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REMARKS ON POSITIVE CONES ASSOCIATED WITH A VON NEUMANN ALGEBRA

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Abstract. Let $\mathscr{M}$ be a von Neumann algebra with a cyclic and separating vector ξ_0 and $\mathscr{P}^\sharp$ (resp. $\mathscr{P}^\flat$) be the closure of $\mathscr{M}_+\xi_0$ (resp. $\mathscr{M}'_+\xi_0$). It is shown that the map: $\xi\in\mathscr{P}^\sharp\mapsto\omega_\xi\in\mathscr{M}_*^+$ is a homeomorphism with respect to the norm topologies. It is also shown that $\mathscr{P}^\sharp$ can be replaced by $\mathscr{P}^\flat$ if $\mathscr{M}$ is finite.

0. Introduction. In [9], Takesaki introduced a $\sharp$-cone $\mathscr{P}^\sharp$ and a $\flat$-cone $\mathscr{P}^\flat$ associated with a σ-finite von Neumann algebra $\mathscr{M}$. Generalizing Sakai's Radon-Nikodym theorem, he showed that the map: $\xi\in\mathscr{P}^\sharp\mapsto\omega_\xi\in\mathscr{M}_*^+$ is bijective. Recently, Skau [7] showed that the map: $\xi\in\mathscr{P}^\flat\mapsto\omega_\xi\in\mathscr{M}_*^+$ is bijective if and only if $\mathscr{M}$ is finite.

These two mappings are clearly continuous with respect to the norm topologies in the Hilbert space and the predual $\mathscr{M}_*$. In the paper, we show that their inverses are also continuous. Thus, these two mappings are actually homeomorphisms.

1. Notations and main results. Fixing our notations, we state our main results. Throughout the paper, $\mathscr{M}$ is a von Neumann algebra on a Hilbert space $\mathscr{H}$ with a unit cyclic and separating vector ξ_0 with the vector state $\varphi_0=\omega_{\xi_0}$. We denote the modular operator and modular conjugation associated with the pair $(\mathscr{M},\varphi_0)$ by Δ and J respectively.

DEFINITION 1.1 ([1], [9]). Let $\mathscr{P}^\sharp$ (resp. $\mathscr{P}^\natural$, $\mathscr{P}^\flat$) be the closure of the positive cone $\mathscr{M}_+\xi_0$ (resp. $\Delta^{1/4}\mathscr{M}_+\xi_0$, $\Delta^{1/2}\mathscr{M}_+\xi_0=\mathscr{M}'_+\xi_0$) in the Hilbert space $\mathscr{H}$.

It is known ([9]) that

$$\mathscr{P}^\sharp=J\mathscr{P}^\flat=(\mathscr{P}^\flat)'\ ,\quad\text{the dual cone}\ .$$

The "natural cone" $\mathscr{P}^\natural$ is neutral in many aspects. For example, $\mathscr{P}^\natural$ is selfdual and fixed pointwise under J. More importantly, the map: $\xi\in\mathscr{P}^\natural\mapsto\omega_\xi\in\mathscr{M}_*^+$ is a homeomorphism, [1], due to the Powers-Størmer inequality.

Our first main result is:

THEOREM 1.2. *The map*: $\xi\in\mathscr{P}^{\sharp}\mapsto\omega_{\xi}\in\mathscr{M}_{*}^{+}$ *is a homeomorphism with respect to the norm topologies. Here*, $\omega_{\xi}\in\mathscr{M}_{*}^{+}$ *is given by* $\omega_{\xi}(x)=(x\xi\,|\,\xi)$, $x\in\mathscr{M}$.

COROLLARY 1.3 (Continuity of Sakai's Radon-Nikodym derivatives). *Let* $\{\varphi_n\}$ *be a sequence in* $\mathscr{M}_{*}^{+}$ *such that* $\varphi_n\leqq l\varphi_0$ *with some* $l>0$, *and* h_n, $n=1,2,\cdots$, *be a unique operator in* $\mathscr{M}_{+}$ *satisfying* $\varphi_n=h_n\varphi_0h_n=\omega_{h_n\xi_0}$ *(Sakai's Radon-Nikodym derivative). If* $\{\varphi_n\}$ *tends to* φ, *whose Radon-Nikodym derivative with respect to* φ_0 *is* h, *in norm, then* $\{h_n\}$ *tends to* h *in the strong operator topology.*

PROOF. It is known that $h_n\xi_0$ (resp. $h\xi_0$) is a unique implementing vector for φ_n (resp. φ) in $\mathscr{P}^{\sharp}$, [9]. It follows from the theorem that $\{h_n\xi_0\}$ converges to $h\xi_0$ in the norm of $\mathscr{H}$. Since $\|h_n\|\leqq l^{1/2}$ (due to $\varphi_n\leqq l\varphi_0$), the result follows from the cyclicity of ξ_0 for $\mathscr{M}'$. q.e.d.

As stated in the introduction, the map: $\xi\in\mathscr{P}^{\flat}\mapsto\omega_{\xi}\in\mathscr{M}_{*}^{+}$ is a bijection provided that $\mathscr{M}$ is finite. As the second main result, we shall prove:

THEOREM 1.4. *For a finite von Neumann algebra* $\mathscr{M}$, *the map*: $\xi\in\mathscr{P}^{\flat}\mapsto\omega_{\xi}\in\mathscr{M}_{*}^{+}$ *is a homeomorphism with respect to the norm topologies.*

2. Proof of Theorem 1.2. We begin with some lemmas. Throughout the section, for each $\psi\in\mathscr{M}_{*}^{+}$, we denote a unique implementing vector in $\mathscr{P}^{\sharp}$ (resp. $\mathscr{P}^{\natural}$) by ζ_{ψ} (resp. ξ_{ψ}).

LEMMA 2.1. *To prove Theorem 1.2, it suffices to show that* $\{\zeta_{\varphi_n}\}$ *converges to* ζ_{φ} *in the weak topology of* $\mathscr{H}$ *whenever* $\{\varphi_n\}$ *tends to* φ *in norm.*

PROOF. Since $\|\zeta_{\varphi_n}\|=(\varphi_n(1))^{1/2}$ tends to $\|\zeta_{\varphi}\|=\varphi(1)^{1/2}$, we may and do assume that $\|\varphi_n\|=\|\varphi\|=\|\zeta_{\varphi_n}\|=\|\zeta_{\varphi}\|=1$. To show the norm convergence of $\{\zeta_{\varphi_n}\}$ to ζ_{φ}, it suffices to show that $\|(1/2)(\zeta_{\varphi_n}+\zeta_{\varphi})\|$ tends to 1 due to the parallelogram law. However, the weak convergence implies:

$$(1\geqq)\,\|(1/2)(\zeta_{\varphi_n}+\zeta_{\varphi})\|\geqq|((1/2)(\zeta_{\varphi_n}+\zeta_{\varphi})\,|\,\zeta_{\varphi})|\to(\zeta_{\varphi}\,|\,\zeta_{\varphi})=1\,. \qquad \text{q.e.d.}$$

LEMMA 2.2. *For* $\psi\in\mathscr{M}_{*}^{+}$, let $\chi_{\psi}\in\mathscr{M}_{*}$ *be a functional determined by* $\chi_{\psi}(x)=(x\xi_{\psi}\,|\,\xi_0)$, $x\in\mathscr{M}$, *with the polar decomposition* $\chi_{\psi}=u_{\psi}|\chi_{\psi}|$. *A unique implementing vector* ζ_{ψ} *for* ψ *in* $\mathscr{P}^{\sharp}$ *is* $Ju_{\psi}^{*}\xi_{\psi}$.

PROOF. We compute, for each $x\in\mathscr{M}$,

$$\begin{aligned}(xJu_\psi^*\xi_\psi \mid Ju_\psi^*\xi_\psi) &= (u_\psi^*\xi_\psi \mid JxJu_\psi^*\xi_\psi) \\ &= (u_\psi u_\psi^*\xi_\psi \mid JxJ\xi_\psi) \qquad (u_\psi \in \mathscr{M},\ JxJ \in \mathscr{M}') \\ &= (\xi_\psi \mid Jx\xi_\psi) = (x\xi_\psi \mid J\xi_\psi) = (x\xi_\psi \mid \xi_\psi) = \psi(x)\ ,\end{aligned}$$

so that $Ju_\psi^*\xi_\psi$ is certainly an implementing vector for ψ.

By the known bijectivity of the map: $\xi \in \mathscr{P}^\sharp \mapsto \omega_\xi \in \mathscr{M}_*^+$, it suffices to check $Ju_\psi^*\xi_\psi \in \mathscr{P}^\sharp$, or equivalently, $u_\psi^*\xi_\psi \in \mathscr{P}^\flat$. For each $x \in \mathscr{M}_+$, we simply compute

$$(u_\psi^*\xi_\psi \mid x\xi_0) = (xu_\psi^*\xi_\psi \mid \xi_0) = (u_\psi^*\chi_\psi)(x) = |\chi_\psi|(x) \geqq 0\ ,$$

so that $u_\psi^*\xi_\psi$ belongs to the dual cone $\mathscr{P}^\flat$ of $\mathscr{P}^\sharp$. q.e.d.

PROOF OF THE THEOREM. We assume that a sequence $\{\varphi_n\}$ tends to φ in norm. Since $\{\zeta_{\varphi_n}\}$ is bounded and $\mathscr{M}'\xi_0$ is dense, it suffices to show that $(x'\xi_0 \mid Ju_{\varphi_n}^*\xi_{\varphi_n})$ tends to $(x'\xi_0 \mid Ju_\varphi^*\xi_\varphi)$ for each fixed $x' \in \mathscr{M}'$ due to the above two lemmas.

For each $x' \in \mathscr{M}'$, one computes

$$(x'\xi_0 \mid Ju_\varphi^*\xi_\varphi) = (x'J\xi_0 \mid Ju_\varphi^*\xi_\varphi) = (Jx'^*Ju_\varphi^*\xi_\varphi \mid \xi_0) = (u_\varphi^*\chi_\varphi)(Jx'^*J) = |\chi_\varphi|(Jx'^*J)\ ,$$

and $(x'\xi_0 \mid Ju_{\varphi_n}^*\xi_{\varphi_n}) = |\chi_{\varphi_n}|(Jx'^*J)$.

Due to the Powers-Størmer inequality, $\{\xi_{\varphi_n}\}$ in $\mathscr{P}^\natural$ tends to ξ_φ in norm so that $\{\chi_{\varphi_n}\}$ tends to χ_φ in the norm of $\mathscr{M}_*$. It is known that the "absolute value part" map: $\psi \in \mathscr{M}_* \mapsto |\psi| \in \mathscr{M}_*^+$ is norm-continuous (See Prop. III, 4, 10, [10]) so that $\{|\chi_{\varphi_n}|\}$ tends to $|\chi_\varphi|$ in norm. The above computation thus shows that $(x'\xi_0 \mid Ju_{\varphi_n}^*\xi_{\varphi_n})$ tends to $(x'\xi_0 \mid Ju_\varphi^*\xi_\varphi)$. q.e.d.

3. Proof of Theorem 1.4. We fix a tracial (normal) state τ on a finite von Neumann algebra $\mathscr{M}$. All (densely-defined) closed operators affiliated with $\mathscr{M}$ are τ-measurable in the sense of Segal [4], [6], as $\mathscr{M}$ being finite. For such operators a, b, we denote their strong product simply by ab, [6]. In other words, we omit a closure sign.

Let $L^2(\mathscr{M};\tau)$ (resp. $L^1(\mathscr{M};\tau)$) be a set of all closed operators affiliated with $\mathscr{M}$ satisfying $\tau(|a|^2) < \infty$ (resp. $\tau(|a|) < \infty$), which is known to be a Hilbert space (resp. a Banach space) under the inner product $(a|b) = \tau(b^*a)$ (resp. the norm $\|a\|_1 = \tau(|a|)$). (See [6].) For each $\psi \in \mathscr{M}_*^+$, k_ψ denotes the Radon-Nikodym derivative relative to τ, that is, k_ψ is a unique positive operator affiliated with $\mathscr{M}$ satisfying $\psi = \tau(k_\psi\,\cdot)$. The predual $\mathscr{M}_*$ is isometrically isomorphic to $L^1(\mathscr{M};\tau)$ via $\varphi = u|\varphi| \mapsto uk_{|\varphi|}$.

It is easy to show that $(\mathscr{M}, L^2(\mathscr{M};\tau), {}^*, L^2(\mathscr{M};\tau)_+)$ is a standard form, [2]. Here, $L^2(\mathscr{M};\tau)_+$ is the positive part (as operators) of $L^2(\mathscr{M};\tau)$ and $\mathscr{M}$ should be understood to act on $L^2(\mathscr{M};\tau)$ by left multiplications. It is again easy to see that a unique implementing "vector" ξ_φ for $\varphi \in \mathcal{M}_*^+$ in $L^2(\mathcal{M}; \tau)_+$ is exactly $k_\varphi^{1/2}$. By the universality (uniqueness) of a standard form, [1], [2], we may and do assume that $(\mathcal{M}, \mathcal{H}, J, \mathcal{P}^\natural) = (\mathcal{M}, L^2(\mathcal{M}; \tau), {}^*, L^2(\mathcal{M}; \tau)_+)$. For simplicity, we shall write the Radon-Nikodym derivative of φ_0 for τ by k_0, that is, $k_0^{1/2} = \xi_0$.

The next result was shown in [3]. However, for the sake of completeness we present a proof.

LEMMA 3.1. *For each $\psi \in \mathcal{M}_*^+$, a unique implementing vector in $\mathcal{P}^\flat$ is $k_0^{1/2}|k_\psi^{1/2}k_0^{-1/2}| = k_0^{1/2}(k_0^{-1/2}k_\psi k_0^{-1/2})^{1/2} \in L^2(\mathcal{M}; \tau)$.*

PROOF. For each $x \in \mathcal{M}$, we compute

$$\begin{aligned}(xk_0^{1/2}|k_\psi^{1/2}k_0^{-1/2}| \,|\, k_0^{1/2}|k_\psi^{1/2}k_0^{-1/2}|) &= \tau(|k_\psi^{1/2}k_0^{-1/2}|k_0^{1/2}xk_0^{1/2}|k_\psi^{1/2}k_0^{-1/2}|)\\ &= \tau(|k_\psi^{1/2}k_0^{-1/2}|^2k_0^{1/2}xk_0^{1/2}) = \tau(k_0^{-1/2}k_\psi k_0^{-1/2}k_0^{1/2}xk_0^{1/2}) = \tau(k_\psi x) = \psi(x)\ .\end{aligned}$$

Thus, it suffices to show that $k_0^{1/2}|k_\psi^{1/2}k_0^{-1/2}|$ belongs $\mathcal{P}^\flat$. Clearly $\mathcal{M}_+\xi_0 = \mathcal{M}_+k_0^{1/2}$ is dense in $\mathcal{P}^\sharp$, the dual cone of $\mathcal{P}^\flat$, and we notice that

$$(k_0^{1/2}|k_\psi^{1/2}k_0^{-1/2}| \,|\, xk_0^{1/2}) = \tau(k_0^{1/2}xk_0^{1/2}|k_\psi^{1/2}k_0^{-1/2}|) \geqq 0\ . \qquad \text{q.e.d.}$$

PROOF OF THE THEOREM. We assume that $\{\varphi_n\}$ tends to φ in norm and prove that $\eta_{\varphi_n} = J(k_0^{1/2}|k_{\varphi_n}^{1/2}k_0^{-1/2}|) = |k_{\varphi_n}^{1/2}k_0^{-1/2}|k_0^{1/2}$ tends to $\eta_\varphi = |k_{\varphi_n}^{1/2}k_0^{-1/2}|k_0^{1/2}$ in the L^2-norm. (See the above lemma.) By the Powers-Størmer inequality, $k_{\varphi_n}^{1/2}$ tends to $k_\varphi^{1/2}$ in the L^2-norm, hence in measure, [8]. The trace being finite, η_{φ_n} tends to η_φ in measure due to [5, Application 2, p. 363], and [4, Theorem 1].

We now choose and fix a positive ε. Ignoring first several terms, we may and do assume that $\|\varphi_n - \varphi_1\| < \varepsilon/3$ for all n. Then we pick up a positive $a \in \mathcal{M}$ such that $\|\varphi_1 - \tau a\| < \varepsilon/3$ and set $\delta = \varepsilon/3\|a\|$. For each projection p with $\tau(p) < \delta$ (and any n), we have $(0 \leqq)\ \varphi_n(p) < \varepsilon$, hence $\varphi(p) \leqq \varepsilon$. In fact, we estimate

$$\varphi_n(p) \leqq |(\varphi_n - \varphi_1)(p)| + |\varphi_1(p) - \tau(ap)| + |\tau(ap)| < \varepsilon/3 + \varepsilon/3 + \|a\|\delta < \varepsilon\ .$$

Since η_{φ_n} tends to η_φ in measure, for n large enough there always exists a projection p in $\mathcal{M}$ (depending upon n) such that $\|(\eta_{\varphi_n} - \eta_\varphi)(1-p)\|_\infty < \varepsilon$ and $\tau(p) < \delta$. We then estimate

$$\begin{aligned}\|\eta_{\varphi_n} - \eta_\varphi\|_2 &\leqq \|(\eta_{\varphi_n} - \eta_\varphi)(1-p)\|_2 + \|(\eta_{\varphi_n} - \eta_\varphi)p\|_2\\ &\leqq \|(\eta_{\varphi_n} - \eta_\varphi)(1-p)\|_\infty + \|\eta_{\varphi_n}p\|_2 + \|\eta_\varphi p\|_2 \quad (\tau(1) = 1)\\ &< \varepsilon + \tau(pk_{\varphi_n}p)^{1/2} + \tau(pk_\varphi p)^{1/2}\\ &\qquad (|\eta_\varphi p|^2 = (\eta_\varphi p)^*(\eta_\varphi p) = p\eta_\varphi^*\eta_\varphi p = pk_\varphi p)\\ &= \varepsilon + \varphi_n(p)^{1/2} + \varphi(p)^{1/2} \leqq \varepsilon + 2\varepsilon^{1/2}\ ,\end{aligned}$$

by what we proved above. q.e.d.

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