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ALMOST PERIODIC GROSS-SUBSTITUTE DYNAMICAL SYSTEMS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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1. Introduction. In this note we shall study tâtonnement processes with time-dependent almost periodic coefficients. The model process is given by a system of ordinary differential equations

$$(0) \qquad \frac{dp_i}{dt} = \lambda_i E_i(p, t) , \quad i = 1, 2, \cdots, n$$

where $p = (p_i)$ is a price-vector, $E_i(p, t)$ is the excess demand function for the ith good and λ_i is a positive constant. These equations form a mathematical model for the classical law of supply and demand. We shall assume below that the system (0) is a gross-substitute system that satisfies Walras' law and that $E(p, t)$ is almost periodic in t. An example of the system we consider is given by $\lambda_i = 1$ for all i and

$$E_i(p, t) = \left(\sum_{\alpha=1}^{M} \sum_{j=1}^{n} a_{ij}^{\alpha}(t) p_j^{\alpha}\right) \Big/ p_i ,$$

where a_{ij}^{α} is almost periodic in t, $a_{ij}^{\alpha} \geqq 0$ when $i \neq j$ and $\sum_{i=1}^{n} a_{ij}^{\alpha}(t) \equiv 0$ for all j and α.

Autonomous tâtonnement processes have been studied extensively in the econometrica literature, cf. [8, 10] for example. The stability and limiting behavior of these systems is well understood, cf. [1-3, 6, 8, 10, 12]. However if one wishes to build a theory of such economic models which reflects changes due to seasonal adjustments, then it is important to study time-dependent or nonautonomous systems. The theory we describe here is adequate to describe the limiting behavior of systems with almost periodic seasonal adjustments. In the example above such systems would occur if the coefficients $a_{ij}^{\alpha}(t)$ are periodic with incommensurable periods, cf. [5, 13].

This paper is a generalization of the periodic theory presented in Nakajima [7]. In particular we will show that any "positively compact" solution of (0) is asymptotically almost periodic, cf. [13]. As we shall

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see, the positive compactness of solutions will guarantee stability. In order to show that the limiting behavior is almost periodic we will use the lifting theory of skew-product flows in Sacker and Sell [9]. In particular we use the result which asserts that the omega-limit set of a positively compact uniformly stable solution of an almost periodic differential equation is a distal minimal set. As a technical point we note that the given positively compact solution need not be asymptotically stable. But nevertheless, because of the strong structure of gross-substitute systems, this solution is asymptotically almost periodic, cf. [5, 9, 13].

2. Gross-substitute systems. Let R^n denote the real n-dimensional Euclidean space with norm $|x| = \sum_{i=1}^n |x_i|$, where $x = (x_1, \cdots, x_n) \in R^n$. Define

$$P = \{x \in R^n : x_i > 0, \quad 1 \leqq i \leqq n\} .$$

Let $F = (F_1, \cdots, F_n): P \times R \to R^n$ be a continuous function. The differential system

$$x' = F(x, t) \tag{1}$$

on $P \times R$ is called a *gross-substitute system* if the following three hypotheses are satisfied:

(H 1) For every compact set $K \subseteq P$ there is a constant $L = L(K) > 0$ such that $|F(x, t) - F(y, t)| \leqq L|x - y|$ for all $x, y \in K$ and $t \in R$.

(H 2) For any $i = 1, \cdots, n$ one has $F_i(x, t) \leqq F_i(y, t)$ for any $x, y \in P$ with $x_i = y_i$ and $x_j \leqq y_j$ $(1 \leqq j \leqq n)$.

(H 3) One has $\sum_{i=1}^n F_i(x, t) = 0$ for all $x \in P$ and $t \in R$.

In this paper we shall be interested in almost periodic gross-substitute systems, which means that in addition to the above, F satisfies:

(H 4) $F(x, t)$ is uniformly almost periodic in t.

REMARK 1. The inequality (H 2) is the standard defining relationship for a gross-substitute system [7]. The equality (H 3) is basically Walras' law. (In terms of Equation (0), Walras' law is sometimes stated as $\sum_{i=1}^n p_i E_i(p, t) = 0$. However the change of variables $x_i = p_i^2/\lambda_i$ shows that the latter is equivalent to (H 3).) In economic theory Walras' law is an assertion of the equality of supply and demand. Since we shall consider only equations that satisfy all four of the above conditions, we have lumped these sins together under the single title of an almost periodic gross-substitute system.

A solution $x(t)$ for a gross-substitute system is said to be *positively compact* if there are positive constants $0 < \alpha \leqq \beta$ such that $\alpha \leqq x_i(t) \leqq \beta$, $1 \leqq i \leqq n$ for all $t \geqq t_0$.

The object of this note is to prove the following result:

THEOREM 1. *Let* (1) *be an almost periodic gross-substitute system. If there exists a positively compact solution* $x(t)$, *then there exists an almost periodic solution* $\phi(t)$ *that satisfies*

$$|x(t) - \phi(t)| \to 0 , \quad as \quad t \to +\infty ,$$

i.e., $x(t)$ *is positively almost periodic* [13].

3. Skew-product flows. Let (1) be an almost periodic gross-substitute system. Define the translate F_τ by $F_\tau(x, t) = F(x, \tau + t)$, where $\tau \in R$. Next define the *hull*

$$\mathscr{F} = \mathrm{Cl}\,\{F_\tau : \tau \in R\} ,$$

where the closure is taken in the topology of uniform convergence on compact sets. It is known that $\mathscr{F}$ is an almost periodic minimal set [11, 13]. It is easily seen that *every* $G \in \mathscr{F}$ is an almost periodic gross-substitute system. For each $x \in P$ and $G \in \mathscr{F}$ we let $\varphi(x, G, t)$ denote the maximally defined solution of $x' = G(x, t)$ that satisfies $\varphi(x, G, 0) = x$. It is known that

$$\pi(x, G, \tau) = (\varphi(x, G, \tau), G_\tau) \tag{2}$$

describes a (local) skew-product flow on $P \times \mathscr{F}$, cf. [9].

A solution $\varphi(x, F, t)$ of (1) is said to be *uniformly stable* if it is defined for all $t \geqq 0$ and there exists a strictly increasing function $\beta(r)$, defined for $0 \leqq r < r_0$ with $\beta(0) = 0$, that satisfies $|\varphi(x, F, \tau + t) - \varphi(y, F_\tau, t)| \leqq \beta(|\varphi(x, F, \tau) - y|)$ for all $t, \tau \geqq 0$ and all y with $|\varphi(x, F, \tau) - y| < r_0$. Notice that one has $\varphi(x, F, \tau + t) = \varphi(\varphi(x, F, \tau), F_\tau, t)$ thus both $\varphi(x, F, \tau + t)$ and $\varphi(y, F_\tau, t)$ are solutions of the translated equation $x' = F_\tau(x, t)$.

We shall use the following lemma, which is easily verified:

LEMMA 1. *Let* $\varphi(\hat{x}, F, T)$ *be a positively compact solution. Assume that for all* $x, y \in P$ *and* $G \in \mathscr{F}$ *one has* $D^+|\varphi(x, G, t) - \varphi(y, G, t)| \leqq 0$, *where* D^+ *denotes the right-hand derivative. Then* $\varphi(\hat{x}, F, t)$ *is uniformly stable.*

The following theorem is an immediate consequence of [9, Theorems 2, 5]:

THEOREM A. *Let* π *be the skew-product flow* (2) *on* $P \times \mathscr{F}$ *generated by the almost periodic gross-substitute system* (1). *Let* $\varphi(\hat{x}, F, t)$ *be a positively compact uniformly stable solution of* (1) *and let* Ω *denote the* ω*-limit set of the motion* $\pi(\hat{x}, F, t)$. *Then* Ω *is a nonempty compact* *connected distal minimal set. Furthermore if for some* $G \in \mathscr{F}$ *the section*

$$\Omega(G) = \{x \in P \colon (x, G) \in \Omega\}$$

has only finitely many points, then Ω *is an almost periodic minimal set, and for each* $(x, G) \in \Omega$ *the solution* $\varphi(x, G, t)$ *is almost periodic in* t.

Recall that a compact invariant set M is minimal if and only if every trajectory is dense in M. The fact that $\mathscr{F}$ is compact and $\varphi(\hat{x}, F, t)$ is positively compact insures that Ω (and therefore every section $\Omega(G)$, $G \in \mathscr{F}$) is compact. Since $\mathscr{F}$ is minimal, every section $\Omega(G)$ is nonempty. The distal property insures that the cardinality of $\Omega(G)$ is constant over $\mathscr{F}$. As we shall see below, the section $\Omega(F)$ contains a single point. Consequently Ω and $\mathscr{F}$ are homeomorphic and the homeomorphism preserves the respective flows on Ω and $\mathscr{F}$, cf. [9].

The fact that Ω is minimal implies that if $x, y \in \Omega(F)$ then there is a sequence $t_n \to +\infty$ such that $\varphi(x, F, t_n) \to y$ and $\varphi(y, F, t_n) \to z$, where $z \in \Omega(F)$.

4. **Preliminaries.** Let $x(t) = \varphi(x, G, t)$ and $y(t) = \varphi(y, G, t)$ be two solutions of a gross-substitute system $x' = G(x, t)$. Assume that both these solutions are defined on a common interval I. (At this point we do not require that $G(x, t)$ be uniformly almost periodic in t.) For $t \in I$ we define the following five subsets of $\{i \colon 1 \leqq i \leqq n\}$.

$$\begin{aligned}
P_t &= \{i \colon x_i(t) \geqq y_i(t)\} \\
Q_t &= \{i \colon x_i(t) \leqq y_i(t)\} \\
A_t &= \{i \colon \exists h_i > 0 \quad \text{with} \quad x_i(s) > y_i(s) \quad \text{for} \quad t < s < t + h_i\} \\
B_t &= \{i \colon \exists h_i > 0 \quad \text{with} \quad x_i(s) < y_i(s) \quad \text{for} \quad t < s < t + h_i\} \\
C_t &= \{1, \cdots, n\} - (A_t \cup B_t) \,.
\end{aligned}$$

Next define the $(n \times n)$ matrix $A(t) = (a_{ik}(t))$, $1 \leqq i, k \leqq n$, by

$$\begin{aligned}
a_{ik}(t) = {} & G_i(x_1(t), \cdots, x_{k-1}(t), x_k(t), y_{k+1}(t), \cdots, y_n(t), t) \\
& - G_i(x_1(t), \cdots, x_{k-1}(t), y_k(t), y_{k+1}(t), \cdots, y_n(t), t) \,.
\end{aligned}$$

Notice that these five sets and the terms $a_{ik}(t)$ depend on t *and* the ordered pair $(x(\cdot), y(\cdot))$.

LEMMA 2. *The following statements are valid:*

(A) $\quad k \in C_t \Rightarrow x_k(t) = y_k(t),\ x'_k(t) = y'_k(t)$ *and* $a_{ik}(t) = 0$ *for all* i .

(B) $\quad k \in A_t \Rightarrow a_{ik}(t) \geqq 0$ *for all* $i \neq k$.

(C) $\quad k \in B_t \Rightarrow a_{ik}(t) \leqq 0$ *for all* $i \neq k$.

(D) $$\sum_{i=1}^{n} a_{ik}(t) = 0 \quad \textit{for all} \quad k .$$

(E) $$k \in A_t \Rightarrow \sum_{i \in A_t} a_{ik}(t) \leqq 0 .$$

(F) $$k \in B_t \Rightarrow \sum_{i \in B_t} a_{ik}(t) \geqq 0 .$$

(G) $$x_i'(t) - y_i'(t) = \sum_{k \in A_t} a_{ik}(t) + \sum_{k \in B_t} a_{ik}(t) \quad \textit{for all} \quad i .$$

(H) $$\sum_{i,k \in A_t} a_{ik}(t) \leqq 0 \quad \textit{and} \quad \sum_{i,k \in B_t} a_{ik}(t) \geqq 0 .$$

(I) $$\sum_{i \in A_t, k \in B_t} a_{ik}(t) \leqq 0 \quad \textit{and} \quad \sum_{i \in B_t, k \in A_t} a_{ik}(t) \geqq 0 .$$

PROOF. (A) follows immediately from the definition of C_t. (B) and (C) are direct consequences of (H 2). (D) follows from (H 3). If $k \in A_t$ then (B) implies that $\sum_{i \in B_t} a_{ik}(t) + \sum_{i \in C_t} a_{ik}(t) \geqq 0$. Statement (E) then follows from (D). Statement (F) is proved similarly. It is easily seen that $x_i'(t) - y_i'(t) = \sum_{k=1}^{n} a_{ik}(t)$ for all i. Statement (G) then follows from (A). Statement (H) follows immediately from (E) and (F). Finally since A_t and B_t are disjoint, statement (I) follows from (B) and (C). q.e.d.

LEMMA 3. *One has* $D^+|x(t) - y(t)| \leqq 0$ *on* I.

PROOF. We use Lemma 2 (A, G, H, I).

$$\begin{aligned} D^+|x(t) - y(t)| &= \sum_{i=1}^{n} D^+|x_i(t) - y_i(t)| \\ &= \sum_{i \in A_t} [x_i'(t) - y_i'(t)] - \sum_{i \in B_t} [x_i'(t) - y_i'(t)] \\ &= \sum_{i \in A_t} \Big[\sum_{k \in A_t} a_{ik}(t) + \sum_{k \in B_t} a_{ik}(t) \Big] - \sum_{i \in B_t} \Big[\sum_{k \in A_t} a_{ik}(t) + \sum_{k \in B_t} a_{ik}(t) \Big] \leqq 0 . \end{aligned}$$

LEMMA 4. *Assume that one has* $D^+|x(t) - y(t)| = 0$ *on* I. *Then the following statements are valid*:

(A) $$\sum_{i,k \in A_t} a_{ik}(t) = 0 \quad \textit{and} \quad \sum_{i,k \in B_t} a_{ik}(t) = 0 .$$

(B) $$\sum_{i \in A_t, k \in B_t} a_{ik}(t) = 0 \quad \textit{and} \quad \sum_{i \in B_t, k \in A_t} a_{ik}(t) = 0 .$$

(C) $$i \in A_t,\ k \in B_t \Rightarrow a_{ik}(t) = a_{ki}(t) = 0 .$$

(D) $$i \in A_t \Rightarrow x_i'(t) - y_i'(t) = \sum_{k \in A_t} a_{ik}(t) \geqq a_{ii}(t) .$$

(E) $$i \in B_t \Rightarrow x_i'(t) - y_i'(t) = \sum_{k \in B_t} a_{ik}(t) \leqq a_{ii}(t) .$$

(F) $$\sum_{i \in A_t} [x_i'(t) - y_i'(t)] = 0 \quad \textit{and} \quad \sum_{i \in B_t} [x_i'(t) - y_i'(t)] = 0 .$$

PROOF. We will use (2A), (2B), etc. to refer to the corresponding statements of Lemma 2. In the proof of Lemma 3 it was shown that $D^+|x(t) - y(t)|$ can be written as the sum of four nonpositive terms, viz $\sum_{i,k \in A_t} a_{ik}(t)$, $-\sum_{i,k \in B_t} a_{ik}(t)$, $\sum_{i \in A_t, k \in B_t} a_{ik}(t)$ and $-\sum_{i \in B_t, k \in A_t} a_{ik}(t)$. Since $D^+|x(t) - y(t)| = 0$ each of these terms must be zero, which proves (A) and (B). Statement (C) follows from (B), (2 B) and (2 C). Statement (D) follows from (2 G), (C) and (2 B). Likewise statement (E) follows from (2G), (C) and (2C). Finally statement (F) follows from (A), (D) and (E). q.e.d.

LEMMA 5. *Assume that* $D^+|x(t) - y(t)| = 0$ *on* I. *If* $i \in A_{t_0}$ *then* $x_i(t) - y_i(t) > 0$ *and* $i \in A_t$ *for all* $t > t_0$. *Likewise if* $i \in B_{t_0}$, *then* $x_i(t) - y_i(t) < 0$ *and* $i \in B_t$ *for all* $t > t_0$.

PROOF. We shall prove the statement concerning A_t. The argument for B_t is similar.

If $i \in A_{t_0}$, then there is an $h > 0$ such that $x_i(t) > y_i(t)$ for $t_0 < t < t_0 + h$. Now define

$$t_1 = \sup\{t \in I\colon x_i(s) > y_i(s) \quad \text{for all} \quad s\,, \quad t_0 < s < t\}\,.$$

It will suffice to show that $t_1 \notin I$. If $t_1 \in I$, then one has $x_i(t_1) = y_i(t_1)$ and $x_i(s) - y_i(s) > 0$ for $t_0 < s < t_1$. However from Lemma (4 D) and Hypothesis (H 1) one has $x_i'(t) - y_i'(t) \geqq a_{ii}(t) \geqq -L\{x_i(t) - y_i(t)\}$ for $t_0 < t < t_1$. The Gronwall inequality then implies that $[x_i(t) - y_i(t)] \geqq e^{-L(t-s)}[x_i(s) - y_i(s)]$ for all $t_0 \leqq s < t$. If s is chosen so that $t_0 < s < t_0 + h$, then $[x_i(s) - y_i(s)] > 0$. Hence $[x_i(t) - y_i(t)] > 0$ for all $t > t_0$, which contradicts the fact that $x_i(t_1) = y_i(t_1)$. q.e.d.

LEMMA 6. *Assume that* $D^+|x(t) - y(t)| = 0$ *on* I. *Pick* $s, t \in I$ *with* $s \leqq t$. *Then one has*

$$A_s \subseteqq A_t\,, \quad B_s \subseteqq B_t\,, \quad A_t \subseteqq P_s\,, \quad B_t \subseteqq Q_s\,.$$

PROOF. The inequalities $A_s \subseteqq A_t$ and $B_s \subseteqq B_t$ follow from Lemma 5. If $i \notin P_s$, then $x_i(s) < y_i(s)$ and $i \in B_s$. Consequently, one has $i \in B_t$ by Lemma 5. Hence $i \notin A_t$ since A_t and B_t are disjoint. In other words, one has $A_t \subseteqq P_s$. The proof that $B_t \subseteqq Q_s$ is similar. q.e.d.

REMARKS 2. One can prove some otherr elationships under the assumption that $D^+|x(t) - y(t)| = 0$ on I. Specifically the following statements are valid:

(A) $$k \in A_t \Rightarrow \sum_{i \in A_t} a_{ik}(t) = 0\,.$$

(B) $$k \in B_t \Rightarrow \sum_{i \in B_t} a_{ik}(t) = 0\,.$$

(C) $$i \in C_t \quad \text{and} \quad k \in A_t \cup B_t \Rightarrow a_{ik}(t) = 0\,.$$

3. It is also possible to show that $D^+|x(t) - y(t)| = 0$ on I if and only if statement (C) of Lemma 4 is valid on I.

5. **Proof of main theorem.** We now turn to the proof of Theorem 1. Let $\varphi(\hat{x}, F, T)$ be a positively compact solution of (1). It follows from Lemmas 1 and 3 that $\varphi(\hat{x}, F, t)$ is uniformly stable. Let Ω denote the ω-limit set of the corresponding motion $\pi(\hat{x}, F, t)$ in $P \times \mathscr{F}$. Then by Theorem A, Ω is a nonempty compact minimal set. Therefore every section $\Omega(G) = \{x \in P: (x, G) \in \Omega\}$ is a nonempty compact set in P. We will now show that the section $\Omega(F)$ contains a single point, x. (It will then follow from Theorem A that the solution $\varphi(x, F, t)$ is almost periodic in t.)

Pick $x \in \Omega(F)$. Define $U: \Omega(F) \to R$ and $V: \Omega(F) \to R$ by

$$U(y) = \sum_{i=1}^{n} \max(x_i - y_i, 0)$$

$$V(y) = \sum_{i=1}^{n} \min(x_i - y_i, 0) .$$

U and V are continuous functions defined on $\Omega(F)$. Furthermore one has $V(y) \leqq 0 \leqq U(y)$ for all $y \in \Omega(F)$. We shall use the following fact:

LEMMA 7. *The set $\Omega(F)$ contains the single point x if and only if one has $U(y) = V(y) = 0$ for all $y \in \Omega(F)$.*

Since U and V are continuous functions on a compact set, they assume their maximum and minimum values on $\Omega(F)$. Thus there are values $y, z \in \Omega(F)$ such that

(i) $0 \leqq U(\xi) \leqq U(y)$, and

(ii) $V(z) \leqq V(\xi) \leqq 0$,

for all $\xi \in \Omega(F)$.

Let $U_0 = U(y)$. We will now show that $U_0 = 0$, by contradiction. (A similar argument shows that $V(z) = 0$. Then by Lemma 7 one has $\Omega(F) = \{x\}$.) Let $x(t) = \varphi(x, F, t)$ and $y(t) = \varphi(y, F, t)$ be the corresponding solutions of (1). Since both $x(t)$ and $y(t)$ remain in a compact set in P for all t, they are defined for all $t \in R$. Now define the corresponding five sets P_t, Q_t, A_t, B_t and C_t as well as the terms $a_{ik}(t)$, $1 \leqq i, k \leqq n$.

Let us now assume the validity of the following

LEMMA 8. *One has $D^+|x(t) - y(t)| = 0$ on R.*

Since A_t is monotone in t (Lemma 6), it follows that there is a set $A \subseteqq \{i: 1 \leqq i \leqq n\}$ and a $T \geqq 0$ such that $A_t = A$ for all $t \geqq T$. It also follows from Lemma 6 that $A \subseteqq P_t$ for all $t \in R$.

Let $w(t) = \sum_{i \in A_t} [x_i(t) - y_i(t)]$. Then $w'(t) = 0$ by Lemma 4F. Hence $w(t) = w(0)$ for all $t \geqq 0$. Since $\{i: x_i > y_i\} \subseteqq A_0$ it follows from our choice of y that $w(0) = U_0$. Now choose a sequence $t_n \to +\infty$ such that $x(t_n) \to y$ and $y(t_n) \to \xi$, where $\xi \in \Omega(F)$. Since $x_i(t_n) - y_i(t_n) > 0$ for $i \in A$ (Lemma 5), it follows that $y_i - \xi_i \geqq 0$ for all $i \in A$. Since $A \subseteqq P_0$ (Lemma 6) it follows that $x_i - y_i \geqq 0$ for all $i \in A$. Since $w(t_n) = w(0) = U_0$, it follows that

$$\sum_{i \in A} (y_i - \xi_i) = U_0 \,. \tag{3}$$

Next since $A_0 \subseteqq A \subseteqq P_0$ one has

$$\sum_{i \in A} (x_i - y_i) = U_0 \,. \tag{4}$$

By adding (3) and (4) together one has $\sum_{i \in A} (x_i - \xi_i) = 2U_0$. However $x_i - \xi_i \geqq 0$ for $i \in A$. Therefore one has $2U_0 = \sum_{i \in A} (x_i - \xi_i) \leqq U(\xi) \leqq U(y) = U_0$, which is impossible if $U_0 > 0$. Hence $U_0 = 0$.

It then follows from Theorem A that $\varphi(x, F, t)$ is almost periodic. In order to show that $|\varphi(\hat{x}, F, t) - \varphi(x, F, t)| \to 0$ as $t \to +\infty$, we shall use Lemma 3. Since one has $D^+|\varphi(\hat{x}, F, t) - \varphi(x, F, t)| \leqq 0$ for all $t \geqq 0$, define β by

$$\beta = \lim_{t \to +\infty} |\varphi(\hat{x}, F, t) - \varphi(x, F, t)| \,.$$

Now choose a sequence $t_n \to +\infty$ so that $\pi(\varphi(x, F, t_n) \to (x, F)$ and $\varphi(\hat{x}, F, t_n) \to \xi$. Since $F_{t_n} \to F$ it follows that $\xi \in \Omega(F)$ and consequently $\xi = x$. Consequently one has $\beta = 0$.

It only remains to verify Lemma 8. There are several ways to do this. Perhaps the simplest argument is based on the fact that Ω is a distal minimal set. It then follows from Ellis' Theorem [4] that the product flow on $\Omega \times \Omega$ is the union of minimal sets. What this implies is that for every pair of points $x, y \in \Omega$ there is a sequence $t_n \to +\infty$ such that the solutions $x(t_n) \to x$ and $y(t_n) \to y$. Now if $D^+|x(t) - y(t)| \not\equiv 0$ it follows from Lemma 1 that there is a $\tau \in R$ (say $\tau > 0$) such that $|x(t) - y(t)| \leqq |x(\tau) - y(\tau)| < |x(0) - y(0)|$, $t \geqq \tau$. Now choose $t_n \to +\infty$ so that $x(t_n) \to x(0)$ and $y(t_n) \to y(0)$. One then has the contradiction $|x(0) - y(0)| = \lim |x(t_n) - y(t_n)| < |x(0) - y(0)|$. If $\tau \leqq 0$ one simply repeats the above argument with a suitable translate of $x(t)$ and $y(t)$. This then completes the proof.

REMARK 4. Since Ω is a 1-fold covering of the base space $\mathscr{F}$, it can easily be shown that the frequency module of the almost periodic solution $\varphi(x, F, t)$ is contained in the frequency module of F, cf. [5, 13]. We shall omit these details since they are based on standard arguments.

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