pi = \hat{\pi} \circ F$.

In the second half of this paper, as applications of the above results, in §6, we deal with a uniformization theorem of two dimensional projective algebraic manifolds, which supplements Griffiths's uniformization theorem in [5]. In §7, we study a compactification of two dimensional Stein manifolds with holomorphic fibration of type (g, n) with $2g-2+n>0$ over finite Riemann surfaces and discuss a condition for an analytic automorphism of C^2 to be a polynomial map.

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1. Terminologies and notations. 1. Let G be a finitely generated Fuchsian group, acting on the upper half-plane U, of the first kind with no elliptic elements such that the quotient space $S=U/G$ is a finite Riemann surface of type (g, n) with $2g-2+n>0$. Denote by $Q_{\mathrm{norm}}(G)$ the set of all quasiconformal automorphisms w of U such that $w(0)=0$, $w(1)=1$, $w(\infty)=\infty$ and $wGw^{-1}\subset SL'(2; \boldsymbol{R})$, where $SL'(2; \boldsymbol{R})$ is the set of all real Möbius transformations. Two elements w_1 and w_2 of $Q_{\mathrm{norm}}(G)$ are called equivalent if $w_1=w_2$ on the real axis. The Teichmüller space $T(G)$ of G is the set of all equivalence classes obtained by classifing $Q_{\mathrm{norm}}(G)$ by the above equivalence relation. We denote by $[w]$ the equivalence class represented by an element w of $Q_{\mathrm{norm}}(G)$. Let $B_2(L, G)$ be the space of all bounded holomorphic quadratic differentials for G on the lower half-plane L, that is, the set of all holomorphic functions ϕ on L such that

$$\phi(g(z))g'(z)^2=\phi(z)$$

for every z of L and for every g of G, and such that its norm

$$||\phi||=\sup_{z\in L}(\operatorname{Im} z)^2|\phi(z)|$$

is finite. This space $B_2(L, G)$ is a $(3g-3+n)$ dimensional complex vector space. For any element w_μ of $Q_{\mathrm{norm}}(G)$ with a Beltrami coefficient μ on U, there is a unique quasiconformal automorphism w^μ of the Riemann sphere $\hat{C}$ with $w(0)=0$, $w(1)=1$, $w(\infty)=\infty$ such that w^μ has the Beltrami coefficient μ on U and is conformal on L. Denote by ϕ_μ the Schwarzian derivative of w^μ on L. Then the Teichmüller space $T(G)$ is canonically identified with a holomorphically convex bounded domain of $B_2(L, G)$ by the mapping sending $[w_\mu]$ into ϕ_μ. We associate with every ϕ of $B_2(L, G)$ a uniquely determined solution $W_\phi(z)=\eta_1(z)/\eta_2(z)$ of the Schwarzian differential equation on L

$$(w''/w')'-\frac{1}{2}(w''/w')^2=\phi\ ,$$

where η_1 and η_2 are solutions of the linear differential equation on L

$$2\eta''(z) + \phi(z)\eta(z) = 0$$

normalized by the conditions $\eta_1 = \eta_2' = 1$ and $\eta_1' = \eta_2 = 0$ at $z = -i$. For every point ϕ of $T(G)$, the mapping W_ϕ defined as above is conformal on L and has a quasiconformal extension of $\hat{C}$ onto itself, which is denoted by the same notation. If we set $G_\phi = W_\phi \circ G \circ W_\phi^{-1}$ and $D_\phi = W_\phi(U)$, then G_ϕ is a quasi-Fuchsian group and the definitions are legitimate since D_ϕ is the complement of the closure of $W_\phi(L)$ and since $W_\phi | L$ depends only on ϕ. Koebe's one-quarter theorem shows $D_\phi \subset (|w| < 2)$ for every ϕ of $T(G)$.

2. Let $(\mathscr{S}, \pi, D^*)$ be a holomorphic family of Riemann surfaces of type (g, n) with $2g - 2 + n > 0$. For a fixed point t_0 of D^*, let ψ be a quadratic differential in $T(G)$ such that the quotient space D_ψ/G_ψ is conformally equivalent to the fibre S_{t_0} of $\mathscr{S}$ over t_0. In a sufficiently small neighborhood δ of t_0 in D^*, there is a uniquely determined holomorphic mapping Φ_0 of δ into $T(G)$ with $\Phi_0(t_0) = \psi$ such that $D_{\Phi_0(t)}/G_{\Phi_0(t)}$ is conformally equivalent to S_t for every t of δ. Moreover, Φ_0 can be continued analytically along every path in D^*. Thus we have an analytic mapping Φ of D^* into $T(G)$, but this is not necessarily single-valued.

Let $\tilde{D}$ be the unit disc $|\tau| < 1$ in the complex τ-plane. We regard $\tilde{D}$ as a universal covering space of D^* with the covering map $p(\tau) = \exp[2\pi(\tau + 1)/(\tau - 1)]$, whose covering transformation group is generated by the transformation

$$\gamma(\tau) = \frac{(1 - 2i)\tau - 1}{\tau - (1 + 2i)} .$$

Take a point τ_0 of D with $p(\tau_0) = t_0$. Since $\tilde{D}$ is simply connected, there is a single-valued analytic mapping $\tilde{\Phi}$ of $\tilde{D}$ into $T(G)$ with $\tilde{\Phi}(\tau_0) = \psi$ such that the quotient space $D_{\tilde{\Phi}(\tau)}/G_{\tilde{\Phi}(\tau)}$ is conformally equivalent to $S_{p(\tau)}$ for every τ of $\tilde{D}$. We call $\tilde{\Phi}$ a representation of $(\mathscr{S}, \pi, D^*)$ into $T(G)$. Moreover, there exists an element $\mathscr{M}$ of the modular group $\mathrm{Mod}(G)$ of G with $\tilde{\Phi}(\gamma(\tau)) = \mathscr{M}(\tilde{\Phi}(\tau))$ for every τ of $\tilde{D}$. This element $\mathscr{M}$ is called the homotopic monodromy of $(\mathscr{S}, \pi, D^*)$ with respect to $\tilde{\Phi}$.

3. Let (S, f, S') be a marked Riemann surface, that is, $S = U/G$ is as before, S' is an arbitrary finite Riemann surface of type (g, n) and f is a quasiconformal mapping of S onto S'. Two marked Riemann surfaces (S, f, S') and (S, g, S'') are said to be equivalent if there exists a conformal mapping h of S' onto S'' such that $g^{-1} \circ h \circ f$ is homotopic to

the identity mapping of S. By using this equivalence relation, we classify all the marked Riemann surfaces (S, f, S') for a given S and we denote by $[S, f, S']$ the equivalence class represented by a marked Riemann surface (S, f, S'). We call the space $T(S)$ of all equivalence classes $[S, f, S']$ the Teichmüller space of S. For each element $[S, f, S']$ of $T(S)$, we can uniquely determine the element $[w]$ of $T(G)$ such that a quasiconformal mapping w of $Q_{\text{norm}}(G)$ gives the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{w} & U \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & S' . \end{array}$$

Thus the point $[S, f, S']$ of $T(S)$ uniquely determines a quadratic differential ϕ of $T(G)$. This mapping sending $[S, f, S']$ into ϕ is a biholomorphic mapping of $T(S)$ onto $T(G)$. Therefore, we may identify $T(S)$ with $T(G)$.

2. Deformation spaces of Riemann surfaces with nodes. 1. In order to discuss the behavior of $\tilde{\Phi}(\tau)$ as τ tends to 1, we need the deformation spaces and moduli spaces of Riemann surfaces with nodes and punctures. In this section, we briefly explain them along Bers's line. (See [2] and [3].)

A Riemann surface S' with nodes is a connected complex space, every point P of which has either a fundamental system of neighborhoods isomorphic to the unit disc $\{z \in \boldsymbol{C} \mid |z| < 1\}$ or a fundamental system of neighborhoods isomorphic to the set $\{(z_1, z_2) \in \boldsymbol{C}^2 \mid z_1 z_2 = 0, |z_1| < 1 \text{ and } |z_2| < 1\}$. In the latter case, P is called a node. Each component of the complement of the set of nodes is called a part of S' and S' is called stable if every part of S' has the upper half-plane as its universal covering space.

In the following, by a Riemann surface S' of finite type we mean a stable Riemann surface with or without nodes such that S' is compact except for n punctures, where n is a non-negative integer. A puncture can never be at a node. In this case, S' has finitely many parts $\Sigma_1, \cdots, \Sigma_r$, and each part Σ_j is of genus g_j and is compact except for n_j punctures with $2g_j - 2 + n_j > 0$ and $\sum_{j=1}^{r} n_j = 2k + n$, where k is the number of nodes of S' and n is the number of punctures of S'. The total Poincaré area of S' equals $A = 2\pi \sum_{j=1}^{r} (2g_j - 2 + n_j)$. The genus g of S' is defined by the relation $A = 2\pi(2g - 2 + n)$. The pair (g, n) is called the type of S'.

2. Let S_0' and S' be two Riemann surfaces of the same type (g, n).

A continuous surjection $\alpha: S' \to S_0'$ is called a deformation if the inverse image of every node of S_0' is either a node of S' or a Jordan curve on a part of S', if, for every part Σ of S_0', the restriction $\alpha^{-1}|\Sigma$ is an orientation-preserving homeomorphism of Σ onto $\alpha^{-1}(\Sigma)$, and if every puncture of S' corresponds to a puncture of S_0' under α.

Once and for all we choose an integer ν (>3) which will be fixed throughout the following discussion.

Two deformations $\alpha: S' \to S_0'$ and $\beta: S'' \to S_0'$ are called equivalent to each other if there exists a homeomorphism f of S' onto S'' with $\alpha = \beta \circ f$ such that f is homotopic to a product of ν-th powers of Dehn twists about Jordan curves on S' mapped by α into nodes, followed by an isomorphism. Denote by $\langle\alpha\rangle = \langle S', \alpha, S_0'\rangle$ the equivalence class determined by a deformation $\alpha: S' \to S_0'$. For a given Riemann surface S_0' of type (g, n), the deformation space $X(S_0')$ is the set of equivalence classes $\langle\alpha\rangle$ obtained from all deformations $\alpha: S' \to S_0'$.

3. In order to reduce the case $n > 0$ to the case $n = 0$, we shall associate with every Riemann surface S' of type (g, n) a Riemann surface $a(S')$ of type $(g+g_0, 0)$, that is, a compact Riemann surface of genus $g+g_0$ with nodes, where $g_0 = (4g+8)n$. This is accomplished by attaching to each puncture P_j on S' a "tagging" Riemann surface V_j. For a deformation $\alpha: S' \to S_0'$, we denote by $a(\alpha)$ the unique deformation of $a(S')$ onto $a(S_0')$ such that $a(\alpha)|S' = \alpha$ and that $a(\alpha)|V_j$ is an isomorphism for $j = 1, \cdots, n$. We observe $a(\mathrm{id}) = \mathrm{id}$ and $a(\alpha\circ\beta) = a(\alpha)\circ a(\beta)$. Furthermore, we see for a fixed type (g, n) that two Riemann surfaces $a(S')$ and $a(S'')$ are isomorphic if and only if S' and S'' are isomorphic and that two deformations $a(\alpha)$ and $a(\beta)$ are equivalent if and only if α and β are equivalent. Every deformation $a(S') \to a(S_0')$ is equivalent to one of the forms $a(\alpha)$. Therefore, the deformation space $X(S_0')$ of a Riemann surface S_0' of type (g, n) is identified with the intersection of $n + k_0$ distinguished subsets of $X(a(S_0'))$, where $k_0 = (12g+22)n$ is the number of nodes on $V_1, \cdots, V_n$, and for a given deformation $g: S_1' \to S_0'$, the allowable mapping $g_*: X(S_1') \to X(S_0')$ sending $\langle S', \alpha, S_1'\rangle$ into $\langle S', g\circ\alpha, S_0'\rangle$ is the restriction of the allowable mapping $a(g)_*: X(a(S_1')) \to X(a(S_0'))$. Thus we see that the case $n > 0$ is reduced to the case $n = 0$.

4. We assume that a Riemann surface S_0' of finite type (g, n) has r parts $\Sigma_1, \cdots, \Sigma_r$ and k nodes and that each part Σ_j has genus g_j and n_j punctures. The associated compact Riemann surface $a(S_0')$ has genus $g+g_0$, $r+r_0$ parts $\Sigma_1, \cdots, \Sigma_{r+r_0}$ and $k+n+k_0$ nodes $P_1, \cdots, P_{k+n+k_0}$, where $g_0 = (4g+8)n$, $r_0 = (8g+15)n$ and $k_0 = (12g+22)n$.

In order to construct the parametrization space $X'(a(S_0'))$ consisting of all Riemann surfaces with nodes which can be deformed to a given Riemann surface $a(S_0')$, we choose $r+r_0$ Fuchsian groups $H_1, \cdots, H_{r+r_0}$ acting on discs $\Delta_1, \cdots, \Delta_{r+r_0}$ with disjoint closures as follows: i) Each H_j, $j=1, \cdots, r$, has n_j non-conjugate maximal elliptic cyclic subgroups with the same fixed order ν (>3) and, for Δ_j' obtained from Δ_j by removing all elliptic fixed points, the Riemann surface Δ_j'/H_j is conformally equivalent to Σ_j, ii) each H_j, $j=r+1, \cdots, r+r_0$, has 3 non-conjugate maximal elliptic cyclic subgroups with the same fixed order ν and the Riemann surface Δ_j'/H_j is of type $(0, 3)$ and iii) $H_1, \cdots, H_{r+r_0}$ generate a Kleinian group H which is their product and has an invariant component Δ_0.

We assign to each node P_i of $a(S_0')$ two non-conjugate maximal elliptic cyclic subgroups Γ_i', Γ_i'' of H so that, if P_i joins Σ_j to Σ_l, then $\Gamma_i' \subset H_j$ and $\Gamma_i'' \subset H_l$. Two elliptic fixed points not contained in Δ_0 are called related if they are fixed under elliptic cyclic subgroups conjugate to either Γ_i' or Γ_i''. These Γ_i are chosen in such a way that the union of Δ_j/H_j, with the images of any two related elliptic fixed points identified, is isomorphic to $a(S_0')$.

A point $\langle S', \alpha, S_0' \rangle$ in $X(S_0')$ is represented by a point $(\xi, \eta) = (\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_k, 0, \cdots, 0)$ in $X'(a(S_0'))$, where every ξ_j, $j=1, \cdots, r$, is a point in the Teichmüller space $T(H_j)$ and every η_i, $i=1, \cdots, k$, is a complex number. Let $H(\xi, \eta)$ be the Kleinian group determined by the point (ξ, η) in $X'(a(S_0'))$. Denote by $\Omega(\xi, \eta)$ the part of the region of discontinuity of $H(\xi, \eta)$ corresponding to S' and denote by $\Omega'(\xi, \eta)$ the complement, in $\Omega(\xi, \eta)$, of all elliptic fixed points. Then the quotient space $\Omega(\xi, \eta)/H(\xi, \eta)$, with the images of any two related elliptic fixed points identified, is a Riemann surface $S_{\xi,\eta}$ with nodes of type (g, n) and is isomorphic to S'. This surface $S_{\xi,\eta}$ is equipped with a canonical deformation $\alpha_{\xi,\eta}$ with $\langle S_{\xi,\eta}, \alpha_{\xi,\eta}, S_0' \rangle = \langle S', \alpha, S_0' \rangle$, up to equivalence.

3. Holomorphic families of Riemann surfaces. Riemann's moduli space $R(g, n)$ is the set of isomorphism classes $[S']$ of all Riemann surfaces S' without nodes and with signature $(g, n; \infty, \cdots, \infty)$. The moduli space $M(g, n)$ is the set of isomorphism classes $[S']$ of all Riemmann surfaces S' with nodes of signature $(g, n; \infty, \cdots, \infty)$. We recall the fact stated in §2 and §3 that the case $n>0$ can be reduced to the case $n=0$. Then we see that $R(g, n)$ can be regarded as a Zariski-open subset of $M(g, n)$.

For a holomorphic family $(\mathscr{S}, \pi, D^*)$ of Riemann surfaces of type (g, n) with $2g-2+n>0$, the holomorphic mapping $J: D^* \rightarrow R(g, n)$ sending t into $[S_t]$ has a holomorphic extension $\hat{J}: D \rightarrow M(g, n)$. This fact can be proved by a reasoning similar to that in the proof of Lemma 1 in [6].

Now let $S = U/G$ be the fixed Riemann surface as before. If we set $\hat{J}(0) = [S_0]$ for a Riemann surface S_0 with or without nodes of signature $(g, n; \infty, \cdots, \infty)$, then there exists a deformation $\alpha : S \to S_0$ such that $\langle S_\tau, \alpha \circ f_\tau^{-1}, S_0 \rangle$ converges uniformly to $\langle \mathrm{id} \rangle$ in $X(S_0)$ as τ tends to 1 through any cusp region Δ at $\tau = 1$ in $\tilde{D}$, where $[S, f_\tau, S_\tau]$ is the point of the Teichmüller space $T(S)$ of S and corresponds to the point $\tilde{\Phi}(\tau)$ in $T(G)$.

Thus we obtain the following theorem, whose proof is similar to that of Theorem 1 in [6] and may be omitted.

THEOREM 1. *Let $(\mathscr{S}, \pi, D^*)$ be a holomorphic family of Riemann surfaces of type (g, n) with $2g - 2 + n > 0$. Then there is an element ϕ_1 in the closure of $T(G)$ in $B_2(L, G)$ such that a representation $\tilde{\Phi}(\tau)$ of $(\mathscr{S}, \pi, D^*)$ into $T(G)$ converges to ϕ_1 uniformly as τ tends to 1 through any cusp region Δ* at $\tau = 1$ *in $\tilde{D}$. The homotopic monodromy $\mathscr{M}$* of *$(\mathscr{S}, \pi, D^*)$ with respect to $\tilde{\Phi}$ is of finite order if and only if $\phi_1 \in T(G)$, and is of infinite order if and only if $\phi_1 \in \partial T(G)$, where $\partial T(G)$ is the boundary of $T(G)$ in $B_2(L, G)$. In the latter case, the boundary group G_1 corresponding to $\phi_1 \in \partial T(G)$ is a regular b-group.*

REMARK. In Theorem 1, let $D_1 = \Omega(G_1) - \Delta(G_1)$, $\Omega(G_1)$ be the region of discontinuity of G_1 and let $\Delta(G_1)$ be the invariant component of G_1. Let S_0 be a Riemann surface with nodes of signature $(g, n; \infty, \cdots, \infty)$ with $\hat{J}(0) = [S_0]$ obtained in the beginning of this section. Then the quotient space $D_1 \cup \{\text{fixed points of accidental parabolic elements of } G_1\}/G_1$ is isomorphic to S_0.

4. Completion of holomorphic families of Riemann surfaces. 1. In order to canonically construct the completion $(\hat{\mathscr{S}}, \hat{\pi}, D)$ of a holomorphic family $(\mathscr{S}, \pi, D^*)$ of Riemann surfaces of type (g, n) with $2g - 2 + n > 0$, we need some preliminaries. We use the previous notations. Let $[S, f_\tau, \tilde{S}_\tau]$ be the point of the Teichmüller space $T(S)$ of the fixed Riemann surface $S = U/G$ corresponding to the point $\tilde{\Phi}(\tau)$ of the Teichmüller space $T(G)$ of G for each τ of $\tilde{D}$. Let S_0 be a Riemann surface with or without nodes of signature $(g, n; \infty, \cdots, \infty)$ obtained in §3. The point $\langle \tilde{S}_\tau, \alpha \circ f_\tau^{-1}, S_0 \rangle$ in $X(S_0)$ is represented by a point $(\xi(\tau), \eta(\tau))$ of the parametrization space $X'(a(S_0))$ for each τ of $\tilde{D}$ and the point $(\xi(\tau), \eta(\tau))$ determines the point $\langle S_{\xi(\tau),\eta(\tau)}, \alpha_{\xi(\tau),\eta(\tau)}, S_0 \rangle$ of $X(S_0)$. We denote by $[\Sigma_j, F_{j,\tau}, \Sigma_{j,\tau}]$ the point of the Teichmüller space $T(\Sigma_j)$ of Σ_j corresponding to the point $\xi_j(\tau)$ in $T(H_j)$ for $j = 1, \cdots, r$. Let $\tilde{H}_j$, $j = 1, \cdots, r$, be the Fuchsian group determined by the universal covering of Σ_j. We regard $[\Sigma_j, F_{j,\tau}, \Sigma_{j,\tau}]$

as a point $\phi_{j,\tau}$ of the Teichmüller space $T(\tilde{H}_j)$ of $\tilde{H}_j$. As was stated in §3, $\langle a(\tilde{S}_\tau), a(\alpha\circ f_\tau^{-1}), a(S_0)\rangle$ converges uniformly to $\langle \mathrm{id}\rangle$ in $X(a(S_0))$ as τ tends to 1 through any cusp region Δ at $\tau=1$ in $\tilde{D}$. Hence we see that holomorphic quadratic differentials $\phi_{j,\tau}$ in $T(\tilde{H}_j)$ converge to zero as τ tends to 1 in Δ. Since $\langle \tilde{S}_\tau, \alpha\circ f_\tau^{-1}, S_0\rangle$ is equal to $\langle S_{\xi(\tau),\eta(\tau)}, \alpha_{\xi(\tau),\eta(\tau)}, S_0\rangle$, by definition, there exists a homeomorphism $g_\tau : S_{\xi(\tau),\eta(\tau)} \to \tilde{S}_\tau$ with the relation $\alpha_{\xi(\tau),\eta(\tau)} = \alpha\circ f_\tau^{-1}\circ g_\tau$ such that g_τ is homotopic to a product d_τ of ν-th powers of Dehn twists about Jordan curves mapped by $\alpha_{\xi(\tau),\eta(\tau)}$ into nodes, followed by an isomorphism. Thus we see that $[S, f_\tau, \tilde{S}_\tau] = [S, d_\tau\circ g_\tau^{-1}\circ f_\tau, S_{\xi(\tau),\eta(\tau)}]$ in $T(S)$.

Let $P_1, \cdots, P_k$ be the nodes of S_0 and let δ_j be a sufficiently small neighborhood of each P_j, and let $\delta = \delta_1 \cup \cdots \cup \delta_k$. We may assume that the deformations α and $\alpha_{\xi(\tau),\eta(\tau)}$ are locally quasiconformal mappings on $S - \bigcup_{j=1}^k \alpha^{-1}(P_j)$ and $S_{\xi(\tau),\eta(\tau)} - \bigcup_{j=1}^k \alpha_{\xi(\tau),\eta(\tau)}^{-1}(P_j)$, respectively. There exists a quasiconformal mapping $h_{\delta,\tau} : S \to S_{\xi(\tau),\eta(\tau)}$ with $[S, f_\tau, \tilde{S}_\tau] = [S, h_{\delta,\tau}, S_{\xi(\tau),\eta(\tau)}]$ and with $h_{\delta,\tau} = d_\tau\circ g_\tau^{-1}\circ f_\tau$ on $S - \alpha^{-1}(\delta)$. Let $\pi_0 : U \to U/G = S$ and $\tilde{\pi}_j : U_j \to U_j/\tilde{H}_j = \Sigma_j$ for each j be the natural projections, where U_j is the upper half z_j-plane. Let $\Delta_{\delta,j}$ be a connected component of $\pi_0^{-1}\circ\alpha^{-1}(\Sigma_j - \delta)$ and let $\tilde{\Delta}_{\delta,j} = \pi_j^{-1}(\Sigma_j - \delta)$. We can lift α to a quasiconformal mapping $A_j : \Delta_{\delta,j} \to \tilde{\Delta}_{\delta,j}$ such that A_j conjugates the subgroup G_j' of G corresponding to the fundamental group of $S_j' = \alpha^{-1}(\Sigma_j)$ into $\tilde{H}_j$.

Let $W_{\delta,\tau} = W_{\phi_\tau}$ be the quasiconformal automorphism of $\hat{C}$ defined in §1 by $\phi_\tau = [S, h_{\delta,\tau}, S_{\xi(\tau),\eta(\tau)}]$, which is induced by $h_{\delta,\tau}$ on the upper half-plane U and is conformal on the lower half-plane L. By the construction, we see that $W_{\delta,\tau} = W_{\delta',\tau}$ on $\Delta_{\delta,j}$ for any δ and δ' with $\delta' \subset \delta$. So we may abbreviate $W_{\delta,\tau}$ to W_τ. We also abbreviate $\Delta_{\delta,j}$ and $\tilde{\Delta}_{\delta,j}$ to Δ_j and $\tilde{\Delta}_j$, respectively. Similarly, let $W_{j,\tau} = W_{\phi_{j,\tau}}$ be the quasiconformal automorphism of $\hat{C}$ defined by $[\Sigma_j, F_{j,\tau}, \Sigma_{j,\tau}]$, which is induced by $F_{j,\tau}$ on the upper half-plane U_j and is conformal on the lower half-plane L_j. We may assume that $W_{j,\tau}(z_j)$ is continuous for (τ, z_j) in $\tilde{D}\times U_j$. Then the mapping

$$V_{j,\tau} = W_\tau\circ A_j^{-1}\circ W_{j,\tau}^{-1} : W_{j,\tau}(\tilde{\Delta}_j) \longrightarrow W_\tau(\Delta_j)$$

is conformal, because W_τ and $W_{j,\tau}\circ A_j$ have the same Beltrami coefficients on Δ_j. Since $\phi_{j,\tau}\to 0$ in $T(\tilde{H}_j)$ as τ tends to 1 in a cusp region Δ at $\tau=1$, we may assume that $W_{j,\tau}$ converges uniformly to the Möbius transformation $W_{j,1}$ sending z_j into $1/(z_j+i)$ on any compact subset of U_j as τ tends to 1 in Δ. Thus, noting that $\{V_{j,\tau}\}_{\tau\in\Delta}$ is a normal family, we see that $\{W_\tau\}_{\tau\in\Delta}$ is a normal family on $U_\delta = U - \pi_0^{-1}\circ\alpha^{-1}(\delta)$ for each δ. By contracting each δ_j to the node P_j, we can prove that W_τ converges uniformly on any compact subset of $U_0 = U - \pi_0^{-1} \circ \alpha^{-1}(P_1 \cup \cdots \cup P_k)$ to a locally quasiconformal mapping defined on U_0 as τ tends to 1 in Δ. In fact, since $\{W_\tau\}_{\tau \in \Delta}$ is a normal family on $U_\delta \cup L$ for each δ, there is a sequence $\{W_{\tau_n}\}_{n=1}^\infty$ of $\{W_\tau\}_{\tau \in \Delta}$ with $\tau_n \to 1$ as n tends to ∞ such that W_{τ_n} converges uniformly on any compact subset of $U_0 \cup L$ to a locally quasiconformal mapping W_1 defined on $U_0 \cup L$ as $n \to \infty$. The Schwarzian derivative of W_1 on L is equal to ϕ_1 in Theorem 1. Let G_1 be a quasi-Fuchsian group or a regular b-group corresponding to ϕ_1. If W_1' is the limit of another sequence $\{W_{\tau_n'}\}_{n=1}^\infty$ with the same properties as $\{W_{\tau_n}\}_{n=1}^\infty$, then $W_1' \circ W_1^{-1}$ is a conformal mapping of the region of discontinuity $\Omega(G_1)$ of G_1 onto itself such that $W_1' \circ W_1^{-1}$ conjugates G_1 into itself. Hence, by a theorem due to Abikoff and Marden, the mapping $W_1' \circ W_1^{-1}$ can be extended to a conformal automorphism of $\hat{C}$. On the other hand, by Theorem 1, we have $W_1 = W_1'$ on L, which implies that $W_1' \circ W_1^{-1}$ is the identity map of $\hat{C}$. Thus we have $W_1 = W_1'$ on $U_0 \cup L$. This shows that $\{W_\tau\}_{\tau \in \Delta}$ converges uniformly on any compact subset of U_0 to the locally quasiconformal mapping W_1 defined on U_0 as τ tends to 1 in a cusp region Δ at $\tau = 1$.

2. Now, for every $\tau \in \tilde{D}$, we set $\phi_\tau = \tilde{\Phi}(\tau)$ and denote by G_τ the quasi-Fuchsian group corresponding to ϕ_τ. Let $\Omega(G_\tau)$ be the region of discontinuity of G_τ and let $\Delta(G_\tau) = W_{\phi_\tau}(L)$. We set $D_\tau = \Omega(G_\tau) - \Delta(G_\tau)$. Similarly, we set $D_1 = \Omega(G_1) - \Delta(G_1)$, where $G_1 = G_{\phi_1}$ is a quasi-Fuchsian group or a regular b-group corresponding to the point ϕ_1 in $\overline{T(G)}$, whose existence is guranteed by Theorem 1.

Let $\mathscr{P}_\tau$ be the set of all parabolic fixed points in ∂D_τ of G_τ for each τ of $\tilde{D}$. Let $\mathscr{P}_1'$ be the set of all non-accidental parabolic fixed points in ∂D_1 of G_1 and let $\mathscr{P}_1''$ be the set of all accidental parabolic fixed points in ∂D_1 of G_1. We set $\tilde{D}_\tau = D_\tau \cup \mathscr{P}_\tau$ for $\tau \in \tilde{D}$, $\mathscr{P}_1 = \mathscr{P}_1' \cup \mathscr{P}_1''$ and $\tilde{D}_1 = D_1 \cup \mathscr{P}_1$. Finally, we set

$$
\begin{aligned}
\mathscr{D} &= \{(\tau, w) \mid \tau \in \tilde{D},\ w \in D_\tau\}\ , \\
\tilde{\mathscr{D}} &= \{(\tau, w) \mid \tau \in \tilde{D},\ w \in \tilde{D}_\tau\}\ , \\
\mathscr{D}_1 &= \{(1, w) \mid w \in D_1\}\ , \\
\tilde{\mathscr{D}}_1 &= \{(1, w) \mid w \in \tilde{D}_1\}\ , \quad \text{and} \\
\hat{\mathscr{D}} &= \tilde{\mathscr{D}} \cup \tilde{\mathscr{D}}_1\ .
\end{aligned}
$$

We will introduce a topology on $\hat{\mathscr{D}}$. Let us define its fundamental system of neighborhoods as follows:

i) If $a = (\tau_0, w_0)$ is in $\mathscr{D}$, then we set $z_0 = W_{\tau_0}^{-1}(w_0)$ and take a disc ε with the center τ_0 in $\tilde{D}$ and a disc K with the center z_0 in the upper half-plane U. The set $\{(\tau, w) \mid \tau \in \varepsilon,\ w \in W_\tau(K)\}$ is an element of the fundamental system $\mathscr{U}_a$ of neighborhoods at a.

ii) If $a=(\tau_0, w_0)$ is in $\tilde{\mathscr{D}}-\mathscr{D}$, then we set $z_0=W_{\tau_0}^{-1}(w_0)$ and take a disc ε with the center τ_0 in $\tilde{D}$. Take a horocycle C at z_0 in U, that is, a Euclidean circle in $\bar{U}$ tangent to the real line at z_0. We set $K=(\operatorname{Int} C)\cup\{z_0\}$, where $\operatorname{Int} C$ denotes the domain bounded by the horocycle C. The set $\{(\tau, w)\,|\,\tau\in\varepsilon,\ w\in W_\tau(K)\}$ is an element of $\mathscr{U}_a$.

From now on, we set $\mathscr{K}=(\operatorname{Int}\mathscr{C})\cup\{1\}$, where $\mathscr{C}$ is a horocycle at $\tau=1$ in $\tilde{D}$.

iii) If $a=(1, w_0)$ is in $\mathscr{D}_1$, then we set $z_0=W_1^{-1}(w_0)$ and take a disc K with the center z_0 in U. The set $\{(\tau, w)\,|\,\tau\in\mathscr{K},\ w\in W_\tau(K)\}$ is an element of $\mathscr{U}_a$.

iv) If $a=(1, w_0)$ with $w_0\in\mathscr{P}_1'$, then we set $z_0=W_1^{-1}(w_0)$ and take a horocycle C at z_0 in U and set $K=(\operatorname{Int} C)\cup\{z_0\}$. In this case, it is observed that W_1 and W_1^{-1} can be continuously extended to the points z_0 and w_0, respectively. The set $\{(\tau, w)\,|\,\tau\in\mathscr{K},\ w\in W_\tau(K)\}$ is an element of $\mathscr{U}_a$.

v) If $a=(1, w_0)$ with $w_0\in\mathscr{P}_1''$, then there exists a node P_0 on S_0 corresponding to w_0. Let δ_0 be a sufficiently small neighborhood of P_0 in S_0 and let K be the connected component of $\pi_0^{-1}\circ\alpha^{-1}(\delta_0)$ such that the closure of $W_1(K\cap U_0)$ contains the point w_0. Then the set

$$\{(\tau, w)\,|\,\tau\in\mathscr{K},\quad w\in W_\tau(K)\}\cup\{a\}$$

is an element of $\mathscr{U}_a$.

We can prove that $\{\mathscr{U}_a\}_{a\in\hat{\mathscr{D}}}$ defined as above satisfies the axioms for a fundamental system of neighborhoods and induces a Hausdorff topology on $\hat{\mathscr{D}}$. In the following, we assume that $\hat{\mathscr{D}}$ is equipped with this Hausdorff topology.

3. Let $N(G)$ be the set of all quasiconformal automorphisms ω of U with $\omega G\omega^{-1}=G$, and let Q_0 be the set of all quasiconformal automorphisms of U which coincide with the identity on the real axis. The modular group $\operatorname{Mod}(G)$ of G is defined as the factor group

$$\operatorname{Mod}(G)=(N(G)/(N(G)\cap Q_0))/G\,,$$

wherc every element of G is regarded as an element of $N(G)$. Every element $\langle\omega\rangle$ of $\operatorname{Mod}(G)$ defined by ω of $N(G)$ induces an automorphism of $T(G)$ sending $[w]$ into $\langle\omega\rangle([w])=[\omega_*(w)]$, where $w\in Q_{\text{norm}}(G)$ and $\omega_*(w)=\lambda\circ w\circ\omega^{-1}\in Q_{\text{norm}}(G)$ with $\lambda\in SL'(2;\boldsymbol{R})$.

Let ω_0 be an element of $N(G)$ inducing the homotopic monodromy $\mathscr{M}$ of $(\mathscr{S}, \pi, D^*)$ with respect to the representation $\tilde{\Phi}$, that is, $\langle\omega_0\rangle=\mathscr{M}$. We set $\omega=g\circ\omega_0^n$ for $g\in G$ and $n\in\boldsymbol{Z}$. Let $w_\tau\in Q_{\text{norm}}(G)$ for $\tau\in\tilde{D}$ be the quasiconformal automorphism of U defined by $[S, h_{\delta,\tau}, S_{\xi(\tau),\eta(\tau)}]$ as in §4.1. We set $\tilde{w}_\tau = \lambda_\tau \circ w_\tau \circ \omega^{-1}$, where $\lambda_\tau \in SL'(2; \boldsymbol{R})$ is taken in such a way that $\tilde{w}_\tau \in Q_{\mathrm{norm}}(G)$. Then $[\tilde{w}_\tau]$ is identical with $[w_{\gamma^n(\tau)}]$ in $T(G)$, where γ is the generator of the covering transformation group of the universal covering $p : \tilde{D} \to D^*$ in §1.2. Let W_τ be the quasiconformal automorphism of $\hat{\boldsymbol{C}}$ which has the same Beltrami coefficient as that of w_τ on U and which is conformal on L such that

$$W_\tau(z) = \frac{1}{z+i} + O(|z+i|)$$

as z tends to $z = -i$. Similarly, let $\tilde{W}_\tau$ be the quasiconformal automorphism of $\hat{\boldsymbol{C}}$ induced by $\tilde{w}_\tau$. We set

$$H_\tau(w) = \tilde{W}_\tau \circ \omega \circ W_\tau^{-1}(w) .$$

Then H_τ is a conformal bijection of D_τ onto $D_{\gamma^n(\tau)}$. Denote by g_n the analytic automorphism of $\mathscr{D}$ sending (τ, w) into $(\gamma^n(\tau), H_\tau(w))$. The set $\mathscr{G} = \{g_n \mid g \in G, n \in \boldsymbol{Z}\}$ is a discrete and fixed-point-free subgroup of the analytic automorphism group of $\mathscr{D}$.

4. Here we will show that every element g_n of $\mathscr{G}$ can be extended naturally to a homeomorphism $\hat{g}_n$ of $\hat{\mathscr{D}}$ onto itself. First we observe that g_n can be extended naturally to a homeomorphism $\tilde{g}_n$ of $\tilde{\mathscr{D}}$ onto itself. By the same reasoning as in §4.1, we can show that $\tilde{W}_\tau$ converges uniformly on any compact subset of $\tilde{U}_0 = \omega(U_0)$ to a locally quasiconformal mapping $\tilde{W}_1$ defined on $\tilde{U}_0$ as τ tends to 1 through any cusp region Δ at $\tau = 1$ in $\tilde{D}$. Hence H_τ converges uniformly on any compact subset of D_1 to a conformal mapping H_1 of D_1 onto itself as τ tends to 1 in Δ. Since G_1 is a quasi-Fuchsian group or a regular b-group, every component of D_1 is bounded by a quasi-circle. Therefore, by Carathéodory's theorem, the conformal mapping H_1 of D_1 onto itself can be extended to a homeomorphism $\hat{H}_1$ of $\bar{D}_1$ onto itself, where $\bar{D}_1$ is the closure of D_1 in $\hat{\boldsymbol{C}}$. To define the extension $\hat{g}_n$ of g_n, we set $\hat{g}_n = \tilde{g}_n$ on $\tilde{\mathscr{D}}$ and we set $\hat{g}_n(1, w) = (1, \hat{H}_1(w))$ on $\tilde{\mathscr{D}}_1$.

We shall show that $\hat{g}_n$ is continuous on $\hat{\mathscr{D}}$. We will prove the continuity of $\hat{g}_n$ at $a = (1, w_0) \in \hat{\mathscr{D}}_1 - \mathscr{D}_1$ with $w_0 \in \mathscr{P}_1''$, since in the other cases, the proof is similar. We set $\hat{w}_0 = \hat{H}_1(w_0)$ and denote by P_0 and by $\hat{P}_0$ the nodes of S_0 corresponding to w_0 and $\hat{w}_0$, respectively. Assume that $\hat{g}_n$ is not continuous at the point a. Then we can choose an infinite sequence $\{\tau_j\}_{j=1}^\infty$ of points of $\tilde{D}$ with $p(\tau_j) \to 0$ as j tends to ∞, a neighborhood δ of $\hat{P}_0$ in S_0 and an infinite sequence $\{\delta_j\}_{j=1}^\infty$ of neighborhoods of P_0 in S_0 with $\delta_j \supset \delta_{j+1}$ for each j and $\bigcap_{j=1}^\infty \delta_j = \{P_0\}$

such that, if K is the connected component of $\pi_0^{-1}\circ\alpha^{-1}(\delta)$ with the property $\hat{w}_0 \in \overline{W_1(K \cap U_0)}$ and if K_j denotes the connected component of $\pi_0^{-1}\circ\alpha^{-1}(\delta_j)$ for each j with the property $w_0 \in \overline{W_1(K_j \cap U_0)}$, then $W_{\gamma^n(\tau_j)}(K)$ does not include $\tilde{W}_{\tau_j}\circ\omega(K_j)$ for each j, where $\omega = g\circ\omega_0^n$ is as before. Since the mapping $\hat{H}_1$ is continuous on $\bar{D}_1$, we may assume that $\hat{H}_1(W_1(K_j \cap U_0))$ is contained in $W_1(K \cap U_0)$. By the same argument as in §4.1, we can prove that $W_{\gamma^n(\tau_j)}$ converges to W_1 uniformly on any compact subset of U_0 as j tends to ∞, and we can also prove that $\tilde{W}_{\tau_j}$ converges to $\tilde{W}_1$ uniformly on any compact subset of $\omega(U_0)$ as j tends to ∞. Thus we may assume that $W_{\gamma^n(\tau_j)}(K)$ intersects $\tilde{W}_{\tau_j}\circ\omega(K_j)$ for each j. Then there are a point ζ_j on the boundary ∂K of K and a point z_j in K_j such that

$$W_{\gamma^n(\tau_j)}(\zeta_j) = \tilde{W}_{\tau_j}\circ\omega(z_j) = H_{\tau_j}\circ W_{\tau_j}(z_j) .$$

Since $W_{\gamma^n(\tau_j)} = \tilde{W}_{\tau_j}$ on L and since, for the hyperbolic transformation h of G which makes K_j invariant, K is invariant under $h' = \omega\circ h\circ\omega^{-1}$, we may assume that ζ_j converges to a point ζ in $\partial K \cap U$ as j tends to ∞. If δ_0 is a sufficiently small neighborhood of ζ in U, then $H_{\tau_j}^{-1}\circ W_{\gamma^n(\tau_j)}$ converges to $H_1^{-1}\circ W_1$ uniformly on δ_0 as j tends to ∞, which implies that

$$H_{\tau_j}^{-1}\circ W_{\gamma^n(\tau_j)}(\zeta_j) = W_{\tau_j}(z_j)$$

converges to the point $w_0' = H_1^{-1}\circ W_1(\zeta)$ in the region of discontinuity of G_1 as j tends to ∞. On the other hand, $W_{\tau_j}(z_j)$ converges to the point w_0 of the limit set of G_1 as j tends to ∞. Hence we have a contradiction.

As was mentioned already, the argument is similar in the other cases. Therefore, $\hat{g}_n$ is a homeomorphism of $\hat{\mathscr{D}}$ onto itself. Denote by $\hat{\mathscr{G}}$ the group of all such topological automorphisms $\hat{g}_n$ of $\hat{\mathscr{D}}$.

5. Next we introduce a normal complex structure on the quotient space $\hat{\mathscr{S}} = \hat{\mathscr{D}}/\hat{\mathscr{G}}$, which will give the completion $(\hat{\mathscr{S}}, \hat{\pi}, D)$ of $(\mathscr{S}, \pi, D^*)$.

Let f_* be an element of the modular group $\mathrm{Mod}(S)$ of the Teichmüller space $T(S)$ corresponding to the homotopic monodromy $\mathscr{M}$ of $(\mathscr{S}, \pi, D^*)$. As is stated in §3, the mapping $J: D^* \to R(g, n)$ sending t into $[S_t]$ has a holomorphic extension $\hat{J}: D \to M(g, n)$ with $\hat{J}(0) = [S_0]$. Further, as is shown by Bers [2], there exist a neighborhood N of $\langle \mathrm{id} \rangle$ in $X(S_0)$ and the (finite) isotropy group $\Gamma_0(S_0)$, in $\Gamma(S_0)$, of the origin $\langle \mathrm{id} \rangle$ of $X(S_0)$ such that the quotient space $N/\Gamma_0(S_0)$ is a neighborhood of $[S_0]$ in $M(g, n)$. Hence there is a positive integer ρ such that f^ρ is homotopic to a product of ν-th powers of Dehn twists about Jordan curves mapped by α into nodes, where $\alpha: S \to S_0$ is a deformation as in §3. Now we set $E = (|\zeta| < 1)$ and $E^* = E - \{0\}$ in the ζ-plane. Let $\kappa: E \to D$ be the mapping sending ζ into ζ^ρ. We consider the holomorphic family $(\mathscr{S}', \pi', E^*)$

constructed from $(\mathscr{S}, \pi, D^*)$ by the relation $t = \zeta^\rho$. Let S'_ζ be the fibre of $\mathscr{S}'$ over ζ in E^* and let $[S, f'_\zeta, S'_\zeta]$ be a point of $T(S)$ corresponding to a point $\tilde{\Phi}(\tau)$ of $T(G)$ for a certain τ in $\tilde{D}$ with $\zeta^\rho = p(\tau)$. Then the analytic mapping $K: E^* \to X'(a(S_0))$ sending ζ into $\langle a(S'_\zeta), a(\alpha \circ f'^{-1}_\zeta), a(S_0)\rangle$ is single-valued. Thus K has a holomorphic extension $\hat{K}: E \to X'(a(S_0))$ with $\hat{K}(0) = \langle \mathrm{id}\rangle$.

Let $H(\zeta)$ be a Kleinian group determined by the point $\hat{K}(\zeta)$ of $X'(a(S_0))$. Let $\Omega(\zeta)$ be the part of the region of discontinuity of $H(\zeta)$ corresponding to S'_ζ for ζ in E, and let $\Omega'(\zeta)$ be the open set obtained from $\Omega(\zeta)$ by deleting all elliptic fixed points of $H(\zeta)$. We set

$$\mathscr{R} = \{(\zeta, [z]) \mid \zeta \in E^*, \quad [z] \in \Omega'(\zeta)/H(\zeta)\}$$

and

$$\hat{\mathscr{R}} = \{(\zeta, [z]) \mid \zeta \in E^*, \quad [z] \in \Omega(\zeta)/H(\zeta)\} .$$

Then, by definition, $\mathscr{R}$ and $\hat{\mathscr{R}}$ are two dimensional complex manifolds and $\mathscr{R}$ is a Zariski open subset of $\hat{\mathscr{R}}$. If $\Pi : \mathscr{R} \to E^*$ and $\hat{\Pi} : \hat{\mathscr{R}} \to E^*$ are canonical projections, then $(\mathscr{R}, \Pi, E^*)$ is a holomorphic family of Riemann surfaces of type (g, n) and $(\hat{\mathscr{R}}, \hat{\Pi}, E^*)$ is a holomorphic family of compact Riemann surfaces of genus g.

Let $\tilde{\Psi}$ and $\tilde{\Psi}_1$ be the representations of $(\mathscr{S}', \pi', E^*)$ and $(\mathscr{R}, \Pi, E^*)$ into $T(G)$ as in §1.2, respectively. For a certain positive integer $\rho = \rho_0$, we see that $\tilde{\Psi} = \tilde{\Psi}_1$. Hence we may assume that $(\mathscr{S}', \pi', E^*)$ and $(\mathscr{R}, \Pi, E^*)$ have the same homotopic monodromy $\mathscr{M}_1$ for a certain positive integer $\rho = \rho_0$. So we can naturally identify $(\mathscr{S}', \pi', E^*)$ with $(\mathscr{R}, \Pi, E^*)$.

For each ζ of E^*, we set $R_\zeta = \Omega'(\zeta)/H(\zeta) = \Pi^{-1}(\zeta)$ and $\hat{R}_\zeta = \Omega(\zeta)/H(\zeta) = \hat{\Pi}^{-1}(\zeta)$. Let $R'_0 = \Omega'(0)/H(0)$ and let R_0 be the union of R'_0 and the images of all elliptic vertices of $H(0)$ corresponding to the nodes of S_0, where all related elliptic vertices are identified. Then, by the construction, R_0 is isomorphic to S_0. We also set $\hat{R}_0 = \Omega(0)/H(0)$ with the images of all related elliptic vertices of $H(0)$ identified, that is, the compactification of R_0. Finally, we set

$$\mathscr{R}_0 = \hat{\mathscr{R}} \cup \{(0, [z]) \mid [z] \in R_0\}$$

and

$$\hat{\mathscr{R}}_0 = \hat{\mathscr{R}} \cup \{(0, [z]) \mid [z] \in \hat{R}_0\} .$$

Then $\mathscr{R}_0$ is a two dimensional complex manifold. By the same reasoning as in the proof of Theorem 4 in [6], we can prove that $\hat{\mathscr{R}}_0$ has a normal complex structure such that its restriction to $\mathscr{R}_0$ is the same one given on $\mathscr{R}_0$ and $\hat{\mathscr{R}}_0 - \mathscr{R}_0$ is a proper analytic subset of $\hat{\mathscr{R}}_0$. Thus the projection $\hat{\Pi} : \hat{\mathscr{R}} \to E^*$ has a holomorphic extension $\hat{\Pi}_0 : \hat{\mathscr{R}}_0 \to E$.

If we take an element $\omega_0 \in N(G)$ with $\langle \omega_0 \rangle = \mathscr{M}$ and set $\omega_1 = \omega_0^{\rho_0}$, then $\langle \omega_1 \rangle = \mathscr{M}_1$. It should be noted that the action of $\hat{I}_{\rho_0} \in \hat{\mathscr{G}}$ on $\tilde{\mathscr{D}}_1$ is trivial, where $I \in G$ is the identity and $\tilde{\mathscr{D}}_1$ is the one defined in §4.2. Let $\mathscr{G}_1$ be the subgroup of $\mathscr{G}$ generated by ω_1 and G. Denote by $\hat{\mathscr{G}}_1$ the subgroup of $\hat{\mathscr{G}}$ induced by $\mathscr{G}_1$. Let $p_1 : \tilde{D} \to E^*$ be the holomorphic mapping with the relation $p(\tau) = p_1(\tau)^{\rho_0}$, where $p : \tilde{D} \to D^*$ is as in §1.2.

We will canonically construct a biholomorphic mapping $\mathscr{F} : \mathscr{D}/\mathscr{G}_1 \to \mathscr{R}$. For that purpose, let $\pi_0 : U \to S = U/G$ be the canonical projection. For a point τ of $\tilde{D}$ and for a point $[z]$ of D_τ/G_τ, we set

$$F_\tau([z]) = h_{\delta,\tau} \circ \pi_0 \circ W_\tau^{-1}(z) ,$$

where $h_{\delta,\tau}$ and W_τ are those in §4.1. Then the mapping $F_\tau : D_\tau/G_\tau \to R_{p_1(\tau)}$ is conformal. If two points (τ, z) and (τ', z') in $\mathscr{D}$ are equivalent under $\mathscr{G}_1$, then $F_\tau([z]) = F_{\tau'}([z'])$. Thus these mappings $\{F_\tau\}_{\tau \in D^*}$ induce a biholomorphic mapping $\mathscr{F} : \mathscr{D}/\mathscr{G}_1 \to \mathscr{R}$. This mapping $\mathscr{F}$ can be extended to a homeomorphism $\hat{\mathscr{F}} : \hat{\mathscr{D}}/\hat{\mathscr{G}}_1 \to \hat{\mathscr{R}}_0$. In fact, by the argument similar to that in the proof of Lemma 2 in [6], we can construct an analytic isomorphism $\hat{F}_1$ of $(D_1 \cup \{\text{parabolic fixed points on } \partial D_1 \text{ of } G_1\})/G_1$ onto $\hat{R}_0$ by using the mappings $V_{j,\tau} = W_\tau \circ A_j^{-1} \circ W_{j,\tau}^{-1}$ appearing in §4.1. For a point $(1, z)$ in $\tilde{\mathscr{D}}_1$, we set $\hat{\mathscr{F}}([1, z]) = (0, \hat{F}_1([z]))$. Since the action $\hat{I}_{\rho_0}$ on $\tilde{\mathscr{D}}_1$ is trivial, the mapping $\hat{\mathscr{F}}$ is well-defined and is bijective. By the definition of topologies of $\hat{\mathscr{D}}/\hat{\mathscr{G}}_1$ and of $\hat{\mathscr{R}}_0$ and by the construction of $\hat{\mathscr{F}}$, we can prove that $\hat{\mathscr{F}}$ is homeomorphic.

6. Let $F(G)$ be the fibre space over the Teichmüller space $T(G)$ and take an element $\langle \omega \rangle = \mathscr{M}$. For every element $[w]$ of $T(G)$, we set $w' = \lambda \circ w \circ \omega^{-1} \in Q_{\mathrm{norm}}(G)$, where λ is a real Möbius transformation. Let ϕ and ϕ' be the quadratic differentials associated with $[w]$ and $[w']$, respectively. We set $\hat{z} = W_{\phi'} \circ \omega \circ W_\phi^{-1}(z)$ for $z \in D_\phi$. Then $[\omega]_*([w], z) = ([w'], \hat{z})$ induces an analytic automorphism $[\omega]_*$ of $F(G)$. This analytic automorphism $[\omega]_*$ induces a finite subgroup Σ of the analytic automorphism group of $\mathscr{D}/\mathscr{G}_1$ and every element σ of Σ can be extended to a homeomorphism $\hat{\sigma}$ of $\hat{\mathscr{D}}/\hat{\mathscr{G}}_1$ onto itself. We set $\hat{\Sigma} = \{\hat{\sigma} \mid \sigma \in \Sigma\}$ and $\hat{\Sigma}_0 = \hat{\mathscr{F}} \circ \hat{\Sigma} \circ \hat{\mathscr{F}}^{-1}$. Since $\hat{\mathscr{R}}_0$ is a normal complex space as mentioned in §4.5, every element of $\hat{\Sigma}_0$ is an analytic automorphism of $\hat{\mathscr{R}}_0$. By Cartan's theorem, $\hat{\mathscr{R}}_0/\hat{\Sigma}_0$ becomes a normal complex space and $\mathscr{F}$ induces an analytic isomorphism of $\mathscr{S}_1 = (\mathscr{D}/\mathscr{G}_1)/\Sigma$ onto $\mathscr{R}/\hat{\Sigma}_0$, which can be extended to a homeomorphism

of $\hat{\mathscr{S}} = (\hat{\mathscr{D}}/\hat{\mathscr{G}}_1)/\hat{\Sigma}$ onto $\hat{\mathscr{R}}_0/\hat{\Sigma}_0$. By this identification, $\hat{\mathscr{S}}$ has a normal complex structure. Let $\pi_1 : \mathscr{S}_1 \to D^*$ and $\hat{\pi} : \hat{\mathscr{S}} \to D$ be the natural projections, respectively. Then $(\mathscr{S}, \pi, D^*)$ is analytically equivalent to $(\mathscr{S}_1, \pi_1, D^*)$ and $(\hat{\mathscr{S}}, \hat{\pi}, D)$ is a completion of $(\mathscr{S}_1, \pi_1, D^*)$. Therefore, $\hat{\mathscr{S}} = \hat{\mathscr{D}}/\hat{\mathscr{G}}$ is a normal complex space and $(\hat{\mathscr{S}}, \hat{\pi}, D)$ is a completion of $(\mathscr{S}, \pi, D^*)$.

Summarizing the results obtained above, we have the following theorem.

THEOREM 2. *For a holomorphic family $(\mathscr{S}, \pi, D^*)$ of Riemann surfaces of type (g, n) with $2g - 2 + n > 0$, a completion $(\hat{\mathscr{S}}, \hat{\pi}, D)$ of $(\mathscr{S}, \pi, D^*)$ is canonically constructed in such a way that $(\hat{\mathscr{S}}, \hat{\pi}, D)$ is a holomorphic family of compact Riemann surfaces of genus g with or without a singular fibre over $t=0$ and that $\hat{\mathscr{S}}$ is a two dimensional normal complex space.*

5. An extension theorem.

THEOREM 3. *Let $\bar{\mathscr{S}}$ be a two dimensional complex manifold and let $\bar{\pi}$ be a proper holomorphic mapping of $\bar{\mathscr{S}}$ onto the unit disc. Assume that there is a one dimensional analytic subset C of $\bar{\mathscr{S}}$ such that, setting $\mathscr{S} = \bar{\mathscr{S}} - \bar{\pi}^{-1}(0) \cup C$, $\pi = \bar{\pi}|\mathscr{S}$ and D^* = the punctured unit disc, $\pi : \mathscr{S} \to D^*$ is a holomorphic family of Riemann surfaces of type (g, n) with $2g - 2 + n > 0$. Let $(\hat{\mathscr{S}}, \hat{\pi}, D^*)$ be the completion of $(\mathscr{S}, \pi, D^*)$ canonically constructed in Theorem 2. Then every holomorphic mapping $F : \mathscr{S} \to \hat{\mathscr{S}}$ with $\pi = \hat{\pi} \circ F$ can be extended to a meromorphic mapping $\hat{F} : \bar{\mathscr{S}} \to \hat{\mathscr{S}}$.*

PROOF. By the construction of $\hat{\mathscr{S}}$, the argument similar to that in the proof of Lemma 1 in [6] which uses Kobayashi's extension theorem shows that F can be extended to a holomorphic mapping $\tilde{F} : (\bar{\mathscr{S}} - A) \to \hat{\mathscr{S}}$, where A is the set of singular points of $\bar{\pi}^{-1}(0) \cup C$. If $\hat{S}_0$ is the fibre of $\hat{\mathscr{S}}$ over $t = 0$, then the graph $\Gamma = \{(P, \tilde{F}(P)) \mid P \in \bar{\mathscr{S}} - A\}$ of $\tilde{F}$ is an analytic subset of $(\bar{\mathscr{S}} \times \hat{\mathscr{S}}) - (A \times \hat{S}_0)$. Since $\dim(A \times \hat{S}_0) = 1$ and $\dim_x(\Gamma) = 2$ for every point x of Γ, Remmert-Stein's theorem implies that the closure $\bar{\Gamma}$ of Γ in $\bar{\mathscr{S}} \times \hat{\mathscr{S}}$ is an analytic subset of $\bar{\mathscr{S}} \times \hat{\mathscr{S}}$. Further, for the canonical projections $\bar{\Pi} : \bar{\mathscr{S}} \times \hat{\mathscr{S}} \to \bar{\mathscr{S}}$ and $\hat{\Pi} : \bar{\mathscr{S}} \times \hat{\mathscr{S}} \to \hat{\mathscr{S}}$, the mappings $\bar{\Pi}|\bar{\Gamma}$ and $\hat{\Pi}|\bar{\Gamma}$ are both proper holomorphic mappings. Thus $\tilde{F}$ can be extended to a meromorphic mapping $\hat{F} : \bar{\mathscr{S}} \to \hat{\mathscr{S}}$. This completes the proof of Theorem 3.

6. Uniformization of algebraic surfaces. 1. In the first place, we explain the uniformization theorem due to Griffiths [5] along Bers's line [1] and [4].

Let X be a two dimensional irreducible non-singular projective algebraic variety over $\boldsymbol{C}$ and let X_1 be a non-empty Zariski open subset of X. Assume that X is embedded in the N-dimensional projective space $\boldsymbol{P}_N$ for some N. We can find two homogeneous polynomials F_0 and F_1 of the same degree in $N+1$ variables and two non-empty Zariski open subsets $Y(\subset X_1)$ and $Z(\subset \boldsymbol{P}_1)$ such that the mapping π of $\boldsymbol{P}_N$ onto $\boldsymbol{P}_1$ sending ζ into $(F_0(\zeta), F_1(\zeta))$ is a well-defined mapping of Y onto Z and is of maximal rank at all points of Y and such that for every z of Z, the fibre $S_z = \pi^{-1}(z) \cap Y$ of Y over z is a Riemann surface of (fixed) finite type (g, n) with $2g-2+n>0$. Replacing Z by a smaller Zariski open subset if necessary, we may assume that the universal covering space $\tilde{Z}$ of Z is the unit disc in the complex plane. Let $p: \tilde{Z} \to Z$ be the universal covering.

Let G be a finitely generated Fuchsian group of the first kind with no elliptic elements acting on the upper half-plane U such that the quotient space $S = U/G$ is of type (g, n). By the same argument as in §1.2, we see that there is a holomorphic mapping $\tilde{\Phi}: \tilde{Z} \to T(G)$ such that $D_{\tilde{\Phi}(\tau)}/G_{\tilde{\Phi}(\tau)}$ is conformally equivalent to $S_{p(\tau)}$ for every $\tau \in \tilde{Z}$.

Let $\mathscr{D}$ be the set of all pairs (τ, w) with $\tau \in \tilde{Z}$ and $w \in D_{\tilde{\Phi}(\tau)}$. Then $\mathscr{D}$ is a bounded Bergman domain in $\boldsymbol{C}^2$. The group G operates on $\mathscr{D}$ as a discrete and fixed-point-free group of analytic automorphisms by the rule

$$g(\tau, w) = (\tau,\ W_{\tilde{\Phi}(\tau)} \circ g \circ W_{\tilde{\Phi}(\tau)}^{-1}(w))$$

for $(\tau, w) \in \mathscr{D}$ and $g \in G$, where $W_{\tilde{\Phi}(\tau)}$ is the quasiconformal automorphism of $\hat{\boldsymbol{C}}$ defined by $\tilde{\Phi}(\tau)$ as in §1.1. The quotient space $\mathscr{D}/G$ is a two dimensional complex manifold and the canonical projection $\mathscr{D} \to \mathscr{D}/G$ is a universal covering. A point of $\mathscr{D}/G$ may be regarded as a pair (τ, a) with $\tau \in Z$ and $a \in S_{p(\tau)}$ such that $\pi(a) = p(\tau)$. Follow the canonical projection $\mathscr{D} \to \mathscr{D}/G$ by the holomorphic mapping which sends (τ, a) to a. Then the composed mapping $\mathscr{D} \to Y$ is considered to be a universal covering. Hence the universal covering space $\tilde{Y}$ of Y is the bounded Bergman domain $\mathscr{D}$ in $\boldsymbol{C}^2$. This is Griffiths's uniformization theorem of algebraic surfaces.

2. As an application of the completion of holomorphic families of Riemann surfaces of type (g, n) stated in §4, we can give a supplement to the above uniformization theorem.

Let Γ be the covering transformation group of the universal covering $p:\tilde{Z}\to Z$. The group Γ acting on the unit disc $\tilde{Z}$ is a finitely generated Fuchsian group of the first kind. Denote by $\mathscr{C}$ the set of all parabolic fixed points of Γ. For each point τ_0 in $\mathscr{C}$, we set $\tilde{\Phi}(\tau_0)=\lim_{\tau\to\tau_0}\tilde{\Phi}(\tau)$, where the limit is taken in a cusp region at τ_0 in $\tilde{Z}$. For each $\tau\in\tilde{Z}\cup\mathscr{C}$, we set $G_\tau=G_{\Phi(\tau)}$, $\Omega(G_\tau)=$ the region of discontinuity of G_τ, $\Delta(G_\tau)=$ the invariant component of G_τ corresponding to the lower half-plane, and $D_\tau=\Omega(G_\tau)-\Delta(G_\tau)$. We denote by $\mathscr{P}_\tau$ the set of all parabolic fixed points on ∂D_τ of G_τ for every $\tau\in\tilde{Z}\cup\mathscr{C}$. We set

$$\hat{\mathscr{D}}=\{(\tau,w)\,|\,\tau\in\tilde{Z}\cup\mathscr{C},\quad w\in D_\tau\cup\mathscr{P}_\tau\}\,.$$

Each point of $\hat{\mathscr{D}}-\mathscr{D}$ is called a cusp point of $\mathscr{D}$. We can canonically introduce a Hausdorff topology on $\hat{\mathscr{D}}$ as in §4.2.

3. As was stated in §4.3, every element of Γ or G induces an analytic automorphism of $\mathscr{D}$, which can be extended to a topological automorphism of $\hat{\mathscr{D}}$. Denote by $\mathscr{G}$ the discrete and fixed-point-free group of all such analytic automorphisms of $\mathscr{D}$ induced by Γ and G, and denote by $\hat{\mathscr{G}}$ the group of all topological automorphisms of $\hat{\mathscr{D}}$ induced by $\mathscr{G}$. The quotient space $\mathscr{S}=\mathscr{D}/\mathscr{G}$ is a two dimensional manifold and $\mathscr{S}$ is biholomorphically equivalent to Y. By this identification of $\mathscr{S}$ with Y, the canonical projection $\mathscr{D}\to\mathscr{S}$ is the universal covering of Y and its covering transformation group is $\mathscr{G}$. By Theorem 2, the quotient space $\hat{\mathscr{S}}=\hat{\mathscr{D}}/\hat{\mathscr{G}}$ is a two dimensional compact normal complex space. Further, by Theorem 3, X is bimeromorphically equivalent to $\hat{\mathscr{S}}$.

Thus we have the following theorem.

THEOREM 4. *Let X be a two dimensional, irreducible, non-singular projective algebraic variety over $\mathbf{C}$ and let X_1 be a non-empty Zariski open subset of X. Then there is a non-empty Zariski open subset Y of X_1 such that the universal covering space $\mathscr{D}$ of Y can be canonically constructed and is a bounded Bergman domain in $\mathbf{C}^2$. Moreover, if $\hat{\mathscr{D}}$ is the union of $\mathscr{D}$ and all its cusp points and if $\mathscr{G}$ is its covering transformation group, then $\hat{\mathscr{D}}$ has a natural Hausdorff topology and every element of $\mathscr{G}$ can be extended to a topological automorphism of $\hat{\mathscr{D}}$. If $\hat{\mathscr{G}}$ is the group of all topological automorphisms of $\hat{\mathscr{D}}$ induced by $\mathscr{G}$, then the quotient space $\hat{\mathscr{S}}=\hat{\mathscr{D}}/\hat{\mathscr{G}}$ is a two dimensional compact normal complex space and is bimeromorphically equivalent to X.*

7. Compactification of two dimensional Stein manifolds with holomorphic fibration. 1. We consider the compactification of a two dimensional Stein manifold with a certain holomorphic fibration. Let M be a compact analytic space and T be an analytic subset of M. We call M a compactification of a complex manifold X if $M - T$ and X are biholomorphically equivalent.

We can prove the following.

THEOREM 5. *Let $\mathscr{S}$ be a two dimensional Stein manifold, let R_0 be a compact Riemann surface and let R be a non-empty Zariski open subset of R_0. Assume that there exists a holomorphic mapping $\pi : \mathscr{S} \to R$ such that*

1) *π is of maximal rank at every point of $\mathscr{S}$, and*

2) *for every point t of R, the fibre $S_t = \pi^{-1}(t)$ of $\mathscr{S}$ over t is an irreducible analytic subset of $\mathscr{S}$ and is of fixed finite type (g, n) with $2g - 2 + n > 0$ as a Riemann surface.*

Then a compactification $\hat{\mathscr{S}}$ of $\mathscr{S}$ can be canonically constructed and $\hat{\mathscr{S}}$ is normal and is bimeromorphically equivalent to a projective algebraic surface. Moreover, every compactification of $\mathscr{S}$ is bimeromorphically equivalent to $\hat{\mathscr{S}}$.

PROOF. We can construct a two dimensional complex manifold $\bar{\mathscr{S}}$ such that $\mathscr{S}$ can be regarded as a Zariski open subset of $\bar{\mathscr{S}}$ and that $C = \bar{\mathscr{S}} - \mathscr{S}$ is a non-singular one dimensional analytic subset. (See Theorem II in Nishino [10].) The mapping $\pi : \mathscr{S} \to R$ can be extended to a proper holomorphic mapping $\bar{\pi} : \bar{\mathscr{S}} \to R$. Hence Theorem 2 implies that a completion $(\hat{\mathscr{S}}, \hat{\pi}, R_0)$ of $(\mathscr{S}, \pi, R)$ can be canonically constructed and $\hat{\mathscr{S}}$ is a two dimensional compact normal complex space. This space $\hat{\mathscr{S}}$ is a compactification of $\mathscr{S}$.

In order to prove that $\hat{\mathscr{S}}$ is bimeromorphically equivalent to a projective algebraic surface, it is sufficient to show that the algebraic dimension $a(\hat{\mathscr{S}})$ of $\hat{\mathscr{S}}$ is equal to 2. (See Theorem 3.1 in Kodaira [8].) In our case, obviously $a(\hat{\mathscr{S}}) = 1$ or 2. The set $C = \bar{\mathscr{S}} - \mathscr{S}$ can be regarded as a one dimensional non-singular analytic subset $\mathscr{C}$ of $\hat{\mathscr{S}}_0 = \hat{\mathscr{S}} - \hat{\pi}^{-1}(R_0 - R)$. By the same reasoning as in the proof of Theorem 3, we can prove that $\mathscr{C}$ has an analytic extension $\hat{\mathscr{C}}$, that is, $\hat{\mathscr{C}}$ is an analytic subset of $\hat{\mathscr{S}}$ with $\mathscr{C} = \hat{\mathscr{C}} \cap \hat{\mathscr{S}}_0$. Thus the one dimensional compact analytic subset $\hat{\mathscr{C}}$ intersects every fibre $\hat{S}_t = \hat{\pi}^{-1}(t)$ of $\hat{\mathscr{S}}$. Therefore, Kodaira's theorem implies that $a(\hat{\mathscr{S}})$ is not equal to 1. (See Theorem 4.3 in Kodaira [9].) So we have $a(\hat{\mathscr{S}}) = 2$ and we see that $\hat{\mathscr{S}}$ is bimeromorphically equivalent to a projective algebraic surface.

For a compactification M of $\mathscr{S}$ with the inclusion map j', we set $\mathscr{S}' = j'(\mathscr{S})$ and $\pi' = \pi \circ j'^{-1}$. The triple $(\mathscr{S}', \pi', R)$ is a holomorphic family of Riemann surfaces of type (g, n). Let j be the inclusion map of $\mathscr{S}$ into $\hat{\mathscr{S}}$ and let $J = j \circ j'^{-1}$. Denote by A the one dimensional analytic subset $M - \mathscr{S}'$ of M. In the same manner as in the proof of Theorem 3, we can prove that J can be extended to a holomorphic mapping $\tilde{J} : M - \mathrm{Sing}(A) \to \hat{\mathscr{S}}$, where $\mathrm{Sing}(A)$ is the set of singular points of A. By the relation $\pi' = \hat{\pi} \circ J$ on $\mathscr{S}'$, the mapping $\pi' : \mathscr{S}' \to R$ can be extended to a holomorphic mapping $\Pi : M - \mathrm{Sing}(A) \to R_0$. Since the codimension of $\mathrm{Sing}(A)$ is not less than 2 and since the compact Riemann surface R_0 is a projective algebraic curve, Levi's extension theorem implies that Π can be extended to a meromorphic mapping of M onto R_0. There exists a finite succession $\sigma : \tilde{M} \to M$ of σ-processes centered at the points of $\mathrm{Sing}(A)$ such that $\tilde{\Pi} = \Pi \circ \sigma$ of $\tilde{M}$ onto R_0 is a proper holomorphic mapping. Hence we have a holomorphic family $(\sigma^{-1}(\mathscr{S}'), \tilde{\Pi}, R)$ of Riemann surfaces of type (g, n). Theorem 3 implies that the holomorphic mapping $J \circ \sigma$ of $\sigma^{-1}(\mathscr{S}')$ into $\hat{\mathscr{S}}$ with the relation $\tilde{\Pi} = \hat{\pi} \circ (J \circ \sigma)$ on $\sigma^{-1}(\mathscr{S}')$ can be extended to a bimeromorphic mapping of $\tilde{M}$ onto $\hat{\mathscr{S}}$. Since M is bimeromorphically equivalent to $\tilde{M}$, we see that M is also bimeromorphically equivalent to $\hat{\mathscr{S}}$. This completes the proof of Theorem 5.

REMARK. If $2g - 2 + n \leqq 0$, then there is a two dimensional Stein manifold $\mathscr{S}$ with a holomorphic fibration $(\mathscr{S}, \pi, R)$ of type (g, n) such that a compactification $\hat{\mathscr{S}}$ of $\mathscr{S}$ is not bimeromorphically equivalent to a projective algebraic surface. We shall give an example. Let T be a linear automorphism of $\boldsymbol{C}^2$ sending (z, w) into $\{(1/2)(z + w), (1/2)w\}$ and let G be the group generated by T. Since G is a properly discontinuous group with no fixed points in $\boldsymbol{C}^2 - \{0\}$, the quotient space $\hat{\mathscr{S}} = (\boldsymbol{C}^2 - \{0\})/G$ is a two dimensional compact complex manifold. Such a surface $\hat{\mathscr{S}}$ is called a Hopf surface. Since $\hat{\mathscr{S}}$ is diffeomorphic to $S^1 \times S^3$, the first Betti number is odd and is equal to 1, which implies that $\hat{\mathscr{S}}$ is not a Kähler manifold. Thus $\hat{\mathscr{S}}$ is not algebraic. Moreover, we can prove that there is no meromorphic functions on $\hat{\mathscr{S}}$ other than constant functions. We can also prove that there is no one dimensional analytic subset of $\hat{\mathscr{S}}$ except for a non-singular elliptic curve $C = \{(z, 0) \mid z \in \boldsymbol{C} - \{0\}\}/G$. If we set $\mathscr{S} = \hat{\mathscr{S}} - C$ and $\Pi([z], [w]) = \exp(2\pi i z/w)$ for (z, w) of $\boldsymbol{C}^2 - \{w = 0\}$, then $\mathscr{S}$ is biholomorphically equivalent to $\boldsymbol{C}^* \times \boldsymbol{C}^*$ by the mapping sending $[z, w]$ into $\{\exp(2\pi i z/w), w \exp((z/w) \log 2)\}$ and the triple $(\mathscr{S}, \Pi, \boldsymbol{C}^*)$ is a holomorphic family of Riemann surfaces of type $(0, 2)$, where $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$.

Finally, as an application of Theorems 3 and 5, we will prove the following Theorem 6 which is due to Kizuka [7].

Let $P(x, y)$ be a non-constant polynomial of two complex variables x and y. For any complex number c, each irreducible component S_c of the analytic subset $\{(x, y) \in \boldsymbol{C}^2 \mid P(x, y) = c\}$ of $\boldsymbol{C}^2$ is called a prime surface of P with value c. If $\tilde{S}_c$ is the desingularization of S_c and if $\tilde{S}_c$ is of type (g, n) as a Riemann surface, we say that S_c is of type (g, n). For all values of c except for a finite number of values, every prime surface S_c of P is non-singular and is of fixed finite type (g_0, n_0). If $2g_0 - 2 + n_0 > 0$, then the polynomial P is said to be of general type.

THEOREM 6. *Let T be an analytic automorphism of $\boldsymbol{C}^2$. If there exists a polynomial P of general type such that $P \circ T$ is also a polynomial, then T is a polynomial map.*

PROOF. For the polynomial P, there exists a polynomial $P_0(x, y)$ and a polynomial $\phi(z)$ of a complex variable z such that $P(x, y) = \phi(P_0(x, y))$ and that, for all values except for a finite number of values, the analytic subset $\{(x, y) \mid P_0(x, y) = c\}$ of $\boldsymbol{C}^2$ is non-singular, irreducible and of order 1. So we may assume that the analytic subset $\{(x, y) \mid P(x, y) = c\}$ is non-singular, irreducible and of order 1 for all values except for a finite number of values.

We set $Q = P \circ T$. There are two one dimensional analytic subsets C_1 and C_2 of the two dimensional complex projective space $\boldsymbol{P}_2$ such that, if we set $\mathscr{S}_1 = \boldsymbol{P}_2 - C_1$, $\mathscr{S}_2 = \boldsymbol{P}_2 - C_2$, $\pi_1 = P|\mathscr{S}_1$, $\pi_2 = Q|\mathscr{S}_2$ and $R =$ a Zariski open subset of $\boldsymbol{P}_1$, then $(\mathscr{S}_1, \pi_1, R)$ and $(\mathscr{S}_2, \pi_2, R)$ are holomorphic families of Riemann surfaces of type (g_0, n_0) with $2g_0 - 2 + n_0 > 0$. The analytic automorphism T of $\boldsymbol{C}^2$ induces a biholomorphic mapping T_0 of $\mathscr{S}_1$ onto $\mathscr{S}_2$ with $\pi_1 = \pi_2 \circ T_0$.

Let $(\hat{\mathscr{S}}_1, \hat{\pi}_1, \boldsymbol{P}_1)$ and $(\hat{\mathscr{S}}_2, \hat{\pi}_2, \boldsymbol{P}_1)$ be the completions of $(\mathscr{S}_1, \pi_1, R)$ and $(\mathscr{S}_2, \pi_2, R)$ constructed canonically in Theorem 5, respectively. If $j_1 : \mathscr{S}_1 \to \hat{\mathscr{S}}_1$ and $j_2 : \mathscr{S}_2 \to \hat{\mathscr{S}}_2$ are the inclusion mappings, then Theorem 3 implies that j_1 and j_2 have bimeromorphic extensions $J_1 : \boldsymbol{P}_2 \to \hat{\mathscr{S}}_1$ and $J_2 : \boldsymbol{P}_2 \to \hat{\mathscr{S}}_2$, respectively. Similarly, the biholomorphic mapping $j_2 \circ T_0 \circ j_1^{-1} : j_1(\mathscr{S}_1) \to j_2(\mathscr{S}_2)$ has a bimeromorphic extension of $\hat{\mathscr{S}}_1$ onto $\hat{\mathscr{S}}_2$. Thus the biholomorphic mapping $T : \boldsymbol{C}^2 \to \boldsymbol{C}^2$ has a bimeromorphic extension $\hat{T} : \boldsymbol{P}_2 \to \boldsymbol{P}_2$, which implies that $\hat{T}$ is a rational map. Since $\hat{T}|\boldsymbol{C}^2$ is holomorphic, T is a polynomial map. This completes the proof of Theorem 6.

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