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A PRIORI ESTIMATES FOR STATIONARY SOLUTIONS OF AN ACTIVATOR-INHIBITOR MODEL DUE TO GIERER AND MEINHARDT

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Introduction. This paper is concerned with *stationary* solutions of the following reaction-diffusion system which was proposed by Gierer and Meinhardt [3] as a model of biological pattern formation:

$$\text{(G-M)}\qquad \begin{cases} \dfrac{\partial a}{\partial t} = D_a \Delta a - \tilde{\mu} a + c\tilde{\rho}\dfrac{a^2}{h} + \rho_0 \tilde{\rho}\ , \\ \dfrac{\partial h}{\partial t} = D_h \Delta h - \nu h + c'\rho' a^2\ , \end{cases} \qquad \text{for } t > 0,\ x \in \Omega\ .$$

Here, D_a, D_h, $\tilde{\mu}$, ν, c, c', $\tilde{\rho}$ and ρ' are positive constants; ρ_0 is a non-negative constant; Δ stands for the Laplace operator and Ω is a bounded domain in $\boldsymbol{R}^n$. The activator $a(t, x)$ and the inhibitor $h(t, x)$ represent the concentrations of some substances, and hence both are to be positive.

Under the homogeneous Neumann boundary condition, the system (G-M) has a unique constant stationary solution, which is interpreted as a homogeneous state of cells or tissues.

It is numerically observed that, for appropriate values of $D_a, D_h, \cdots, \rho_0$, the system also has stable stationary solutions exhibiting spatially wavy patterns, which is considered to account for biological pattern formation. In addition, it seems that, as D_a tends to 0, the amplitude of such solutions becomes unbounded. See, for example, [5] and [8].

There are some rigorous results on the system (G-M). For example, Keener [7] considered the stationary problem in the case that $D_h \to \infty$. In the vicinity of the constant solution, bifurcation theory is effective to obtain the existence of non-constant stationary solutions and to study their stability (see [12]).

However, little is known about the solutions with large amplitude. The shape and the stability of stationary solutions seem to depend on D_a in a fairly complicated way, as is suggested in Fujii, Mimura and Nishiura [2] where global structure of the solution set is investigated for some reaction-diffusion systems with saturation.

The purpose of this paper is to estimate the range of existence of stationary solutions by giving *a priori* bounds and by showing the uniqueness of the solution for sufficiently large values of $D = D_a/D_h$. *A priori* bounds for stationary solutions of (G-M) appeared first in Hadeler, Rothe and Vogt [4]; in fact, they constructed an invariant set for a modified evolutional problem. We take a different approach to obtain more explicit estimates, especially to clarify the dependence on D.

The outline of this paper is as follows: Main results are stated in Section 1; Theorems 1 and 2 give *a priori* bounds, while Theorem 3 is concerned with the uniqueness. We shall verify the *a priori* estimates in Section 2. The uniqueness theorem will be proved in Section 3. Lastly, as an application of these theorems, we shall discuss in Section 4 the global behavior of bifurcating branches in the case that the spatial dimension is one (Theorem 4).

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1. Main results. First of all, we normalize the system (G-M) to simplify the notations. For our purpose we need only the equations satisfied by stationary solutions. Thus we consider the system

$$D\Delta u(x') - u(x')/m + u(x')^2/v(x') + \rho = 0 \tag{1.1a}$$

$$\Delta v(x') - v(x') + u(x')^2 = 0 \tag{1.1b}$$

for $x' \in \Omega'$ under the boundary condition

$$\partial u/\partial \nu = \partial v/\partial \nu = 0 \qquad \text{on } \partial\Omega' , \tag{1.2}$$

where

$$\begin{cases} x' = \sqrt{\nu/D_h}\,x, \quad \Omega' = \{x'; x \in \Omega\} , \\ u(x') = c'\rho'(c\tilde{\rho})^{-1}a(x), \quad v(x') = c'\rho'\nu(c\tilde{\rho})^{-2}h(x) \end{cases} \tag{1.3}$$

and

$$D = D_a/D_h, \quad m = \nu/\tilde{\mu}, \quad \rho = (c'\rho'\rho_0)/(c\nu) . \tag{1.4}$$

Note that $D > 0$, $m > 0$ and $\rho \geqq 0$ by definition.

In what follows, we write x and Ω instead of x' and Ω', respectively, for simplicity.

Throughout this paper, we put the following assumptions on the domain Ω and on the regularity of the solutions (u, v):

(H.1) Ω is a bounded domain in $\boldsymbol{R}^n$ with $C^{2+\alpha}$-class boundary $\partial\Omega$ $(0 < \alpha < 1)$;

(H.2) $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$.

We observe that system (1.1)–(1.2) has a unique constant solution

$$(u, v) = (\bar{u}, \bar{v}) ,$$

where

$$\bar{u} = m(1 + \rho) \quad \text{and} \quad \bar{v} = \bar{u}^2 . \tag{1.5}$$

The main object of this paper is to estimate the suprema of $u(x)$, $v(x)$, $|\Delta u(x)|$ and $|\Delta v(x)|$ in terms of the constants D, m, ρ and ω, ω being the volume of Ω. We shall also show the uniqueness of the solution for sufficiently large D (see Theorem 3 below). The proofs will be given in the subsequent sections.

Before stating our main results, we note some fundamental facts.

Let us put

$$\begin{cases} U^* = \operatorname{Max} u(x), & U_* = \operatorname{Min} u(x) , \\ V^* = \operatorname{Max} v(x), & V_* = \operatorname{Min} v(x) . \end{cases} \tag{1.6}$$

Then we have estimates of these four quantities which are valid independent of D.

PROPOSITION 1.1. *Let (u, v) be a solution of* (1.1)-(1.2). *Then, u and v are positive up to the boundary; and we have*

$$V^* \leqq U^{*2} , \tag{1.7a}$$

$$V_* \geqq U_*^2 , \tag{1.7b}$$

$$U^* \leqq mU^{*2}/V_* + m\rho , \tag{1.8a}$$

$$U_* \geqq mU_*^2/V^* + m\rho . \tag{1.8b}$$

The equalities occur only for the constant solution $(\bar{u}, \bar{v})$.

By this proposition the ranges of these four quantities can be limited very roughly as follows:

Suppose for instance that we know an upper bound on U^*. Then V^* is limited from above by (1.7a). This and (1.8b) give us a lower bound of U_*, which in turn limits V_* from below by virtue of (1.7b). (Cf. Figure 1.)

It is also to be noted that from (1.7a) and (1.8b) we have

$$U_* > m\rho \quad \text{and} \quad V_* > (m\rho)^2 , \tag{1.9}$$

which give nontrivial lower bounds if $\rho > 0$.

Next, to obtain bounds of U^* and V^* depending on D, we at first assume that V_* is known and estimate them by means of V_* and D. Using (1.9), we shall have explicit *a priori* bounds when $\rho > 0$.

THEOREM 1. *Suppose that (u, v) is a solution of* (1.1)-(1.2). *Then*

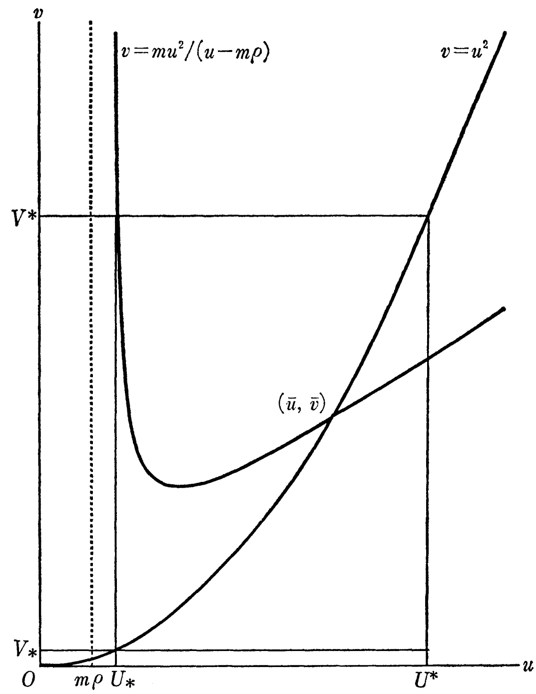

FIGURE 1

we have

$$U^* < \bar{u} + A(m, D, \sqrt{V_*}) \,, \tag{1.10}$$

$$V^* < [\bar{u} + A(m, D, \sqrt{V_*})]^2 \,, \tag{1.11}$$

$$\operatorname{Sup} |\Delta u(x)| < [m + A(m, D, \sqrt{V_*})]/(mD) \,, \tag{1.12}$$

$$\operatorname{Sup} |\Delta v(x)| < [\bar{u} + A(m, D, \sqrt{V_*})]^2 \,, \tag{1.13}$$

where

$$A(m, D, t) = 2mD^{-1}[m^{-1} + t^{-1} + (tD)^{-1}\{t^{-1} + \operatorname{Max}(m^{-1}, t^{-1})\}] \,. \tag{1.14}$$

In particular, if $\rho > 0$, *we have*

$$A(m, D, \sqrt{V_*}) < A(m, D, m\rho) \,. \tag{1.15}$$

In the case of $n = 1$, n being the spatial dimension, Theorem 1 can be improved; and we obtain *a priori* bounds in terms of m, ρ, D and the length of the interval both for $\rho > 0$ and for $\rho = 0$.

THEOREM 2. *Let* $n = 1$ *and* $\Omega = (0, L)$. *Suppose that* (u, v) *is a solution of* (1.1)-(1.2). *Then* (u, v) *is bounded as follows*:

$$m[a(L/\sqrt{mD}) + \rho] < u(x) < \bar{u} + L\sqrt{m/D} \,, \tag{1.16}$$

$$m^2[a(L/\sqrt{mD}) + \rho]^2 < v(x) < (\bar{u} + L\sqrt{m/D})^2 \,, \tag{1.17}$$

where the function $a(t)$ *is defined by*

$$a(t) = (L^{-1} \tanh L)t \operatorname{cosech} t . \tag{1.18}$$

Moreover, we have

$$V^* < e^L V_* . \tag{1.19}$$

REMARK 1.2. It is to be noticed that $a(L/\sqrt{mD})$ is strictly increasing in D; that $a(L/\sqrt{mD}) \uparrow L^{-1} \tanh L$ as $D \uparrow +\infty$; and that $a(L/\sqrt{mD}) \downarrow 0$ as $D \downarrow 0$. Inequality (1.19) implies that the amplitude of v becomes small as L tends to 0.

Using above results, we can derive a uniqueness theorem for sufficiently large D. Here are some notations necessary to state the theorem:

Let l_1 be the smallest positive eigenvalue of the self-adjoint extension of $-\Delta$ under the homogeneous Neumann boundary condition. Define a function $F(s, t)$ by

$$F(s, t) = s[2\{m^2(1 + l_1)\}^{-1} + t^{-2}] + m^{-1} . \tag{1.20}$$

Furthermore, let us put

$$\Phi(D) = \begin{cases} F(\bar{u} + A(m, D, m\rho), m\rho) & \text{if } \rho > 0 , \\ F(\bar{u} + L\sqrt{m/D},\ m[a(L/\sqrt{mD}) + \rho]) & \text{if } n = 1 . \end{cases} \tag{1.21}$$

Then $\Phi(D)$ is strictly decreasing in D because $A(m, D, m\rho)$ and $\sqrt{m/D}$ are both decreasing functions of D, and $a(L/\sqrt{mD})$ is increasing in D.

Therefore the equation

$$l_1 D = \Phi(D) \tag{1.22}$$

for $D > 0$ has a unique solution $D = D^*$.

Now we can state our uniqueness theorem.

THEOREM 3. *Assume either* $\rho > 0$ *or* $n = 1$. *Let* D^* *be the solution of* (1.22). *Then, for* $D > D^*$, *the constant solution* $(\bar{u}, \bar{v})$ *is the unique solution of* (1.1)-(1.2).

In Section 4, Theorems 2 and 3 will be used to study the global behavior of bifurcating branches (see Theorem 4).

2. Proof of a priori estimates. This section is divided into three subsections. In the first paragraph we give basic equalities which play an essential role in proving our *a priori* estimates (particularly Theorem 1). Proposition 1.1 and Theorem 2 are verified in the second subsection by means of the Green function. The last subsection deals with Theorem 1.

Let us put

$$z = u^2/v \quad \text{and} \quad Z^* = \operatorname{Max} z \ . \tag{2.1}$$

As will be seen below, the crucial points of our proofs lie in estimating the term z.

From now on we shall abbreviate $\int_\Omega f(x)dx$ as $\int f$.

2.1. *Basic equalities.* The following equalities (B.1), $\cdots$, (B.7) will be important and used repeatedly.

Multiply both sides of (1.1a) and (1.1b) by $u^p v^q$ and $u^{p+1}v^{q-1}$, respectively, and then integrate them over Ω. By virtue of (1.2), integration by parts leads to

$$\begin{aligned}\int u^{p+1}v^q + pmD\int u^{p-1}v^q|\nabla u|^2 + qmD\int u^p v^{q-1}\nabla u\cdot\nabla v \\ = m\int u^{p+2}v^{q-1} + m\rho\int u^p v^q\end{aligned} \tag{B.1}$$

and

$$\int u^{p+1}v^q + (q-1)\int u^{p+1}v^{q-2}|\nabla v|^2 + (p+1)\int u^p v^{q-1}\nabla u\cdot\nabla v = \int u^{p+3}v^{q-1} \tag{B.2}$$

for each p and $q \in \boldsymbol{R}$.

Eliminate $\int u^p v^{q-1}\nabla u\cdot\nabla v$ to obtain

$$\begin{aligned}(1+p-qmD)\int u^{p+1}v^q + qmD\int u^{p+3}v^{q-1} + p(p+1)mD\int u^{p-1}v^q|\nabla u|^2 \\ = (p+1)m\left(\int u^{p+2}v^{q-1} + \rho\int u^p v^q\right) + q(q-1)mD\int u^{p+1}v^{q-2}|\nabla v|^2 \ .\end{aligned} \tag{B.3}$$

In particular, setting $p = q = 0$ in (B.1) and $p = -1$, $q = 0$ in (B.2), we have

$$\int u = m\int z + m\rho\omega \tag{B.4}$$

and

$$\int z + \int v^{-2}|\nabla v|^2 = \omega \ . \tag{B.5}$$

(Recall that $z = u^2/v$ and ω is the volume of Ω.) Eliminating $\int z$, we also have

$$\int u + m\int v^{-2}|\nabla v|^2 = \omega\bar{u} \ . \tag{B.6}$$

Lastly, (B.2) with $p = -1$ and $q = 1$ yields

(B.7)
$$\int v = \int u^2 .$$

2.2. *Proofs of Proposition* 1.1 *and Theorem* 2. For $\alpha > 0$, let $G(x, \xi; \alpha)$ be the Green function for the boundary value problem

(2.2)
$$\begin{cases} w(x) - \alpha \Delta w(x) = f(x) & \text{in } \Omega , \\ \partial w/\partial \nu = 0 & \text{on } \partial\Omega . \end{cases}$$

Then we have

(2.3)
$$G(x, \xi; \alpha) > 0$$

for $(x, \xi) \in \Omega \times \Omega \setminus \{(x, x);\ x \in \Omega\}$ and

(2.4)
$$\int_\Omega G(x, \xi; \alpha) d\xi = 1 \qquad \text{for all } x \in \Omega ;$$

moreover, the solution of (2.2) is given by

$$w(x) = \int_\Omega G(x, \xi; \alpha) f(\xi) d\xi .$$

Using the Green function, we can convert (1.1)-(1.2) into the following equivalent integral equations:

(2.5a)
$$u(x) = m\int_\Omega G(x, \xi; mD) z(\xi) d\xi + m\rho ,$$

(2.5b)
$$v(x) = \int_\Omega G(x, \xi; 1) u(\xi)^2 d\xi ,$$

where we have used (2.4).

The positivity of u and v is now clear from (2.3) and (2.5). Moreover, (2.5b) and (2.4) imply that $U_*^2 \leqq v(x) \leqq U^{*2}$, which verifies (1.7a) and (1.7b). Since $U_*^2/V^* \leqq z(\xi) \leqq U^{*2}/V_*$ for all $\xi \in \Omega$, we obtain (1.8a) and (1.8b) from (2.5b) and (2.3). Therefore, Proposition 1.1 has been proved.

To demonstrate Theorem 2, we need the explicit form of the Green function. Let $n = 1$ and $\Omega = (0, L)$. Then $G(x, \xi; \alpha)$ is given by

(2.7)
$$G(x, \xi; \alpha) = \begin{cases} b(L/\sqrt{\alpha}) \cosh ((L - x)/\sqrt{\alpha}) \cosh (\xi/\sqrt{\alpha}) \\ \qquad\qquad\qquad \text{if } 0 \leqq \xi \leqq x \leqq L , \\ b(L/\sqrt{\alpha}) \cosh (x/\sqrt{\alpha}) \cosh ((L - \xi)/\sqrt{\alpha}) \\ \qquad\qquad\qquad \text{if } 0 \leqq x \leqq \xi \leqq L , \end{cases}$$

where $b(t) = L^{-1} t \operatorname{cosech} t$.

It is easy to see that

(2.8) $$b(L/\sqrt{\alpha}) \leqq G(x, \xi; \alpha) \leqq (1/\sqrt{\alpha}) \coth (L/\sqrt{\alpha}) .$$

Hence from (2.5a) and (2.8) with $\alpha = mD$ it follows that

$$u(x) \leqq m\left[(1/\sqrt{mD}) \coth (L/\sqrt{mD})\int z + \rho\right] .$$

Since $\int z \leqq L$ by (B.5) and $t \coth t < 1 + t$ for $t > 0$, we have

$$u(x) < m(1 + L/\sqrt{mD}) + m\rho = \bar{u} + L\sqrt{m/D} .$$

On the other hand, putting $\alpha = 1$ in (2.8), we see from (2.5b) that $v(x) \leqq \coth L \cdot \int u^2$. Thus it follows from (2.5a) that

$$u(x) \geqq mb(L/\sqrt{mD})\int u^2 \Big/ \left(\coth L \cdot \int u^2\right) + m\rho = m[\tanh L \cdot b(L/\sqrt{mD}) + \rho] .$$

Therefore (1.16) is verified since $a(L/\sqrt{mD}) = \tanh L \cdot b(L/\sqrt{mD})$. Inequality (1.17) is an immediate consequence of (1.16) and (1.7).

We now pass to the proof of (1.19). Let $x_0 \in [0, L]$ be such that $v(x_0) = V_*$. Then by Schwarz' inequality

$$\begin{aligned} \log v(x) - \log v(x_0) &= \int_{x_0}^{x} v'(\xi)/v(\xi) d\xi \\ &\leqq |x - x_0|^{1/2} \left| \int_{x_0}^{x} [v'(\xi)/v(\xi)]^2 d\xi \right|^{1/2} \\ &\leqq L^{1/2}\left[\int_0^L v^{-2} v'^2 d\xi\right]^{1/2} . \end{aligned}$$

Because $\int v^{-2} v'^2 < L$ by (B.5), we have

$$\log v(x) - \log V_* < L \quad \text{for all } x \in [0, L] ,$$

whence follows $V^*/V_* < e^L$. q.e.d.

2.3. *Proof of Theorem* 1. Our plan of proving Theorem 1 is as follows: We at first assume that V_* is known. Then we have Lemma 2.1 below which permits us to estimate Z^* by means of V_*. Next, (2.5a) and (2.4) imply

(2.9) $$U^* \leqq m(Z^* + \rho) .$$

The estimate of U^* gives us an upper bound on V^* because of (1.7a). This also yields

$$\operatorname{Max} |\Delta v| \leqq U^{*2}$$

since $|\Delta v| = |v - u^2| \leqq \operatorname{Max} \{v, u^2\}$.

On the other hand, we have

$$D \operatorname{Max} |\Delta u| \leqq Z^* \tag{2.10}$$

by virtue of Lemma 2.2 below.

Therefore, the infimum V_* of v gives us all the estimates stated in Theorem 1.

We now state key lemmas:

LEMMA 2.1. *Let* (u, v) *be a solution of* (1.1)-(1.2). *Then, for each natural number* k, *we have*

$$\int z^k \leqq \omega M^{k-1}, \tag{2.11}$$

where

$$M = [1 + \{1 + V_* D(D + 2/m)\}^{1/2}]^2/(V_* D^2). \tag{2.12}$$

In particular,

$$Z^* \leqq M.$$

LEMMA 2.2. *Suppose that* (u, v) *satisfies* (1.1)-(1.2). *Then for any natural number* k

$$D^k \int |\Delta u|^k \leqq 2 \int z^k. \tag{2.13}$$

Consequently, we have

$$D |\Delta u| \leqq Z^*.$$

Before proving these lemmas, we majorize M. From (2.12) we see

$$M = 1 + 2(mD)^{-1} + 2(\sqrt{V_*}D)^{-1}[(\sqrt{V_*}D)^{-1} + \{1 + 2(mD)^{-1} + (\sqrt{V_*}D)^{-2}\}^{1/2}].$$

Noting that $1 + 2(mD)^{-1} + (\sqrt{V_*}D)^{-2} \leqq (1 + D^{-1} \max\{m^{-1}, \sqrt{V_*}^{-1}\})^2$, we find

$$M \leqq 1 + m^{-1} A(m, D, \sqrt{V_*}).$$

Especially, if $\rho > 0$, then $(m\rho)^2 > 0$ limits V_* from below. Therefore we have (1.15) since $A(m, D, t)$ is strictly decreasing in t.

Thus it remains only to demonstrate Lemmas 2.1 and 2.2.

PROOF OF LEMMA 2.1. Substitute $p = 2k - 1$ and $q = -k$ $(k \geqq 1)$ into (B.3) and (B.1) to have

$$(mD + 2)\int z^k + 2(2k-1)mD\int z^k u^{-2}|\nabla u|^2 = mD\int z^{k+1} + (k+1)mD\int z^k v^{-2}|\nabla v|^2 + 2m\left(\int z^{k+1}u^{-1} + \rho\int z^k u^{-1}\right) \tag{2.14}$$

and

$$(2.15)\qquad \int z^k + (2k-1)mD\int z^k u^{-2}|\nabla u|^2$$
$$= m\int z^{k+1}u^{-1} + m\rho\int z^k u^{-1} + kmD\int z^k u^{-1}v^{-1}\nabla u\cdot\nabla v\ ,$$

respectively. Since $\int z^k - m\rho\int z^k u^{-1} = \int z^k u^{-1}(u - m\rho) > 0$ by (1.9), we have from (2.15) that

$$(2k-1)mD\int z^k u^{-2}|\nabla u|^2 < m\int z^{k+1}u^{-1} + kmD\left[\int z^k u^{-2}|\nabla u|^2\cdot\int z^k v^{-2}|\nabla v|^2\right]^{1/2},$$

whence follows, for $0 < \varepsilon < 2k-1$,

$$(2.16)\quad (2k-1-\varepsilon)mD\int z^k u^{-2}|\nabla u|^2 < m\int z^{k+1}u^{-1} + k^2 mD(4\varepsilon)^{-1}\int z^k v^{-2}|\nabla v|^2\ .$$

Eliminating $\int z^k u^{-2}|\nabla u|^2$ from (2.14) and (2.16), we obtain

$$(2.17)\qquad mD\int z^{k+1} < (mD+2)\int z^k + 2m\varepsilon(2k-1-\varepsilon)^{-1}\int z^{k+1}u^{-1}$$
$$+ [k^2(2k-1)\{2\varepsilon(2k-1-\varepsilon)\}^{-1} - (k+1)]mD\int z^k v^{-2}|\nabla v|^2\ .$$

We may eliminate $\int z^k v^{-2}|\nabla v|^2$ from (2.17) by choosing

$$\varepsilon = 2^{-1}[2k-1-\{(2k-1)(k-1)/(k+1)\}^{1/2}]\ ,$$

which satisfies $0 < \varepsilon < 2k-1$ and $\varepsilon(2k-1-\varepsilon)^{-1} \leqq 1$.

Therefore, we have on the one hand

$$mD\int z^{k+1} < (mD+2)\int z^k + 2m\int z^{k+1}u^{-1}\ ;$$

on the other hand,

$$\int z^{k+1}u^{-1} = \int z^{k+1/2}v^{-1/2} \leqq V_*^{-1/2}\int z^{k+1/2} \leqq 2^{-1}\varepsilon\int z^{k+1} + (2\varepsilon V_*)^{-1}\int z^k\ .$$

Consequently we are led to

$$m(D-\varepsilon)\int z^{k+1} < [mD + 2 + m(\varepsilon V_*)^{-1}]\int z^k$$

for $0 < \varepsilon < D$. Minimize $[mD + 2 + m(\varepsilon V_*)^{-1}]/[m(D-\varepsilon)]$ to have $\int z^{k+1} < M\int z^k$, so that

$$\int z^{k+1} < M^k\int z\ ,$$

where $M = [1 + \{1 + V_* D(D + 2/m)\}^{1/2}]^2/(V_* D^2)$. In view of (B.5), we finally obtain (2.11). q.e.d.

PROOF OF LEMMA 2.2. Let us put

$$w = u - m\rho \,.$$

Then we see $w > 0$ because of (1.9). Since $mD\Delta u = w - mz$ and z is also positive, we have $mD|\Delta u| \leqq \mathrm{Max}\{w, mz\}$. Suppose that $w \geqq mz$ on $E_1 \subset \Omega$ and $w \leqq mz$ on $E_2 \subset \Omega$. Then

$$(mD)^k \int |\Delta u|^k \leqq \int_{E_1} w^k + m^k \int_{E_2} z^k \leqq \int_\Omega w^k + m^k \int_\Omega z^k \,,$$

that is,

$$(mD)^k \int |\Delta u|^k \leqq \int w^k + m^k \int z^k \,. \tag{2.18}$$

In view of $\Delta u = \Delta w$, we see $mD\Delta w = w - mz$. Integrate both sides of this equation over Ω after multiplication by w^{k-1} to find

$$\int w^k + (k-1)mD \int w^{k-2} |\nabla w|^2 = m \int z w^{k-1} \,,$$

so that

$$\int w^k \leqq m \int z w^{k-1} \qquad \text{for each } k \geqq 1 \,.$$

By Hölder's inequality, we have

$$\int w^k \leqq m^k \int z^k \,. \tag{2.19}$$

Combining (2.18) and (2.19) proves (2.13), as desired. q.e.d.

3. Proof of uniqueness. Let us decompose u and v as follows:

$$u = u_0 + \phi \quad \text{and} \quad v = v_0 + \psi \,, \tag{3.1}$$

where

$$u_0 = \omega^{-1} \int u \,, \qquad v_0 = \omega^{-1} \int v \,. \tag{3.2}$$

Then we have

$$\int \phi = \int \psi = 0 \,. \tag{3.3}$$

By a simple computation, we see that (ϕ, ψ) is a solution of the system of equations

$$(3.4a)\quad D\Delta\phi - (1/m - 2a)\phi - a^2\psi + (\phi - a\psi)^2/v = u_0/m - \rho - u_0^2/v_0\,,$$

$$(3.4b)\quad \Delta\psi - \psi + 2u_0\phi + \phi^2 = v_0 - u_0^2\,,$$

where $a = u_0/v_0$.

We wish to show that $\phi = \psi = 0$ if D is greater than some value D^*. Theorem 3 follows at once from this, because (u_0, v_0) is then a constant solution of (1.1)-(1.2), hence is equal to $(\bar{u}, \bar{v})$ by the uniqueness of constant solutions.

Our proof is divided into three steps. The first is to show that

$$(3.5)\quad \int\psi^2 \leqq \beta^2\int\phi^2$$

for a constant β. The second is to derive an inequality of type

$$(3.6)\quad (D - A)\int\phi^2 \leqq B\int\phi^2$$

with some constants A and B. Then we have the desired result $\phi = \psi = 0$ if $D > A + B$. Our third step is to show that A and B are majorized by decreasing functions of D, whence follows $D > A + B$ for sufficiently large D.

Our main tool is the following Lemma 3.1. Let l_1 be the smallest positive eigenvalue of the self-adjoint extension of $-\Delta$ under the homogeneous Neumann boundary condition. Then by the eigenfunction expansion, we have the following inequality of Poincaré type:

LEMMA 3.1. *If $w(x) \in C^1(\bar{\Omega})$, then*

$$(3.7)\quad \int w^2 \leqq l_1^{-1}\int|\nabla w|^2 + \omega^{-1}\left(\int w\right)^2 .$$

We now pass to the first step of the proof. After multiplying both sides of (3.4b) by ψ, we integrate over Ω to have

$$\int|\nabla\psi|^2 + \int\psi^2 = \int(u_0 + u)\phi\psi\,.$$

The right hand side does not exceed $2U^*\int|\phi\psi|$, where $U^* = \operatorname{Max} u$; while the left hand side is greater than $(l_1 + 1)\int\psi^2$ by virtue of Lemma 3.1. Therefore we have (3.5) with

$$(3.8)\quad \beta = 2U^*/(1 + l_1)\,.$$

Let us proceed to the second step. Integrate both sides of (3.4a) over Ω. Then we see

$$\text{(3.9)} \quad \omega^{-1}\int(\phi - a\psi)^2 v^{-1} = u_0/m - \rho - u_0^2/v_0 = (u_0 - \bar{u})/m + (v_0 - u_0^2)/v_0$$
$$\leqq (v_0 - u_0^2)/v_0 = (\omega v_0)^{-1}\int\phi^2 ,$$

where we have used (B.4) and (B.5) which imply $u_0 \leqq \bar{u}$. Therefore,

$$\text{(3.10)} \qquad \int(\phi - a\psi)^2 v^{-1} \leqq v_0^{-1}\int\phi^2 .$$

Next, let us integrate over Ω both sides of (3.4a) multiplied by ϕ to obtain

$$D\int|\nabla\phi|^2 + (1/m - 2a)\int\phi^2 + a^2\int\phi\psi = \int\phi(\phi - a\psi)^2 v^{-1} .$$

We remark that $|\phi|$ does not exceed U^*, because $0 < u_0 \leqq U^*$ and $U_* - u_0 \leqq \phi \leqq U^* - u_0$. Hence the right hand side of the above equation is not greater than $(U^*/v_0)\int\phi^2$ by virtue of (3.10). Using Lemma 3.1 again to see that $\int|\nabla\phi|^2 \geqq l_1\int\phi^2$, we have

$$\text{(3.11)} \quad (Dl_1 + 1/m - 2a - U^*/v_0)\int\phi^2 \leqq a^2\int|\phi\psi| \leqq 2a^2U^*(1 + l_1)^{-1}\int\phi^2 ,$$

where we have used (3.5) and (3.10). This is nothing but an inequality of type (3.6) if we can majorize a and $1/v_0$ by some constants.

Recalling that $U_* = \operatorname{Min} u$, we see that $1/v_0 = \omega\Big/\int v = \omega\Big/\int u^2 \leqq 1/U_*^2$, and that from (3.9) $u_0/m - \rho - u_0^2/v_0 \geqq 0$ hence $a = u_0/v_0 \leqq 1/m - \rho/u_0 \leqq 1/m$. That is,

$$\text{(3.12)} \qquad a \leqq 1/m \quad \text{and} \quad 1/v_0 \leqq 1/U_*^2 .$$

Therefore,

$$2a^2U^*/(1 + l_1) + U^*/v_0 + 2a \leqq U^*[2/\{m^2(1 + l_1)\} + 1/U_*^2] + 2/m .$$

Consequently, if D satisfies

$$l_1D > U^*[2/\{m^2(1 + l_1)\} + 1/U_*^2] + 1/m ,$$

then we have $\int\phi^2 = \int\psi^2 = 0$.

In the case of $\rho > 0$, we estimate U^* by (1.10) and (1.15) of Theorem 1 and U_* by (1.9); if $n = 1$, then we can apply Theorem 2 to obtain an upper bound of U^* and a lower bound of U_*. This is enough to prove Theorem 3. q.e.d.

4. Remarks on behavior of bifurcating branches. We consider in this section the structure of the solution set of (1.1)-(1.2) in the case that the spatial dimension is one.

Regarding D as a parameter and the unknowns as a column vector $U = {}^t(u, v)$, we define

$$\mathscr{E} = \boldsymbol{R}_+ \times E\,, \tag{4.1}$$

where $\boldsymbol{R}_+$ denotes the set of all positive real numbers and

$$E = \{U = {}^t(u, v);\ u, v \in C^2[0, L],\ u' = v' = 0 \text{ at } x = 0, L\}\,. \tag{4.2}$$

Here and hereafter, the prime stands for d/dx. A pair $(D_0, U_0) \in \mathscr{E}$ is said to be a solution of (1.1)-(1.2) if $U = U_0$ satisfies (1.1) for $D = D_0$. Let us put

$$\Gamma = \{(D, \bar{U});\ D \in \boldsymbol{R}_+\}, \quad \bar{U} = {}^t(\bar{u}, \bar{v})\,.$$

Observing that $(D, \bar{U})$ is always a solution for $D > 0$, we call Γ the *trivial branch.*

Under assumption (H.3) on ρ below, it is shown by bifurcation theory that there exists a sequence $\{D_j\}_{j=1}^{\infty} \subset \boldsymbol{R}_+$ such that (i) $D_j \to 0$ as $j \to \infty$, and (ii) for each j, $(D_j, \bar{U}) \in \Gamma$ is a bifurcation point, i.e., in a neighborhood of $(D_j, \bar{U})$ there is a one-parameter family of nonconstant solutions passing through $(D_j, \bar{U})$ (see Proposition 4.1 below).

We are interested in the behavior of the bifurcating branch when it is far from the bifurcation point $(D_j, \bar{U})$. It is observed numerically that (1.1)-(1.2) has wavy solutions with extremely large amplitude if D is very small (so-called "striking patterns"). Therefore, it is a fundamental problem to prove the existence of such solutions and clarify the connection between bifurcating branches and striking patterns. As a first step in attacking this problem, we shall show in Theorem 4 below that bifurcating branches can be prolonged with respect to D into any small neighborhood of $D = 0$. (Cf. Nishiura [10], where more detailed results are established for some systems with saturation.)

The plan of this section is as follows: First we formulate the problem as an abstract equation (4.4) on $\mathscr{E}$. Secondly we state the existence of local bifurcation in Proposition 4.1. Then the main goal of this section will be asserted in Theorem 4.

We begin by rewriting (1.1) as equations around $\bar{U}$. Put

$$U = \bar{U} + W \quad \text{and} \quad W = {}^t(\phi, \psi)\,.$$

Then W satisfies the following equations:

$$D\phi'' + \mu m^{-1}\phi - \bar{v}^{-1}\psi + (\phi - \bar{u}^{-1}\psi)^2/(\bar{v} + \psi) = 0\,, \tag{4.3a}$$

(4.3b) $$\psi'' - \psi + 2\bar{u}\phi + \phi^2 = 0 ,$$

where

$$\mu = (1 - \rho)/(1 + \rho) .$$

As was pointed out in [12], if $\rho \geqq 1$, then there is no bifurcation, i.e., $\bar{U}$ is an isolated solution for each $D > 0$. Therefore, we assume

(H.3) $$0 \leqq \rho < 1 ,$$

which is equivalent to the condition $0 < \mu \leqq 1$.

Now, for the positive real numbers α and β, let $\mathscr{G}(\alpha, \beta)$ be the Green operator for the boundary value problem

$$\begin{cases} \beta w - \alpha w'' = f & \text{in } (0, L) , \\ w' = 0 & \text{at } x = 0, L . \end{cases}$$

Put

$$G(D) = \mathscr{G}(D, \mu/m) \quad \text{and} \quad G_1 = \mathscr{G}(1, 1) .$$

Then (4.3) can be interpreted as the following integral equation for $(D, W) \in \mathscr{E}$:

(4.4) $$[I - T(D)]W - N(D, W) = 0 ,$$

where I denotes the identity operator on E, $T(D)$ and $N(D, W)$ are defined by

(4.5) $$T(D) = \begin{bmatrix} 2\mu m^{-1}G(D) & -\bar{v}^{-1}G(D) \\ 2\bar{u}G_1 & 0 \end{bmatrix}$$

and

(4.6) $$N(D, W) = {}^t(G(D)[(\phi - \bar{v}^{-1}\psi)^2/(\bar{v} + \psi)],\ G_1[\phi^2])$$

for $W = {}^t(\phi, \psi)$, respectively.

It should be noticed that T and N are compact operators from a domain of $\mathscr{E}$ into E; that $D \mapsto T(D) \in B(E)$, $(D, W) \mapsto N(D, W) \in E$ are analytic, $B(E)$ being the Banach space of bounded linear operators on E; and that $N(D, W) = O(\|W\|^2)$ uniformly on each compact sub-interval of $\boldsymbol{R}_+$.

If $I - T(D_0)$ is invertible, then the implicit function theorem yields that $(D_0, 0) \in \mathscr{E}$ is not a bifurcation point for (4.4). Thus we wish to find a condition on D for $I - T(D_0)$ to be singular (i.e., non-invertible). For this purpose we introduce a function $g(l)$ for $l > 0$:

$$g(l) = (\mu l - 1)/[ml(1 + l)] .$$

Define a sequence $\{D_j\}_{j=1}^{\infty}$ by

$$D_j = g(l_j) \quad \text{for } j = 1, 2, 3, \cdots ,$$

where

$$l_j = (\pi j/L)^2 \quad \text{for } j = 0, 1, 2, \cdots .$$

As will be seen in Appendix, $I - T(D)$ is singular if and only if $D = D_j > 0$ for some j. We say that D_j is a *simple critical value of mode* j if $D_j > 0$ and $D_n \neq D_j$ for all $n \neq j$. (Note that $g(l)$ is positive for $l > \mu^{-1}$. Since $g(l)$ is strictly increasing in the interval $(0, \hat{l})$ and strictly decreasing in the interval $(\hat{l}, +\infty)$ with $\hat{l} = (1 + \sqrt{1+\mu})/\mu$, the equation $g(l) = D_j$ has two roots $l = l_j$ and $l = l_j^*$, where $l_j^* = (1+l_j)/(\mu l_j - 1)$. Therefore, D_j is a simple critical value if and only if $l_n \neq l_j^*$ for all $n \neq j$.) If $D_j > 0$ is not simple, we call it a *double critical value*.

First, we state the existence of local bifurcation.

PROPOSITION 4.1. *Assume that* (H.3) *is satisfied. Let D_j be a simple critical value. Then $(D_j, \bar{U})$ is a bifurcation point. More precisely, in a neighborhood of $(D_j, \bar{U}) \in \mathscr{C}$ there exists a one-parameter family of non-constant solutions $(D_j(\varepsilon), U_j(\varepsilon))$ with $U_j(\varepsilon) = {}^t(u_j(\varepsilon), v_j(\varepsilon))$ such that*

$$\begin{aligned} D_j(\varepsilon) &= D_j + O(\varepsilon^2) , \\ u_j(\varepsilon) &= \bar{u} + \varepsilon \cdot a_1 \cos(\pi j x/L) + O(\varepsilon^2) , \\ v_j(\varepsilon) &= \bar{v} + \varepsilon \cdot a_2 \cos(\pi j x/L) + O(\varepsilon^2) , \end{aligned}$$

where $a_1 = \sqrt{2/L}$ and $a_2 = 2\bar{u}a_1/(1 + l_j)$. Furthermore, the solution set of (1.1) *near $(D_j, \bar{U})$ consists of exactly two curves Γ and $(D_j(\varepsilon), U_j(\varepsilon))$.*

This proposition can be verified by the well-known theorem of Crandall and Rabinowitz [1]. (See, for details, [12].)

Let $\mathscr{S}$ be the closure of the set of non-trivial solutions (D, U), $U \neq \bar{U}$, in $\mathscr{E}$. Moreover, let $\mathscr{C}^{(j)}$ be the connected component of $\mathscr{S}$ containing $(D_j, \bar{U})$. Put

$$\mathrm{Proj}_{\boldsymbol{R}_+}\mathscr{C}^{(j)} = \{D \in \boldsymbol{R}_+;\ (D, U) \in \mathscr{C}^{(j)}\}$$

and

$$\mathrm{Proj}_{E}\mathscr{C}^{(j)} = \{U \in E;\ (D, U) \in \mathscr{C}^{(j)}\} .$$

Our main goal in this section is then stated as follows:

THEOREM 4. *Suppose that* (H.3) *holds and that D_j is a simple critical value. Then $\mathrm{Proj}_{\boldsymbol{R}_+}\mathscr{C}^{(j)}$ is a bounded interval and*

$$\mathrm{Proj}_{\boldsymbol{R}_+}\mathscr{C}^{(j)} \supset (0, D_j] .$$

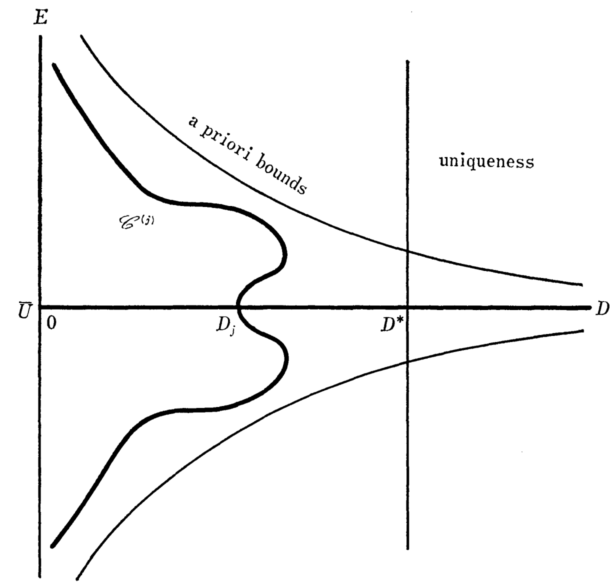

FIGURE 2

This theorem asserts that if we continue the bifurcating branch $(D_j(\varepsilon),\ U_j(\varepsilon))$ with respect to ε, we may take the value of D arbitrarily small. Although numerical analyses suggest that $\mathrm{Proj}_E\mathscr{C}^{(j)}$ is unbounded in E, we have not succeeded in proving it.

In order to demonstrate Theorem 4, we use the following key lemma:

LEMMA 4.2. *Assume that* (H.3) *holds. Let D_j be simple. Then the connected component $\mathscr{C}^{(j)}$ is not compact in $\mathscr{E}$.*

Now, Theorem 4 is verified by combining *a priori* estimates (Theorem 2), uniqueness of the solutions for large values of D (Theorem 3) and Lemma 4.2. First, observe that Theorem 3 ensures the boundedness of the interval $\mathrm{Proj}_{R_+}\mathscr{C}^{(j)}$. Next, assume that $\mathrm{Proj}_{R_+}\mathscr{C}^{(j)} \cap (0, \delta) = \varnothing$ for some $\delta > 0$. Then the *a priori* estimate implies that $\mathrm{Proj}_E\mathscr{C}^{(j)}$ is bounded in E. Therefore $\mathscr{C}^{(j)}$ is bounded in $\mathscr{E}$. The compactness of the operators $T(D)$ and $N(D, W)$ then yields that $\mathscr{C}^{(j)}$ is a compact set in $\mathscr{E}$, which contradicts Lemma 4.2. Consequently, we have $\mathrm{Proj}_{R_+}\mathscr{C}^{(j)} \cap (0, \delta) \neq \varnothing$ for all $\delta > 0$. This is sufficient to show Theorem 4.

It remains to prove Lemma 4.2. The main idea is due to Nishiura [10]. Suppose that $\mathscr{C}^{(j)}$ be compact in $\mathscr{E}$. We are going to show that this assumption reduces to absurdity. We start by claiming that Rabinowitz' results on global bifurcation [11] can be applied to our case:

LEMMA 4.3. *Assume that* (H.3) *is satisfied and* D_j *is simple. If* $\mathscr{C}^{(j)}$ *is compact, then there exists at least one simple critical value* D_n *such that* $D_n \neq D_j$ *and* $(D_n, \bar{U}) \in \mathscr{C}^{(j)}$.

For the proof, see Appendix.

Now, let p be the largest mode of the simple critical values contained in $\mathrm{Proj}_{R_+}\mathscr{C}^{(j)}$. Note that $p > 1$ since $\mathscr{C}^{(j)}$ contains at least two simple bifurcation points.

Following [10], let us pose an auxiliary problem:

$(\tilde{\mathrm{P}})$ Consider (1.1)-(1.2) on the interval $(0, L/p)$.

We refer to our original problem on the interval $(0, L)$ as the problem (P). Define two function spaces $\tilde{E}$ and $\tilde{\mathscr{E}}$ by replacing L with L/p in the definition of E and $\mathscr{E}$. We continue a function $\tilde{U}$ in $\tilde{E}$ by reflection and periodicity to obtain a function $\gamma\tilde{U}$ in E as follows: Put $x_n = nL/p$ for $n = 0, 1, 2, \cdots$, and set

$$(\gamma\tilde{U})(x) = \begin{cases} \tilde{U}(x - x_{2n}) & \text{if } x_{2n} \leqq x \leqq x_{2n+1}\,, \\ \tilde{U}(x_{2n+2} - x) & \text{if } x_{2n+1} \leqq x \leqq x_{2n+2}\,. \end{cases}$$

We also define a mapping $\gamma: \tilde{\mathscr{E}} \to \mathscr{E}$ by $\gamma(D, \tilde{U}) = (D, \gamma\tilde{U})$.

It is to be noticed that, for each solution $(D, \tilde{U})$ of $(\tilde{\mathrm{P}})$, $\gamma(D, \tilde{U})$ is a solution of (P); and that D_j is a simple critical value of mode j for $(\tilde{\mathrm{P}})$ if and only if it is a simple critical value of mode pj for (P).

Therefore, $(D_p, \bar{U}) \in \tilde{\mathscr{E}}$ is a bifurcation point of mode one. Let $\tilde{\mathscr{C}}^{(1)}$ be the connected component of $\tilde{\mathscr{S}}$ containing $(D_p, \bar{U})$, where $\tilde{\mathscr{S}}$ is defined analogously to $\mathscr{S}$. If $\tilde{\mathscr{C}}^{(1)}$ is non-compact in $\tilde{\mathscr{E}}$, then $\gamma\tilde{\mathscr{C}}^{(1)}$ is also non-compact in $\mathscr{E}$. However, this is impossible because $\gamma\tilde{\mathscr{C}}^{(1)} \subset \mathscr{C}^{(j)}$. Hence $\tilde{\mathscr{C}}^{(1)}$ must be compact. Then Lemma 4.3 applied to $(\tilde{\mathrm{P}})$ asserts that $\tilde{\mathscr{C}}^{(1)}$ has to contain a simple bifurcation point $(\tilde{D}_q, \bar{U})$ of mode $q>1$. Thus we see that $\tilde{D}_q$ is a critical value of mode $pq > p$ contained in $\mathrm{Proj}_{R_+}\mathscr{C}^{(j)}$, which is inconsistent with the maximality of p. Consequently $\mathscr{C}^{(j)}$ must be non-compact in $\mathscr{E}$.

Appendix. For the sake of completeness, we prove Lemma 4.3 which is a special version of Rabinowitz' results [11]. He showed the corresponding fact for equations of type (4.4) in the case of $T(D) = DT$, T being a compact linear operator. A close examination of his proof leads to the conclusion that the assertion is valid if $T(D)$ satisfies the following condition at $D = D_j$:

$$\text{ind}\,(I - T(D_j - \varepsilon)) \neq \text{ind}\,(I - T(D_j + \varepsilon)) \tag{A.1}$$

for all sufficiently small $\varepsilon > 0$. Here $\operatorname{ind}(I - T(D))$ denotes the Leray-Schauder index of $I - T(D)$, i.e., $\operatorname{ind}(I - T(D)) = \deg(I - T(D), B_r, 0)$ with $B_r = \{W \in E;\ \|W\| < r\}$.

The general case that $T(D)$ is a non-linear function of D was treated by Ize [6], who gave a sufficient condition for (A.1). However, it seems rather tedious to verify his condition in our case. Instead, we are going to show by a straightforward computation that (A.1) is satisfied if D_j is a simple critical value.

We begin by recalling that

$$\operatorname{ind}(I - T(D)) = (-1)^{\beta(D)}, \tag{A.2}$$

where $\beta(D)$ is the sum of the algebraic multiplicities of all the eigenvalues of $T(D)$ which are greater than 1 (see, e.g., Nirenberg [9]). The function $\beta(D)$ has a simple expression as follows:

LEMMA A.1. *Let $\beta(D)$ be as above. Then*

$$\beta(D) = \#\{n \in N;\ D_n > D\}, \tag{A.3}$$

where $\#S$ denotes the number of the elements of a set S.

Before proving the lemma, we observe that

$$\beta(D_j - \varepsilon) = \begin{cases} \beta(D_j + \varepsilon) + 1, & \text{if } D_j \text{ is simple,} \\ \beta(D_j + \varepsilon) + 2, & \text{if } D_j \text{ is double,} \end{cases}$$

for sufficiently small $\varepsilon > 0$. Therefore, in view of (A.2), we see that (A.1) is satisfied if D_j is simple.

PROOF OF LEMMA A.1. We begin by computing the eigenvalues of $T(D)$. Using the Fourier cosine expansion, we see easily that λ is an eigenvalue of $T(D)$ if and only if λ satisfies the following characteristic equation for some $j \geqq 0$:

$$\det \begin{bmatrix} 2\mu(\mu + mDl_j)^{-1} - \lambda & -m[\bar{v}(\mu + mDl_j)]^{-1} \\ 2\bar{u}(1 + l_j)^{-1} & -\lambda \end{bmatrix} = 0,$$

that is,

$$\lambda^2 - 2\mu(\mu + mDl_j)^{-1}\lambda + 2m[\bar{u}(1 + l_j)(\mu + mDl_j)]^{-1} = 0. \tag{A.4}$$

Here $l_j = (\pi j/L)^2$ is the $(j + 1)$-th eigenvalue of $-d^2/dx^2$ under the Neumann boundary condition.

Since $\operatorname{Ker}(I - T(D))$ is non-trivial if and only if $\lambda = 1$ is an eigenvalue of $T(D)$, we see that $D = D_j$ is necessary and sufficient for $I - T(D)$ to be singular.

Now let $\lambda_0 > 1$ be an eigenvalue of $T(D)$. We wish to compute its algebraic multiplicity. First, note that $\lambda = \lambda_0$ satisfies (A.4) for some $j \geqq 1$. It is immediately seen that the other root $\lambda = \lambda_0'$ of (A.4) is smaller than 1, so that $\lambda = \lambda_0$ is a simple root. Therefore, the algebraic multiplicity of λ_0 coincides with the number of the j's such that l_j satisfies (A.4) with $\lambda = \lambda_0$. (This number does not exceed two because (A.4) is quadratic in l_j.)

Running λ_0 over all the eigenvalues greater than 1, we see that $\beta(D)$ is equal to the number of all the j's for which (A.4) has a root greater than 1. However this occurs if and only if j satisfies $D_j > D$. Consequently we obtain $\beta(D) = \#\{n;\ D_n > D\}$, as required. q.e.d.

REFERENCES

[1] M. G. CRANDALL AND P. H. RABINOWITZ, Bifurcation from simple eigenvalues, J. Functional Analysis 8 (1971), 321-340.

[2] H. FUJII, M. MIMURA AND Y. NISHIURA, A picture of global bifurcation diagram in ecological interacting and diffusing systems, Research Report, No. 79-11, Kyoto Sangyo University, 1979.

[3] A. GIERER AND H. MEINHARDT, A theory of biological pattern formation, Kybernetik 12 (1972), 30-39.

[4] K. P. HADELER, F. ROTHE AND H. VOGT, Stationary solutions of reaction-diffusion equations, Math. Mech. in the Appl. Scien. 1 (1979), 418-431.

[5] H. HAKEN AND H. OLBRICH, Analytical treatment of pattern formation in the Gierer-Meinhardt model of morphogenesis, J. Math. Biol. 6 (1978), 317-331.

[6] J. IZE, Bifurcation theory for Fredholm operators, Mem. Amer. Math. Soc. Vol. 7 No. 174, 1976.

[7] J. P. KEENER, Activators and inhibitors in pattern formation, Studies in Appl. Math. 59 (1978), 1-23.

[8] H. MEINHARDT AND A. GIERER, Applications of a theory of biological pattern formation based on lateral inhibition, J. Cell Sci. 15 (1974), 321-346.

[9] L. NIRENBERG, Topics in Nonlinear Functional Analysis, Lecture Notes, New York University, 1974.

[10] Y. NISHIURA, Global structure of bifurcating solutions of some reaction-diffusion systems, Research Report, No. 79-12, Kyoto Sangyo University, 1979.

[11] P. H. RABINOWITZ, Some global results for nonlinear eigenvalue problems, J. Functional Analysis 7 (1971), 487-513.

[12] I. TAKAGI, Stability of bifurcating solutions of the Gierer-Meinhardt system, Tôhoku Math. J. 31 (1979), 221-246.

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