Tôhoku Math. Journ.
34 (1982), 141-160.

THE DEFECT RELATIONS FOR THE DERIVED CURVES OF A HOLOMORPHIC CURVE IN $P^n(C)$

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(Received July 30, 1981)

1. Introduction. In [2], H. Cartan generalized the defect relation for meromorphic functions obtained by R. Nevanlinna to the case of holomorphic curves in the n-dimensional complex projective space $P^n(C)$, where by a holomorphic curve in $P^n(C)$ we mean a holomorphic map of C into $P^n(C)$. Subsequently, Ahlfors and H. and J. Weyl gave the defect relations for the derived curves ([1] and [11]). Recently, some new proofs of them and certain generalizations of them to the case of several complex variables have been given ([3], [7], [8], [9] etc.). They mainly follow either Cartan's method or Ahlfors-Weyl's method. The former is more elementary than the latter and, moreover, Cartan's result is better in the sense that he defines the defect by counting functions which count each zero of order $\geqq n$ only n times. However, he did not give defect relations for the derived curves.

In this paper, following Cartan's method we shall give a new proof of the defect relations for the derived curves. Also, we improve the defect relation of Ahlfors and Weyl for the derived curves as follows.

Let f be a non-degenerate holomorphic curve in $P^n(C)$ and f_k the k-th derived curve (cf., Definition 2.12) for $0 \leqq k < n$. For a non-zero decomposable $(k+1)$-vector A, we denote the intersection multiplicity of $f_k(C)$ with A at z by $\nu_k(A)(z)$ (cf., Definition 3.1) and set

$$\tilde{\nu}_k(A) = \min(\nu_k(A), (k+1)(n-k)) . \tag{1.1}$$

We define the modified counting function of f_k for A to be

$$\tilde{N}_k(A)(r) = \int_0^r \Big(\sum_{0<|z|\leqq t} \tilde{\nu}_k(z)\Big)\frac{dt}{t} + \tilde{\nu}_k(0)\log r$$

and the modified defect to be

$$\tilde{\tilde{\delta}}_k(A) = \liminf_{r\to\infty}(1 - \tilde{N}_k(A)(r)/T_k(r)) ,$$

where $T_k(r)$ is the order function of f_k (cf., Definition 2.12).

We can prove the following:

THEOREM. *Let $A^0, A^1, \cdots, A^q$ be decomposable $(k+1)$-vectors in* *general position. Then,*

$$\sum_{\nu=0}^{q} \tilde{\delta}_k(A^\nu) \leqq \binom{n+1}{k+1} .$$

The paper is organized as follows. In §2 we shall recall some definitions and known results for later use. Next, in §3 we shall formulate precisely the defect relations mentioned above and give an example which shows that the number $(k+1)(n-k)$ in (1.1) is sharp. To prove the above theorem, we shall give a basic lemma on the Wronskian of meromorphic functions in §4 and an inequality for divisors in §5. After these preparations, we shall complete the proof of the above theorem in §6. In [3], Cowen and Griffiths gave a new proof of the defect relations by using the method of negative curvature. In the last section, we shall give another proof of the above theorem by making use of their method.

2. **Preliminaries.** Let ν be a divisor on $\boldsymbol{C}$, by which we mean an integer-valued function on $\boldsymbol{C}$ such that the support $|\nu| := \{z; \nu(z) \neq 0\}$ has no accumulation points in $\boldsymbol{C}$.

DEFINITION 2.1. The *counting function* of ν is defined as

$$N(r, \nu) := \int_0^r \Big(\sum_{0<|z|\leq t} \nu(z) \Big) \frac{dt}{t} + \nu(0) \log r \quad (r > 0) .$$

For a non-zero meromorphic function φ on $\boldsymbol{C}$, we define the divisors

$$\nu^\infty(\varphi)(z) := \begin{cases} 0 & \text{if } z \text{ is not a pole of } \varphi , \\ m & \text{if } z \text{ is a pole of } \varphi \text{ of order } m , \end{cases}$$

$$\nu^0(\varphi) := \nu^\infty(1/\varphi) , \qquad \nu(\varphi) := \nu^0(\varphi) - \nu^\infty(\varphi) .$$

(2.2) (Jensen's formula, cf., [6, p. 4]). *If φ is a non-zero meromorphic function on $\boldsymbol{C}$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |\varphi(re^{i\theta})| d\theta = N(r, \nu(\varphi)) + \lim_{z\to 0} \log |z^{-\nu(\varphi)(0)} \varphi(z)| \quad (r > 0) .$$

Let f be a holomorphic curve in $P^n(\boldsymbol{C})$. For an arbitrarily fixed homogeneous coordinates $(w_0 : w_1 : \cdots : w_n)$, f has a representation

$$f(z) = (f_0(z) : f_1(z) : \cdots : f_n(z)) \qquad (z \in \boldsymbol{C})$$

with entire functions $f_0, f_1, \cdots, f_n$ such that

$$\{z; f_0(z) = f_1(z) = \cdots = f_n(z) = 0\} = \emptyset .$$

Such a representation of f is referred to as a reduced representation in the following.

Taking a reduced representation $f = (f_0 : \cdots : f_n)$, we set

$$u(z) := \max_{0 \leqq i \leqq n} \log |f_i(z)| .$$

DEFINITION 2.3. The *order function* (in the sense of H. Cartan [2]) of f is defined to be

$$T(r, f) := \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta - u(0) .$$

As is easily seen by (2.2), $T(r, f)$ is uniquely determined independently of a choice of reduced representations of f, and we have only to add a bounded term to $T(r, f)$ if homogeneous coordinates on $P^n(C)$ are changed.

We now consider a hyperplane $H: a^0 w_0 + a^1 w_1 + \cdots + a^n w_n = 0$ in $P^n(C)$ with $f(C) \not\subset H$. Taking a reduced representation $f = (f_0 : f_1 : \cdots : f_n)$, we set

$$F := a^0 f_0 + a^1 f_1 + \cdots + a^n f_n .$$

The divisor $\nu(F)$ is uniquely determined independently of choices of homogeneous coordinates as well as reduced representations of f.

DEFINITION 2.4. We set $\nu(H) = \nu(F)$ and define the *counting function* of f for H to be $N(r, H) = N(r, \nu(H))$.

We can easily show by (2.2)

$$N(r, H) \leqq T(r, f) + O(1) . \tag{2.5}$$

Let φ be a meromorphic function on C.

DEFINITION 2.6. The *proximity function* of φ is defined to be

$$m(r, \varphi) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\varphi(re^{i\theta})| d\theta \qquad (r > 0) ,$$

where $\log^+ |x| = \max (\log |x|, 0)$.

(2.7) (cf., [2, p. 9]). *Regarding φ as a holomorphic map of C into the Riemann sphere $P^1(C)$, we have*

$$T(r, \varphi) = N(r, \nu^\infty(\varphi)) + m(r, \varphi) + O(1) .$$

We consider two hyperplanes

$$H: a^0 w_0 + a^1 w_1 + \cdots + a^n w_n = 0 , \qquad H': b^0 w_0 + b^1 w_1 + \cdots + b^n w_n = 0$$

such that $f(C) \not\subset H'$ for a holomorphic curve f in $P^n(C)$.

(2.8) (cf., [2, p. 10]) *Taking a reduced representation* $f = (f_0 : f_1 : \cdots : f_n)$, *we see*

$$T\left(r, \sum_{i=0}^{n} a^i f_i \Big/ \sum_{i=0}^{n} b^i f_i\right) \leqq T(r, f) + O(1) .$$

Let $\varphi_1, \varphi_2, \cdots, \varphi_q$ be meromorphic functions on $\boldsymbol{C}$ and $R(u_1, \cdots, u_q)$ a rational function such that the composite $R(\varphi_1, \cdots, \varphi_q)$ is well-defined. Then,

$$(2.9) \qquad T(r, R(\varphi_1, \cdots, \varphi_q)) \leqq O\left(\sum_{\nu=1}^{q} T(r, \varphi_\nu)\right) + O(1) .$$

For the proof, see [6, p. 15].

For real-valued functions $t(r)$, $s(r)$ on $[0, +\infty)$, by the notation

$$s(r) \leqq t(r) \qquad \|$$

we mean that $s(r) \leqq t(r)$ on $[0, +\infty)$ except on a set $E \subset [0, +\infty)$ with $\int_E dt/t < +\infty$.

PROPOSITION 2.10 ([6, pp. 62–63 and p. 115]). *Let* φ *be a non-zero meromorphic function on* $\boldsymbol{C}$ *and* l *a non-negative integer. Then,*

(i) $m(r, (\varphi'/\varphi)^{(l)}) = O(\log r) + O(\log T(r, \varphi)) \qquad \| .$

If φ *is rational, then* $m(r, (\varphi'/\varphi)^{(l)}) = O(1)$.

(ii) $T(r, \varphi^{(l)}) = O(T(r, \varphi)) \qquad \| .$

Now, we consider a holomorphic curve in $P^n(\boldsymbol{C})$ which is non-degenerate, namely, whose image is not contained in any hyperplane in $P^n(\boldsymbol{C})$. Setting

$$V = (f_0, \cdots, f_n) , \quad V' = (f'_0, \cdots, f'_n), \cdots , \quad V^{(l)} = (f_0^{(l)}, \cdots, f_n^{(l)}), \cdots$$

for a reduced representation $f = (f_0 : f_1 : \cdots : f_n)$, we define the holomorphic map

$$(2.11) \qquad \Lambda_k = V \wedge V' \wedge \cdots \wedge V^{(k)} : \boldsymbol{C} \to \bigwedge^{k+1} \boldsymbol{C}^{n+1} = \boldsymbol{C}^{N+1} ,$$

where $0 \leqq k < n$ and $N = \binom{n+1}{k+1} - 1$. Take a holomorphic function g on $\boldsymbol{C}$ such that

$$\nu(g) = \min \{\nu(W(f_{i_0}, \cdots, f_{i_k})); 0 \leqq i_0 < \cdots < i_k \leqq n\} ,$$

where $W(f_{i_0}, \cdots, f_{i_k})$ denotes the Wronskian of the functions $f_{i_0}, \cdots, f_{i_k}$. Then, the map $\Lambda_k^* := (1/g)\Lambda_k$ is holomorphic and its image is contained in $\boldsymbol{C}^{N+1} - \{0\}$.

DEFINITION 2.12. We define the *k-th derived curve* of f to be the map

$$f_k := \pi \circ \Lambda_k^* : \boldsymbol{C} \to P^N(\boldsymbol{C}) ,$$

where π denotes the canonical projection of $\boldsymbol{C}^{N+1} - \{0\}$ onto $P^N(\boldsymbol{C})$. By $T_k(r)$ we denote the order function of the holomorphic curve f_k in $P^N(\boldsymbol{C})$ in the sense of Definition 2.3. Particularly, f_0 means the original curve f and $T_0(r) = T(r, f)$.

In [3], [11] and [12], the order function of the k-th derived curve f_k is defined to be

$$T_k^*(r) := \int_0^r \Big(\int_{|z|<\rho} dd^c \log \|\Lambda_k\|^2 \Big) \frac{d\rho}{\rho} ,$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$ and $\|\Lambda_k\|$ denotes the standard norm of the vector $\Lambda_k \in \boldsymbol{C}^{N+1}$.

As is easily seen by the basic integral formula in [3, p. 97], we have $T_k^*(r) = T_k(r) + O(1)$.

Later, we need the following:

PROPOSITION 2.13. *For all k, l $(0 \leqq k, l < n)$,*

$$T_k(r) \leqq O(T_l(r)) + O(1) \qquad \| .$$

For the proof, see [11, p. 160], [12, p. 132] or [3, p. 121].

3. **Defect relations for the derived curves.** Let f be a non-degenerate holomorphic curve in $P^n(\boldsymbol{C})$ and $0 \leqq k < n$. Take arbitrarily a non-zero vector A in $\wedge^{k+1}\boldsymbol{C}^{n+1}$ which is decomposable, namely, written as $A = A_0 \wedge A_1 \wedge \cdots \wedge A_k$ with $k+1$-vectors $A_0, A_1, \cdots, A_k$ in $\boldsymbol{C}^{n+1}$. We consider the hyperplane

$$H := \pi(\{Z \in \overset{k+1}{\wedge} \boldsymbol{C}^{n+1} ; Z \neq 0, \langle Z, \bar{A} \rangle = 0\})$$

in $P^N(\boldsymbol{C})$, where $\langle\ ,\ \rangle$ denotes the canonical hermitian product on $\wedge^{k+1}\boldsymbol{C}^{n+1}$.

DEFINITION 3.1. We define the *intersection multiplicity* $\nu_k(A)(z)$ of $f_k(\boldsymbol{C})$ with A at z to be the integer $\nu(H)(z)$ given in Definition 2.4 for the k-th derived curve f_k in $P^N(\boldsymbol{C})$. We also define the counting function of f_k for A to be $N_k(A)(r) := N(r, \nu_k(A))$ and the defect of f_k for A to be

$$\delta_k(A) := 1 - \limsup_{r\to\infty} N_k(A)(r)/T_k(r) .$$

As is stated in §1, setting

$$\tilde{\nu}_k(A) := \min(\nu_k(A), (k+1)(n-k)) ,$$

we define the modified counting function to be $\tilde{N}_k(A)(r) := N(r, \tilde{\nu}_k(A))$ and the modified defect to be

$$\tilde{\delta}_k(A) := 1 - \limsup_{r\to\infty} \tilde{N}_k(A)(r)/T_k(r) .$$

By (2.5) we see easily

$$0 \leqq \delta_k(A) \leqq \tilde{\delta}_k(A) \leqq 1 . \tag{3.2}$$

The main result is stated as follows.

THEOREM 3.3. *Let f be a non-degenerate holomorphic curve in $P^n(C)$ and $A^0, A^1, \cdots, A^q$ be decomposable vectors in $\wedge^{k+1}C^{n+1}$ located in general position. Then*

$$\sum_{\nu=0}^{q} \tilde{\delta}_k(A^\nu) \leqq N+1 = \binom{n+1}{k+1} .$$

As an immediate consequence of this and (3.2), we have the following defect relation of Ahlfors and Weyl.

COROLLARY 3.4. *Under the same assumption as in Theorem* 3.3,

$$\sum_{\nu=0}^{q} \delta_k(A^\nu) \leqq \binom{n+1}{k+1} .$$

To prove Theorem 3.3, we need the following:

THEOREM 3.5. *Under the same assumption as in Theorem* 3.3,

$$(q-N)T_k(r) \leqq \sum_{\nu=0}^{q} \tilde{N}_k(A^\nu)(r) + S(r) , \tag{3.6}$$

where

$$S(r) = O(\log T_k(r)) + O(\log r) \qquad \| .$$

When f is rational, we have $S(r) = O(1)$.

The proof of Theorem 3.5 will be given in the following sections. We prove here Theorem 3.3 under the assumption that Theorem 3.5 is true. We may rewrite (3.6) as

$$\sum_{\nu=0}^{q} (1 - \tilde{N}_k(A^\nu)(r)/T_k(r)) \leqq N + 1 + S(r)/T_k(r) .$$

If f is not rational, then $\lim_{r\to\infty} \log r/T_k(r) = 0$ and so

$$\liminf_{r\to\infty} S(r)/T_k(r) = \liminf_{r\to\infty} (O(\log T_k(r))/T_k(r) + O((\log r)/T_k(r))) = 0 .$$

When f is rational, we also have

$$\lim_{r\to\infty} S(r)/T_k(r) = \lim_{r\to\infty} O(1/T_k(r)) = 0 .$$

In either case, we can conclude Theorem 3.3. q.e.d.

Take a positive number M smaller than $(k+1)(n-k)$. If we define the modified counting functions and defects by using the divisor $\min(\nu_k(A), M)$ instead of $\tilde{\nu}_k(A)$, then Theorem 3.3 does not hold. We shall give an example which illustrates this fact. We consider the holomorphic curve

$$f(z) = (1: e^z: \cdots : e^{nz}) : \boldsymbol{C} \to P^n(\boldsymbol{C}) . \tag{3.7}$$

Obviously, f is non-degenerate. Let $0 \leqq k < n$ and set

$$\mathfrak{J} = \{(i_0, \cdots, i_k); 0 \leqq i_0 < \cdots < i_k \leqq n\} . \tag{3.8}$$

For each $I = (i_0, \cdots, i_k) \in \mathfrak{J}$, we define the decomposable $(k+1)$-vector $A^I = e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $(e_0, e_1, \cdots, e_n)$ is the canonical basis of $\boldsymbol{C}^{n+1}$. Take another $(k+1)$-vector $A = A_0 \wedge A_1 \wedge \cdots \wedge A_k$ defined by the vectors

$$A_l = \left(\binom{n-l}{0}, \cdots, \binom{n-l}{n-l}, 0, \cdots, 0\right) \qquad (0 \leqq l \leqq k) .$$

It is easily shown that $N+2$ $(k+1)$-vectors A^0 and A^I $(I \in \mathfrak{J})$ are in general position. For each $I = (i_0, \cdots, i_k) \in \mathfrak{J}$, we have

$$\langle \Lambda_k, \bar{A}^I \rangle = \det(i_l^m; 0 \leqq l, m \leqq k) e^{(i_0+\cdots+i_k)z} ,$$

where Λ_k is the map defined by (2.11). This shows that $\nu_k(A^I) = 0$ and so $\delta_k(A^I) = \tilde{\delta}_k(A^I) = 1$. On the other hand, if we set

$$\varphi_l(z) = \sum_{m=0}^{n-l} \binom{n-l}{m} e^{mz} = (1+e^z)^{n-l} ,$$

then we have

$$\langle \Lambda_k, \bar{A}^0 \rangle = \det(\varphi_l^{(m)}; 0 \leqq l, m \leqq k) .$$

By an elementary calculation, we obtain

$$\langle \Lambda_k, \bar{A}^0 \rangle = (-1)^{k(k+1)/2} 1! \cdots k!\, (1+e^z)^{(k+1)(n-k)} e^{k(k+1)z/2} .$$

If we denote the number of zeros of $e^z + 1$ in $\{z; |z| \leqq t\}$ by $n(t)$, then $n(t) = t/\pi + O(1)$. Therefore, for an integer M with $0 < M \leqq (k+1)(n-k)$, we have

$$\sum_{|z|\leqq t} \min(\nu_k(A^0), M)(z) = tM/\pi + O(1)$$

and have $N(r, \min(\nu_k(A^0), M)) = rM/\pi + O(\log r)$.

We shall next evaluate the order function $T_k(r)$. To this end, we recall the following fact.

(3.9) (cf., [11, Chap. II, §5]). *Let $\lambda_0, \cdots, \lambda_n$ be mutually distinct complex numbers and consider the holomorphic curve*

$$f(z) = (e^{\lambda_0 z}: e^{\lambda_1 z}: \cdots : e^{\lambda_n z}) : C \to P^n(C) .$$

If we denote by L_k the length of the circumference of the convex polygon spanned around the points

$$\bar{\lambda}_{i_0} + \bar{\lambda}_{i_1} + \cdots + \bar{\lambda}_{i_k} \qquad ((i_0, \cdots, i_k) \in \mathfrak{J})$$

in C, then $T_k(r) = (L_k/2\pi)r + O(1)$.

Apply this to the case $\lambda_0 = 0, \lambda_1 = 1, \cdots, \lambda_n = n$. Then $L_k = 2(k+1)(n-1)$. For the holomorphic curve (3.7), we obtain

$$T_k(r) = ((k+1)(n-k)/\pi)r + O(1) .$$

Consequently,

$$1 - \limsup_{r\to\infty} N(r, \min(\nu_k(A^0), M))/T_k(r) = 1 - M/(k+1)(n-k) .$$

Theorem 3.3 is valid only when $M = (k+1)(n-k)$.

4. A basic lemma. Let $f_0, f_1, \cdots, f_k$ $(k > 0)$ be meromorphic functions on a subdomain of C which are linearly independent over C. Take $I = (i_0, \cdots, i_r)$ with $0 \leqq i_0 < \cdots < i_r < \infty$ and $J = (j_0, \cdots, j_r)$ with $0 \leqq j_0 < \cdots < j_r \leqq n$, where $0 \leqq r \leqq k$. We set

$$W(I; J) = W(i_0, \cdots, i_r; j_0, \cdots, j_r) := \det(f_{j_m}^{(i_l)}; 0 \leqq l, m \leqq r) .$$

Particularly, $W(0, \cdots, r; j_0, \cdots, j_r)$ means the Wronskian of the functions $f_{j_0}, \cdots, f_{j_r}$.

DEFINITION 4.1. For each $I = (i_0, \cdots, i_k)$ with $0 \leqq i_0 < \cdots < i_k < +\infty$, we define the *weight* of I to be

$$w(I) = (i_0 - 0) + (i_1 - 1) + \cdots + (i_k - k) .$$

Now, we give the following lemma which is basic for the proof of Theorem 3.5.

LEMMA 4.2. *For every $I = (i_0, \cdots, i_k)$ with $0 \leqq i_0 < i_1 < \cdots < i_k < +\infty$, the meromorphic function*

$$W(i_0, \cdots, i_k; 0, \cdots, k)/W(0, \cdots, k; 0, \cdots, k)$$

can be written as a polynomial of some of functions

(4.3) $\qquad (W(0, 1, \cdots, r; j_0, \cdots, j_r)'/W(0, 1, \cdots, r; j_0, \cdots, j_r))^{(l-1)}$

where $0 \leqq r \leqq k$, $l \geqq 1$, $0 \leqq j_0 < j_1 < \cdots < j_r \leqq k$.

If we associate weight l *with the function given by* (4.3), *such a polynomial can be chosen so as to be isobaric of weight* $w(I)$.

PROOF. We shall give the proof of Lemma 4.2 by double induction on k and $w(I)$. We first consider the case $k = 0$. If $w(I) = 0$, we have nothing to prove. Assume that Lemma 4.2 is true for the case $k = 0$ and $w(I) \leqq w$, and so there exists a polynomial $P_w(u_1, \cdots, u_w)$ such that

$$f_0^{(w)}/f_0 = P_w(f_0'/f_0, (f_0'/f_0)', \cdots, (f_0'/f_0)^{(w-1)})$$

and P_w is isobaric of weight w if we associate weight l with each variable u_l, namely, $P_l(u, u^2, \cdots, u^w)$ is homogeneous of degree w as a polynomial in u. Then

$$f_0^{(w+1)}/f_0 = (f_0^{(w)}/f_0)' + (f_0'/f_0)(f_0^{(w)}/f_0) = \sum_{j=1}^{w} (\partial P_w/\partial u_j)(f_0'/f_0)^{(j)} + (f_0'/f_0)P_w \ .$$

Therefore, if we set

$$P_{w+1}(u_1, \cdots, u_{w+1}) = \sum_{j=1}^{w} (\partial P_w/\partial u_j)u_{j+1} + u_1 P_w(u_1, \cdots, u_w) \ ,$$

P_w is isobaric of weight $w + 1$ and we have

$$f_0^{(w+1)}/f_0 = P_{w+1}(f_0'/f_0, (f_0'/f_0)', \cdots, (f_0'/f_0)^{(w)}) \ .$$

This shows that Lemma 4.2 holds in the case $k = 0$ and $w(I) = w + 1$. Lemma 4.2 is proved for the case $k = 0$.

We shall next prove Lemma 4.2 under the assumption that it is true for the case $<k$. If $w(I) = 0$, the proof is trivial because we have necessarily $I = (0, 1, \cdots, k)$. We assume that Lemma 4.2 is true for the case $w(I) < w$ and consider the case $w(I) = w$.

We first study the case $I := (i_0, \cdots, i_{k-1}, i_k) \neq (0, \cdots, k-1, k+w)$. Set

$$F := \begin{vmatrix} f_0 \ , & f_1 \ , & \cdots, & f_k \ , & 0 \ , & \cdots, & 0 \\ & \cdots\cdots & & & \cdots\cdots & & \\ f_0^{(k-1)}, & f_1^{(k-1)}, & \cdots, & f_k^{(k-1)}, & 0 \ , & \cdots, & 0 \\ f_0^{(i_0)} \ , & f_1^{(i_0)} \ , & \cdots, & f_k^{(i_0)} \ , & f_0^{(i_0)} \ , & \cdots, & f_{k-1}^{(i_0)} \\ & \cdots\cdots & & & \cdots\cdots & & \\ f_0^{(i_k)} \ , & f_1^{(i_k)} \ , & \cdots, & f_k^{(i_k)} \ , & f_0^{(i_k)} \ , & \cdots, & f_{k-1}^{(i_k)} \end{vmatrix} .$$

By the Laplace expansion theorem, we get

$$F = \sum_{l=0}^{k} (-1)^{k+l} W(0, \cdots, k-1, i_l; 0, \cdots, k-1, k) \\ \times W(i_0, \cdots, \hat{i}_l, \cdots, i_k; 0, \cdots, k-1) \ ,$$

where $\hat{i}_l$ means that the index i_l is deleted. On the other hand, by subtracting the l-th column from the $(k+l+1)$-th column for each $l=1, \cdots, k$, we obtain

$$F = (-1)^k W(i_0, \cdots, i_k; 0, \cdots, k) W(0, \cdots, k-1; 0, \cdots, k-1) .$$

Therefore,

$$\frac{W(i_0, \cdots, i_k; 0, \cdots, k)}{W(0, \cdots, k; 0, \cdots, k)}$$
$$= \sum_{l=0}^{k} (-1)^l \frac{W(0, \cdots, k-1, i_l; 0, \cdots, k-1, k) W(i_0, \cdots, \hat{i}_l, \cdots, i_k; 0, \cdots, k-1)}{W(0, \cdots, k-1, k; 0, \cdots, k-1, k) W(0, \cdots, k-1; 0, \cdots, k-1)} .$$

Since $w(0, \cdots, k-1, i_l) < w$ $(0 \leqq l \leqq k)$, $W(0, \cdots, k-1, i_l; 0, \cdots, k-1, k)/W(0, \cdots, k; 0, \cdots, k)$ can be written as a polynomical of some of functions given by (4.3) which is isobaric of weight $w(0, \cdots, k-1, i_l) = i_l - k$ according to the induction hypothesis on $w(I)$. On the other hand, we can apply the induction hypothesis on k to each function $W(i_0, \cdots, \hat{i}_l, \cdots, i_k; 0, \cdots, k-1)/W(0, \cdots, k-1; 0, \cdots, k-1)$. It can be written as an isobaric polynomial of some of functions given by (4.3) whose weight is $w(i_0, \cdots, \hat{i}_l, \cdots, i_k) = i_0 + \cdots + \hat{i}_l + \cdots + i_k - (0 + 1 + \cdots + (k-1)) = w(I) - i_l + k$. From these facts, we conclude that $W(i_0, \cdots, i_k; 0, \cdots, k)/(0, \cdots, k; 0, \cdots, k)$ has the desired representation.

It remains to prove Lemma 4.2 for the case $(i_0, \cdots, i_{k-1}, i_k) = W(0, \cdots, k-1, k+w)$. As is easily seen by induction on w, we can write

$$\frac{W(0, \cdots, k; 0, \cdots, k)^{(w)}}{W(0, \cdots, k; 0, \cdots, k)} = \frac{W(0, \cdots, k-1, k+w; 0, \cdots, k-1, k)}{W(0, \cdots, k-1, k; 0, \cdots, k-1, k)} + \sum_{\substack{w(I)=w \\ I:=(i_0, \cdots, i_k) \\ \neq (0, \cdots, k-1, k+w)}} C_I \frac{W(i_0, \cdots, i_k; 0, \cdots, k)}{W(0, \cdots, k; 0, \cdots, k)} ,$$

where C_I are constants depending only on I. The left hand side and, as was shown above, the last term of the right hand side have the desired representation. Accordingly, we obtain the same conclusion for $W(0, \cdots, k-1, k+w; 0, \cdots, k)/W(0, \cdots, k; 0, \cdots, k)$. This completes the proof of Lemma 4.2.

COROLLARY 4.4. *In the same situation as in Lemma 4.2, we have* $\nu(W(I; I_0)) \geqq \nu(W(I_0; I_0)) - w(I)$ *for every* $I = (i_0, \cdots, i_k)$ *and* $I_0 = (0, \cdots, k)$ *in* $\mathfrak{J}$.

PROOF. The function given by (4.3) has no pole of order larger than l. As a result of Lemma 4.2, $W(I; I_0)/W(I_0; I_0)$ has no pole of order larger than $w(I)$. This proves Corollary 4.4. q.e.d.

5. An inequality for divisors. Let f be a non-degenerate holomorphic curve in $P^n(C)$ and take a reduced representation $f=(f_0: f_1: \cdots : f_n)$. Let $0 \leqq k < n$. We attach lavels to all elements in the set $\mathfrak{J}$ given by (3.8) as $I_0 := (0, \cdots, k)$, $I_1, I_2, \cdots, I_N$, where $N = \binom{n+1}{k+1} - 1$. By W we denote the square matrix $(W(I_r; I_s); 0 \leqq r, s \leqq N)$, where $W(I_r; I_s) = \det(f_{j_m}^{(i_l)}; 0 \leqq l, m \leqq k)$ as in the previous section if $I_r = (i_0, \cdots, i_k)$, $I_s = (j_0, \cdots, j_k)$.

As a result of the classical theorem of Sylvester and Franke (e.g., [5, p. 94]), we have

$$\det(W) = W(0, 1, \cdots, n; 0, 1, \cdots, n)^{\binom{n}{k}} \quad (\neq 0) . \tag{5.1}$$

DEFINITION 5.2. We define

$$\nu_k := \min(\nu(W(I_0; I_0)), \nu(W(I_0; I_1)), \cdots, \nu(W(I_0; I_N))) .$$

It is easy to show that ν_k does not depend on a particular choice of a reduced representation of f.

The purpose of this section is to prove the following:

PROPOSITION 5.3. *Let $A^0, A^1, \cdots, A^q$ $(q \geqq N)$ be decomposable $(k+1)$-vectors in general position. Then,*

$$\nu(\det(W)) \geqq \binom{n+1}{k+1}\nu_k + \sum_{\nu=0}^{q} (\nu_k(A^\nu) - (k+1)(n-k))^+ ,$$

where $x^+ = \max(x, 0)$.

To prove Proposition 5.3, we recall the following fact.

LEMMA 5.4 ([6, p. 41]). *Let f be a non-degenerate holomorphic curve in $P^n(C)$ and z_0 be an arbitrary point of C. If we choose suitably homogeneous coordinates on $P^n(C)$, a reduced representation of f and a local coordinate t in a neighborhood of z_0 with $t(z_0)=0$, then f can be written as $f=(f_0: f_1: \cdots : f_n)$ with holomorphic functions f_i $(0 \leqq i \leqq n)$ which are expanded as*

$$f_j = t^{\delta_j} + \sum_{\nu > \delta_j} c_{j\nu} t^\nu \qquad (c_{j\nu} \in C)$$

in a neighborhood of z_0, where $\delta_0 = 0 < \delta_1 < \cdots < \delta_n$.

For the function $f_j = t^{\delta_j} + \cdots$, we have

$$f_j^{(i)}(t) = \delta_j(\delta_j - 1) \cdots (\delta_j - i + 1) t^{\delta_j - i} + \cdots ,$$

where "$\cdots$" indicates the sum of terms of higher degrees. Set

$$\phi_i(\delta_j) = \delta_j(\delta_j - 1) \cdots (\delta_j - i + 1) .$$

Then, for all $I=(i_0, \cdots, i_k)$ and $J=(j_0, \cdots, j_k)$, we have easily

$$W(I; J)=\det(\phi_{i_l}(\delta_{j_m}); 0 \leqq l, m \leqq k) t^{\delta_{j_0}+\cdots+\delta_{j_k}-(i_0+\cdots+i_k)}+\cdots .$$

LEMMA 5.5. $\nu_k(z_0)=(\delta_0-0)+(\delta_1-1)+\cdots+(\delta_k-k)$.

PROOF. We see easily $\nu(W(I_0; I_0))(z_0)=(\delta_0-0)+\cdots+(\delta_k-k)$ and $\nu(W(I_0; I))(z_0)>(\delta_0-0)+\cdots+(\delta_k-k)$ if $I \neq I_0$. As an immediate consequence of Definition 5.2, we have Lemma 5.5. q.e.d.

LEMMA 5.6. *For all $I, J \in \mathfrak{J}$, we have*

$$\nu(W(I; J)) \geqq(\nu_k-w(I)+w(J))^+ .$$

PROOF. For each point $z_0 \in \boldsymbol{C}$, we take $\delta_0=0, \delta_1, \cdots, \delta_k$ as in Lemma 5.4. For $I=(i_0, \cdots, i_k)$ and $J=(j_0, \cdots, j_k)$ in $\mathfrak{J}$,

$$\begin{aligned}\nu(W(I; J)) &\geqq (\delta_{j_0}-i_0)+(\delta_{j_1}-i_1)+\cdots+(\delta_{j_k}-i_k) \\ &=\sum_{l=0}^k(\delta_l-l)-\sum_{l=0}^k(i_l-l)+\sum_{l=0}^k(\delta_{j_l}-\delta_l) \geqq \nu_k(z_0)-w(I)+w(J) ,\end{aligned}$$

because $\delta_{j_l}-\delta_l \geqq j_l-l$ for $l=0,1, \cdots, k$. Since we have always $\nu(W(I; J)) \geqq 0$, Lemma 5.6 holds. q.e.d.

LEMMA 5.7. *Let f_{rs} $(0 \leqq r, s \leqq N)$ be non-zero holomorphic functions on $\boldsymbol{C}$. Assume that, for a non-negative integer m and w_r $(0 \leqq r \leqq N)$,*

$$\nu(f_{rs}) \geqq(m-w_r+w_s)^+$$

at a point $z_0 \in \boldsymbol{C}$ and $\det(f_{rs}) \not\equiv 0$. *Then,*

$$\nu(\det(f_{rs}))(z_0) \geqq m(N+1) .$$

PROOF. By definition, we have $\det(f_{rs})=\sum_\sigma \operatorname{sgn}(\sigma) f_{0 i_0} f_{1 i_1} \cdots f_{N i_N}$, where $\sigma=\begin{pmatrix}0 & \cdots & N \\ i_0 & \cdots & i_N\end{pmatrix}$ runs through all permulations of the letters $0, \cdots, N$. For each function $F_{i_0 \cdots i_N}:=f_{0 i_0} f_{1 i_1} \cdots f_{N i_N}$, we have

$$\nu(F_{i_0 \cdots i_N})=\sum_{r=0}^N \nu(f_{r i_r}) \geqq m(N+1) .$$

This implies Lemma 5.7. q.e.d.

COROLLARY 5.8. $\nu(\det(W)) \geqq\binom{n+1}{k+1} \nu_k$.

This is a direct result of Lemmas 5.6 and 5.7.

PROOF OF PROPOSITION 5.3. Take a point $z_0 \in \boldsymbol{C}$ arbitrarily. For brevity, we set $m_\nu=\nu_k(A^\nu)$ $(0 \leqq \nu \leqq q)$. Changing indices if necessary, we may assume that

$$m_0 \geqq m_1 \geqq \cdots \geqq m_t \geqq (k+1)(n-k) > m_{t+1} \geqq \cdots \geqq m_q .$$

If $t+1=0$, namely, $(k+1)(n-k) > m_\nu$ for all ν, Proposition 5.3 is true because of Corollary 5.8. We may assume $t \geqq 0$. Set

$$F^\nu = \langle \Lambda_k, \bar{A}^\nu \rangle \qquad (\nu = 0, 1, \cdots, q) ,$$

where Λ_k are the maps given by (2.11). Since $A^0, A^1, \cdots, A^N$ are linearly independent over $\boldsymbol{C}$ by assumption, $W(I_0; I_0), \cdots, W(I_0; I_N)$ can be written as linear combinations of $F^0, \cdots, F^N$. If $t \geqq N$, then

$$\nu(W_{I_0 I_r}) \geqq \min(\nu(F^0), \nu(F^1), \cdots, \nu(F^N)) = \nu_k + m_N > \nu_k$$

for $r = 0, 1, \cdots, N$. This contradicts Definition 5.2. So, $t < N$.

Now, we choose $N-t$ vectors $B^{t+1}, \cdots, B^N$ in $\wedge^{k+1}\boldsymbol{C}^{n+1}$ such that $B^0 := A^0, \cdots, B^t := A^t, B^{t+1}, \cdots, B^N$ are linearly independent, where $A^0, \cdots, A^t$ in $\boldsymbol{C}^{N+1}$ are regarded as column vectors. We define the square matrix $\boldsymbol{B} = (B^0, B^1, \cdots, B^N)$ and

$$\boldsymbol{U} \equiv (U_r^s; 0 \leqq r, s \leqq N) \equiv (U^0, U^1, \cdots, U^N) := \boldsymbol{W}\boldsymbol{B} .$$

Then, $\nu(\det \boldsymbol{W}) = \nu(\det \boldsymbol{U})$ because $\det \boldsymbol{B} \neq 0$. Set

$$W_{I_r} := (W_{I_r I_0}, W_{I_r I_1}, \cdots, W_{I_r I_N}) , \qquad W^{I_s} := {}^t(W_{I_0 I_s}, W_{I_1 I_s}, \cdots, W_{I_N I_s}) .$$

We can write $U_r^s = \langle W_{I_r}, \bar{B}^s \rangle$ $(0 \leqq r, s \leqq N)$ and

$$U^s = \sum_{r=0}^{N} b_r^s W^{I_r} \qquad (0 \leqq s \leqq N) , \tag{5.9}$$

where $B^s = {}^t(b_0^s, \cdots, b_N^s)$. By assumption,

$$F^\nu := \langle \Lambda_k, \bar{A}^\nu \rangle \qquad (= \langle W_{I_0}, \bar{A}^\nu \rangle)$$

has a zero of order $m_\nu + \nu_k(z_0)$ at z_0. We claim here that $U_r^s = \langle W_{I_r}, \bar{A}^s \rangle$ $(s = 0, 1, \cdots, t)$ has a zero of order $\geqq \nu_k + m_s - w(I_r)$. To see this, for each ν we choose a system of orthonormal basis $e_0, e_1, \cdots, e_n$ of $\boldsymbol{C}^{n+1}$ such that $A^\nu = ce_0 \wedge e_1 \wedge \cdots \wedge e_k$ $(c \in \boldsymbol{C})$. If we take the reduced representation of f with respect to this, we can write

$$\langle W_{I_r}, \bar{A}^\nu \rangle = cW(i_0, \cdots, i_k; 0, \cdots, k)$$

for each $I_r = (i_0, \cdots, i_k) \in \mathfrak{J}$. Then, we can apply Corollary 4.4 to these functions and obtain the desired conclusion.

On the other hand, since $i_l - l \leqq n - k$ $(l = 0, 1, \cdots, k)$ for all $I = (i_0, \cdots, i_k) \in \mathfrak{J}$, we always have

$$w(I_r) \leqq (k+1)(n-k) . \tag{5.10}$$

Therefore, every component of U^s has a zero of order $\geqq m_s -$

$(k+1)(n-k)+\nu_k$ at z_0 for $s=0, 1, \cdots, t$. Set

$$\tilde{U}^s := (z-z_0)^{(k+1)(n-k)-m_s} U^s \qquad (s=0, 1, \cdots, t)$$

and $\tilde{\boldsymbol{U}} := (\tilde{U}^0, \cdots, \tilde{U}^t, U^{t+1}, \cdots, U^N)$. Then,

$$\det \tilde{\boldsymbol{U}} = (z-z_0)^{\sum_{s=0}^{t}((k+1)(n-k)-m_s)} \det \boldsymbol{U}$$

and the order of the r-th component $\tilde{U}_r^s$ of $\tilde{U}^s$ $(0 \leqq s \leqq t)$ at z_0 is not less than $\nu_k + m_s - w(I_r) + ((k+1)(n-k) - m_s)$ $(=\nu_k + (k+1)(n-k) - w(I_r))$. By virtue of (5.9), we can rewrite

$$\det \tilde{\boldsymbol{U}} = \sum_{0 \leqq i_{t+1} < \cdots < i_N \leqq N} c_{i_{t+1} \cdots i_N} \det (\tilde{U}^0, \cdots, \tilde{U}^t, W^{I_{i_{t+1}}}, \cdots, W^{I_{i_N}}) \tag{5.11}$$

with suitable constants $c_{i_{t+1} \cdots i_N}$. For each $(i_{t+1}, \cdots, i_N)$ we set

$$G := \det (\tilde{U}^0, \cdots, \tilde{U}^t, W^{I_{i_{t+1}}}, \cdots, W^{I_{i_N}}) .$$

We determine the indices $i_0, i_1, \cdots, i_t$ $(0 \leqq i_0 < \cdots < i_t \leqq N)$ so that $\{i_0, \cdots, i_t, i_{t+1}, \cdots, i_N\} = \{0, 1, \cdots, N\}$. For convenience' sake, we set

$$\tilde{W}^{i_0} := \tilde{U}^0, \cdots, \tilde{W}^{i_t} := \tilde{U}^t , \qquad \tilde{W}^{i_{t+1}} := W^{I_{i_{t+1}}}, \cdots, \tilde{W}^{i_N} := W^{I_{i_N}} .$$

$$G = \operatorname{sgn} \begin{pmatrix} 0 \cdots t & t+1 \cdots N \\ i_0 \cdots i_t & i_{t+1} \cdots i_N \end{pmatrix} \det (\tilde{W}^0, \tilde{W}^1, \cdots, \tilde{W}^N) .$$

For each $s=0, 1, \cdots, N$, the r-th component $\tilde{W}_r^s$ of $\tilde{W}^s$ has a zero of order $\geqq \nu_k - w(I_r) + w(I_s)$ at z_0. In fact, if $s \in \{i_{t+1}, \cdots, i_N\}$, this is a result of Lemma 5.6. On the other hand, if $s = i_{s'}$ for some s' with $0 \leqq s' \leqq t$, we see that

$$\nu(\tilde{W}_r^s)(z_0) = \nu(\tilde{U}_r^{s'})(z_0) \geqq \nu_k - w(I_r) + (k+1)(n-k) \geqq \nu_k - w(I_r) + w(I_s)$$

by virtue of (5.10).

We now apply Lemma 5.7 to the matrix $(\tilde{W}^0, \cdots, \tilde{W}^N)$. We can conclude that each term on the right hand side of (5.11) has a zero of order $\geqq (N+1)\nu_k = \binom{n+1}{k+1}\nu_k$ at z_0. So, $\nu(\det \tilde{\boldsymbol{U}}) \geqq \binom{n+1}{k+1}\nu_k$. Consequently,

$$\begin{aligned} \nu(\det \boldsymbol{U})(z_0) &= \nu(\det \tilde{\boldsymbol{U}})(z_0) + \sum_{\nu=0}^{t} (m_\nu - (k+1)(n-k)) \\ &\geqq \binom{n+1}{k+1}\nu_k + \sum_{\nu=0}^{t} (m_\nu - (k+1)(n-k)) \\ &= \binom{n+1}{k+1}\nu_k + \sum_{\nu=0}^{q} (m_\nu - (k+1)(n-k))^+ . \end{aligned}$$

This completes the proof of Proposition 5.3. q.e.d.

6. Proof of the defect relation. In this section, we shall complete the proof of Theorem 3.5. Let $A^0, A^1, \cdots, A^q$ be decomposable $(k+1)$-vectors in general position. We write them as $A^\nu = A_0^\nu \wedge A_1^\nu \wedge \cdots \wedge A_k^\nu$, where $A_l^\nu = (a_{l0}^\nu, a_{l1}^\nu, \cdots, a_{ln}^\nu) \in \boldsymbol{C}^{n+1}$ $(l = 0, \cdots, k)$. For a given non-degenerate holomorphic curve f in $P^n(\boldsymbol{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$ and define $F^\nu = \langle \varLambda_k, \bar{A}^\nu \rangle$, $W(I; J) = \det(f_{j_m}^{(j_l)})$ for $I = (i_0, \cdots, i_k)$, $J = (j_0, \cdots, j_k) \in \mathfrak{J}$, $W = (W(I_r; I_s))$, $\nu_k = \min\{\nu(W(I_0; I)); I \in \mathfrak{J}\}$ and so on as in the previous sections. Choose a non-zero entire function g such that $\nu(g) = \nu_k$.

(6.1) *Any chosen $N+1$ functions among $G^0 := F^0/g, \cdots, G^q := F^q/g$ have no common zero.*

In fact, for any chosen $\alpha_0, \cdots, \alpha_N$ $(0 \leqq \alpha_0 < \cdots < \alpha_N \leqq q)$, $W(I_0; I_0)/g, \cdots, W(I_0; I_N)/g$ can be written as linear combinations of $G^{\alpha_0}, \cdots, G^{\alpha_N}$ because $A^{\alpha_0}, \cdots, A^{\alpha_N}$ are linearly independent. If $G^{\alpha_0}, \cdots, G^{\alpha_N}$ have a common zero, then $W(I_0; I_0)/g, \cdots, W(I_0; I_N)/g$ have also a common zero, which is impossible. Thus, we have the above conclusion.

DEFINITION 6.2. We define

$$v(z) := \max_{0 \leqq \beta_{N+1} < \cdots < \beta_q \leqq q} \log |G^{\beta_{N+1}}(z) \cdots G^{\beta_q}(z)| .$$

LEMMA 6.3. $(q - N) T_k(r) \leqq \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} v(re^{i\theta}) d\theta + O(1).$

PROOF. Take a point $z_0 \in \boldsymbol{C}$. We determine the indices $\alpha_0, \cdots, \alpha_N$, $\beta_{N+1}, \cdots, \beta_q$ so that $\{\alpha_0, \cdots, \alpha_N, \beta_{N+1}, \cdots, \beta_q\} = \{0, \cdots, q\}$ and

$$|G^{\alpha_0}| \leqq \cdots \leqq |G^{\alpha_N}| \leqq |G^{\beta_{N+1}}| \leqq \cdots \leqq |G^{\beta_q}|$$

at z_0. Since $A^{\alpha_0}, \cdots, A^{\alpha_N}$ are linearly independent, we can write

$$W(I_0; I_s) = c_{0s} F^{\alpha_0} + c_{1s} F^{\alpha_1} + \cdots + c_{Ns} F^{\alpha_N} \qquad (0 \leqq s \leqq N) \tag{6.4}$$

with suitable constants c_{rs} depending only on A^ν. Therefore, we can find a positive constant L not depending on z_0 such that

$$|W(I_0; I_r)(z_0)| \leqq L |F^{\alpha_N}(z_0)| .$$

If we set $u := \max_{I \in \mathfrak{J}} \log |W(I_0; I)/g|$, then we have

$$u(z_0) \leqq \log L + \log |G^{\beta_r}(z_0)| \qquad (r = N+1, \cdots, q) .$$

Summing them up, we obtain

$$\begin{aligned} (q - N) u(z_0) &\leqq (q - N) \log L + \log |G^{\beta_{N+1}}(z_0) \cdots G^{\beta_q}(z_0)| \\ &= (q - N) \log L + v(z_0) . \end{aligned}$$

Taking the mean value of each term as a function of z_0 on $\{z; |z| = r\}$, we get the desired inequality.

DEFINITION 6.5. We define

$$w(z) := \max_{0 \leqq \alpha_0 < \cdots < \alpha_N \leqq q} \log |\det(W)(z)/F^{\alpha_0}(z) \cdots F^{\alpha_N}(z)|$$

and

$$H(z) := F^0(z)F^1(z) \cdots F^q(z)/g(z)^{q-N} \det(W)(z) .$$

Then, we have

$$(q-N)T_k(r) \leqq \frac{1}{2\pi}\int_0^{2\pi} \log |H(re^{i\theta})| d\theta + \frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta}) d\theta + O(1) . \tag{6.6}$$

To see this, we choose indices $\alpha_0, \cdots, \alpha_N, \beta_{N+1}, \cdots, \beta_q$ as in the proof of Lemma 6.3 for an arbitrarily fixed point $z_0 \in C$. Then,

$$\begin{aligned} v(z_0) &= \log |G^{\beta_{N+1}}(z_0) \cdots G^{\beta_q}(z_0)| \\ &= \log |F^0(z_0) \cdots F^q(z_0)/g(z_0)^{q-N}(\det W)(z_0)| \\ &\qquad + \log |(\det W)(z_0)/F^{\alpha_0}(z_0) \cdots F^{\alpha_N}(z_0)| \\ &= \log |H(z_0)| + w(z_0) . \end{aligned}$$

This gives

$$\frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta}) d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log |H(re^{i\theta})| d\theta + \frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta}) d\theta ,$$

which concludes (6.6) by the help of Lemma 6.3.

LEMMA 6.7. $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} \log |H(re^{i\theta})| d\theta \leqq \sum_{\nu=0}^{q} \tilde{N}_k(A^\nu)(r) + O(1).$

PROOF. According to (2.2),

$$\frac{1}{2\pi}\int_0^{2\pi} \log |H(re^{i\theta})| d\theta \leqq N(r, \nu(H)) + O(1) .$$

We have only to prove the inequality $\nu(H) \leqq \sum_{\nu=0}^{q} \tilde{\nu}_k(A^\nu)$. Since $\nu_k(A^\nu) = \nu(G^\nu)$ $(0 \leqq \nu \leqq q)$ and we can write $H = G^0G^1 \cdots G^q g^{N+1}/\det W$ and

$$\nu(H) = \sum_{\nu=0}^{q} \nu_k(A^\nu) + (N+1)\nu_k - \nu(\det W) .$$

Accordingly, by virtue of Proposition 5.3, we get

$$\begin{aligned} \nu(H) &\leqq \sum_{\nu=0}^{q} \nu_k(A^\nu) + (N+1)\nu_k \\ &\qquad - \left(\binom{n+1}{k+1}\nu_k + \sum_{\nu=0}^{q} (\nu_k(A^\nu) - (k+1)(n-k))^+\right) \\ &= \sum_{\nu=0}^{q} \nu_k(A^\nu) - (\nu_k(A^\nu) - (k+1)(n-k))^+ \end{aligned}$$

$$= \sum_{\nu=0}^{q} \min (\nu_k(A^\nu), (k+1)(n-k)) .$$

This completes the proof of Lemma 6.7. q.e.d.

For a reduced representation $f = (f_0 : \cdots : f_n)$ of f, setting $\tilde{V} = \tilde{V}^{(0)} := (1, f_1/f_0, \cdots, f_n/f_0)$ and

$$\tilde{V}^{(l)} := (0, (f_1/f_0)^{(l)}, \cdots, (f_n/f_0)^{(l)}) \qquad (l = 1, 2, \cdots) ,$$

we define

$$\tilde{F}_I^\nu := \langle \tilde{V}^{(i_0)} \wedge \tilde{V}^{(i_1)} \wedge \cdots \wedge \tilde{V}^{(i_k)}, \bar{A}^\nu \rangle$$

for each $\nu = 0, 1, \cdots, q$ and $I = (i_0, \cdots, i_k) \in \mathfrak{J}$. We also define

$$\tilde{W}(I_r; I_s) := \det ((f_{j_m}/f_0)^{(i_l)}; 0 \leqq l, m \leqq k)$$

for all $I_r = (i_0, \cdots, i_k)$ and $I_s = (j_0, \cdots, j_k)$ in $\mathfrak{J}$ and the matrix $W = (\tilde{W}(I_r; I_s); 0 \leqq r, s \leqq N)$.

$$w(z) = \max_{0 \leqq \alpha_0 < \cdots < \alpha_N \leqq q} \log |\det W(z)/\tilde{F}_{I_0}^{\alpha_0}(z) \cdots \tilde{F}_{I_0}^{\alpha_N}(z)| . \tag{6.8}$$

PROOF. As is easily seen, $F^{\alpha_l} = f_0^{k+1} \tilde{F}_{I_0}^{\alpha_l}$ and so

$$F^{\alpha_0} F^{\alpha_1} \cdots F^{\alpha_N} = f_0^{(k+1)(N+1)} \tilde{F}_{I_0}^{\alpha_0} \cdots \tilde{F}_{I_0}^{\alpha_N} .$$

On the other hand, the Wronskian of the functions $1, f_1/f_0, \cdots, f_n/f_0$ is equal to the Wronskian of $f_0, f_1, \cdots, f_n$ devided by f_0^{n+1}. According to (5.1), we have $\det \tilde{W} = f_0^{(n+1)\binom{n}{k}} \det W$. Since $(k+1)(N+1) = (k+1)\binom{n+1}{k+1} = (n+1)\binom{n}{k}$, we see

$$(\det W)/F^{\alpha_0} \cdots F^{\alpha_N} = (\det \tilde{W})/\tilde{F}_{I_0}^{\alpha_0} \cdots \tilde{F}_{I_0}^{\alpha_N} .$$

This gives (6.8). q.e.d.

LEMMA 6.9. *There exists a positive constant K_0 such that*

$$w(z) \leqq K_0 \sum_{\substack{0 \leqq \nu \leqq q \\ 1 \leqq r \leqq N}} \log^+ |\tilde{F}_{I_r}^\nu(z)/\tilde{F}_{I_0}^\nu(z)| + K_0 .$$

PROOF. The identity (6.8) implies that

$$w(z) \leqq \sum_{0 \leqq \alpha_0 < \cdots < \alpha_N \leqq q} \log^+ |\det \tilde{W}(z)/\tilde{F}_{I_0}^{\alpha_0}(z) \cdots \tilde{F}_{I_0}^{\alpha_N}(z)| .$$

It suffices to estimate each term $\log^+ |\det \tilde{W}/\tilde{F}_{I_0}^{\alpha_0} \cdots \tilde{F}_{I_0}^{\alpha_N}|$ for each $(\alpha_0, \cdots, \alpha_N)$ with $0 \leqq \alpha_0 \leqq \cdots \leqq \alpha_N \leqq q$. Together with the identity (6.4), we have also

$$\tilde{W}(I_r; I_s) = c_{0s} \tilde{F}_{I_r}^{\alpha_0} + c_{1s} \tilde{F}_{I_r}^{\alpha_1} + \cdots + c_{Ns} \tilde{F}_{I_r}^{\alpha_N}$$

for all $r, s = 0, 1, \cdots, N$. Therefore, we can write

$$(\det \tilde{W})/\tilde{F}_{I_0}^{\alpha_0}\cdots\tilde{F}_{I_0}^{\alpha_N} = \det(c_{rs}) \times \det(\tilde{F}_{I_r}^{\alpha_s}; 0 \leqq r, s \leqq N)/\tilde{F}_{I_0}^{\alpha_0}\tilde{F}_{I_0}^{\alpha_1}\cdots\tilde{F}_{I_0}^{\alpha_N}$$
$$= \det(c_{rs}) \times \det(\tilde{F}_{I_r}^{\alpha_s}/\tilde{F}_{I_0}^{\alpha_s}; 0 \leqq r, s \leqq N) .$$

By the basic formulas

$$\log^+(x+y) \leqq \log^+ x + \log^+ y + \log 2 , \qquad \log^+ xy \leqq \log^+ x + \log^+ y ,$$

we easily conclude Lemma 6.9. q.e.d.

To complete the proof of Theorem 3.5, it suffices to prove the following lemma because of (6.6) and Lemma 6.7.

LEMMA 6.10. $$\frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta})d\theta = S(r) ,$$

where

$$S(r) = O(\log T_k(r)) + O(\log r)$$

and, if f is rational, then $S(r) = O(1)$.

PROOF. We first estimate $\log^+|\tilde{F}_{I_r}^{\nu}/\tilde{F}_{I_0}^{\nu}|$ for each fixed ν $(0 \leqq \nu \leqq q)$ and $I_r = (i_0, \cdots, i_k) \in \mathfrak{J}$. Since A^ν is a decomposable vector, we can choose an orthonormal basis $\{e_0, e_1, \cdots, e_n\}$ of C^{n+1} such that $A^\nu = ce_0 \wedge e_1 \wedge \cdots \wedge e_n$ $(c \in C)$. Using this basis, we represent f as $f = (g_0: \cdots : g_n)$, where $g_0, \cdots, g_n$ are linear combinations of $f_1/f_0, \cdots, f_n/f_0$. Then,

$$\text{(6.11)} \qquad \begin{aligned} \tilde{F}_{I_r}^{\nu} &= \langle \tilde{V}^{(i_0)} \wedge \cdots \wedge \tilde{V}^{(i_k)}, \bar{A}^\nu \rangle = c\det(g_m^{(i_l)}; 0 \leqq l, m \leqq k) . \\ \tilde{F}_{I_0}^{\nu} &= \langle \tilde{V} \wedge \cdots \wedge \tilde{V}^{(k)}, \bar{A}^\nu \rangle = c\det(g_m^{(l)}; 0 \leqq l, m \leqq k) . \end{aligned}$$

By Lemma 4.2, the function $\tilde{F}_{I_r}^{\nu}/\tilde{F}_{I_0}^{\nu}$ can be represented as a polynomial of some of the functions

$$(W(g_{j_0}, \cdots, g_{j_r})'/W(g_{j_0}, \cdots, g_{j_r}))^{(l-1)} ,$$

where $W(g_{j_0}, \cdots, g_{j_r})$ denotes the Wronskian of $g_{j_0}, \cdots, g_{j_r}$ and $0 \leqq r \leqq k$, $l \geqq 1$, $0 \leqq j_0 < \cdots < j_r \leqq k$. Each $W(g_{j_0}, \cdots, g_{j_r})$ is a polynomial of $(f_1/f_0)^{(l)}, \cdots, (f_n/f_0)^{(l)}$ $(l = 0, 1, \cdots)$, which we denote by φ_i $(i = 0, 1, \cdots, i_0)$. By Lemma 6.9 and (6.11), there exists a constant K_1 such that

$$w(z) \leqq K_1\Big(\sum_{\substack{l\geqq 1\\ i=1,2,\cdots,i_0}} \log^+|(\varphi_i'/\varphi_i)^{(l-1)}(z)|\Big) + K_1 ,$$

and so

$$\frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta})d\theta \leqq K_1\sum_{l,i} m(r, (\varphi_i'/\varphi_i)^{(l-1)}) + K_1 .$$

It then follows from Proposition 2.10 that

$$\frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta})d\theta \leqq O(\log r) + O(\log T(r, \varphi_i)) \qquad \| ,$$

where the right hand side is replaced by $O(1)$ if f is rational. On the other hand, by (2.8), (2.9) and Proposition 2.13,

$$T(r, \varphi_i) \leqq O(T_0(r)) \leqq O(T_k(r)) \qquad \| .$$

From these facts, we easily conclude Lemma 6.10. q.e.d.

7. **Appendix.** In [3], Cowen and Griffiths gave a new proof of the defect relations for the derived curves of a holomorphic curve in $P^n(\boldsymbol{C})$ by using the method of negative curvature. We can give another proof of Theorem 3.3 in this way. In this section, we shall state its outline. We shall use freely notations and results in [3] and the previous sections of this paper except in §6.

Cowen and Griffiths gave the following result.

THEOREM 7.1 ([3, p. 152]). *Let f be a non-degenerate holomorphic curve in $P^n(\boldsymbol{C})$ and $\{A^\nu\}_{\nu=0}^q$ be decomposable $(k+1)$-vectors in general position. Then, for every $\varepsilon > 0$,*

$$\sum_{\nu=0}^{q} N_k(A^\nu)(r) \geqq \sum_{h=0}^{k}\sum_{j=h}^{n-1+h-k} p_k(j, h)N_j(r) + \left(q + 1 - \binom{n+1}{k+1} - \varepsilon\right)T_k(r) \qquad \| ,$$

where

$$p_k(j, h) := \sum_{l=k-h}^{k}\binom{n-j}{l+1}\binom{j+1}{k-l} \qquad (j \geqq h,\ k \geqq h) .$$

As in [3, pp. 117-118], we denote by $a_k(z)$ the order of ramification of f_k at z. Then, we see

$$a_k = \nu_{k-1} + \nu_{k+1} - 2\nu_k . \tag{7.2}$$

We have also

LEMMA 7.3. $\displaystyle\sum_{h=0}^{k}\sum_{j=h}^{n-1+h-k} p_k(j, h)a_j = \binom{n}{k}\nu_n - \binom{n+1}{k+1}\nu_k .$

The proof is given by the same calculation as in the proof of [3, Proposition, p. 147]. In the calculation, we have only to replace the terms $\mathrm{Ric}\,\Omega_j$ and Ω_j by a_j and ν_j, respectively, word for word except that we have to attend to the fact $\Omega_n \equiv 0$ but $\nu_n \not\equiv 0$.

According to (5.1), we see $\binom{n}{k}\nu_n = \nu(\det W)$. Using Proposition 5.3, we obtain

$$\begin{aligned}\sum_{\nu}\tilde{\nu}_k(A) &= \sum_{\nu}\min\,(\nu_k(A^{\nu}),\,(k+1)(n-k))\\ &= \sum_{\nu}\nu_k(A^{\nu}) - \sum_{\nu}(\nu_k(A^{\nu}) - (k+1)(n-k))^{+}\\ &\geqq \sum_{\nu}\nu_k(A^{\nu}) - \left(\binom{n}{k}\nu_n - \binom{n+1}{k+1}\nu_k\right)\\ &= \sum_{\nu}\nu_k(A^{\nu}) - \sum_{h=0}^{k}\sum_{j=h}^{n-1+h-k} p_k(j,\,h)a_j\,.\end{aligned}$$

By the monotonicity of integral, we have

$$\sum_{\nu}\tilde{N}_k(A^{\nu})(r) \geqq \sum_{\nu}N_k(A^{\nu})(r) - \sum_{h=0}^{k}\sum_{j=h}^{n-1+h-k} p(j,\,h)N_j(r)\,.$$

For every $\varepsilon > 0$ we conclude by Theorem 7.1

$$\sum_{\nu}\tilde{N}_k(A^{\nu})(r) \geqq \left(q + 1 - \binom{n+1}{k+1} - \varepsilon\right)T_k(r) \qquad \|\,.$$

We can easily prove Theorem 3.3 by the method similar to its proof in §3.

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