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ON GENERALIZED SIEGEL DOMAINS WITH EXPONENT $(c_1, c_2, \cdots, c_s)$, II

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Introduction. This is a continuation of our previous paper [3], and we retain the terminology and notations there.

As a natural generalization of the notion of generalized Siegel domains in $C^n \times C^m$ with exponent c due to Kaup, Matsushima and Ochiai [2], we introduced in [3] the notion of generalized Siegel domains in $C^n \times C^{m_1} \times C^{m_2} \times \cdots \times C^{m_s}$ with exponent $(c_1, c_2, \cdots, c_s)$. For a domain D in C^N, we shall denote by $\mathrm{Aut}(D)$ the group of all holomorphic transformations of D onto itself. Then we say that D is a sweepable domain if there exist a subgroup Γ of $\mathrm{Aut}(D)$ and a compact subset K of D such that $\Gamma \cdot K = D$. In [5], Vey investigated the structure of generalized Siegel domains in $C^n \times C^m$ with exponent c and gave an interesting characterization of Siegel domains of the first or the second kind in the sense of Pjateckii-Sapiro [4] among generalized Siegel domains. His results may be stated as follows:

THEOREM (Vey [5]). (A) *Let $\mathscr{D}$ be a sweepable generalized Siegel domain in $C^n \times C^m$ with exponent c. Then we have the following*:

(A-1) *If $c \neq 0$, then $\mathscr{D}$ is a Siegel domain of the first or the second kind according as $m = 0$ or $m > 0$.*

(A-2) *If $c = 0$, then $\mathscr{D}$ is the direct product $\mathscr{D}_1 \times \mathscr{D}_2$, where $\mathscr{D}_1$ is a Siegel domain of the first kind in C^n and $\mathscr{D}_2$ is a homogeneous bounded circular domain in C^m containing the origin.*

(B) *Let $\mathscr{D}$ be a generalized Siegel domain in $C^n \times C^m$ with exponent c. Suppose that $\mathscr{D}$ admits a discrete subgroup Γ of* $\mathrm{Aut}(\mathscr{D})$ *such that $\mathscr{D}/\Gamma$ is compact. Then $\mathscr{D}$ is symmetric.*

As a generalization of (A-1) of Vey's theorem, we proved the following theorem in [3]:

THEOREM I (Kodama [3]). *A sweepable generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2} \times \cdots \times C^{m_s}$ with exponent $(c_1, c_2, \cdots, c_s)$ with $c_i \neq 0$ for*

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$1 \leqq i \leqq s$ is a Siegel domain of the first or the second kind according as $m_1 + m_2 + \cdots + m_s = 0$ or >0.

The purpose of this paper is to extend the above results (A-2) and (B) by Vey to our generalized Siegel domains in $\boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \boldsymbol{C}^{m_2} \times \cdots \times \boldsymbol{C}^{m_s}$ with exponent $(c_1, c_2, \cdots, c_s)$. Given a permutation σ of the set $\{1, 2, \cdots, s\}$, we shall define a linear transformation $\mathscr{L}_\sigma$ of $\boldsymbol{C}^{n+m_1+m_2+\cdots+m_s}$ onto itself by

$$\mathscr{L}_\sigma(z, w_1, w_2, \cdots, w_s) = (z, w_{\sigma(1)}, w_{\sigma(2)}, \cdots, w_{\sigma(s)}) .$$

Under this notation we have the following:

THEOREM II. *Let $\mathscr{D}$ be a sweepable generalized Siegel domain in $\boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \boldsymbol{C}^{m_2} \times \cdots \times \boldsymbol{C}^{m_s}$ with exponent $(c_1, c_2, \cdots, c_s)$. Suppose that some of the exponents, say, $c_{i_1}, c_{i_2}, \cdots, c_{i_k}$ $(1 \leqq k \leqq s)$ are equal to zero and the others are not. Then, putting $\tilde{m}_2 = m_{i_1} + m_{i_2} + \cdots + m_{i_k}$ and $\tilde{m}_1 = (m_1 + m_2 + \cdots + m_s) - \tilde{m}_2$, we have that $\mathscr{D}$ is the direct product $\mathscr{D}_1 \times \mathscr{D}_2$ up to a suitable linear transformation $\mathscr{L}_\sigma$, where $\mathscr{D}_1$ is a Siegel domain of the first or the second kind in $\boldsymbol{C}^n \times \boldsymbol{C}^{\tilde{m}_1}$ according as $\tilde{m}_1 = 0$ or $\tilde{m}_1 > 0$ (i.e., $k = s$* or *$1 \leqq k < s$) and $\mathscr{D}_2$ is a homogeneous bounded circular domain in $\boldsymbol{C}^{\tilde{m}_2}$ containing the origin.*

THEOREM III. *Let $\mathscr{D}$ be a generalized Siegel domain in $\boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \boldsymbol{C}^{m_2} \times \cdots \times \boldsymbol{C}^{m_s}$ with exponent $(c_1, c_2, \cdots, c_s)$. Suppose that $\mathscr{D}$ admits a discrete subgroup Γ of* Aut$(\mathscr{D})$ *with compact quotient $\mathscr{D}/\Gamma$. Then $\mathscr{D}$ is symmetric.*

The idea of the proofs is due to Kaup, Matsushima and Ochiai [2] and also Vey [5].

This paper is organized as follows. In Section 1 we investigate the structure of the Lie algebra $\mathfrak{g}(\mathscr{D})$ in the case where $\mathscr{D}$ is a generalized Siegel domain in $\boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \boldsymbol{C}^{m_2}$ with exponent $(1/2, 0)$. And we give the proofs of our Theorems II and III in this special case.

In Section 2, as a preparation for the next section, we study holomorphic vector fields belonging to $\mathfrak{g}(\mathscr{D})$ which are independent of $z_1, z_2, \cdots, z_n$.

In the final Section 3 we first prove that our problem can be reduced to the special case where $\mathscr{D}$ is a generalized Siegel domain in $\boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \boldsymbol{C}^{m_2}$ with exponent $(1/2, 0)$. After that, the proofs of Theorems II and III will be obtained as an immediate consequence of Section 1.

1. The structure of generalized Siegel domains in $\boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \boldsymbol{C}^{m_2}$ with exponent $(1/2, 0)$. The purpose of this section is to prove the following theorems:

THEOREM 1.1. *Let $\mathscr{D}$ be a sweepable generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2}$ with exponent* (1/2, 0). *Then $\mathscr{D}$ is the direct product $\mathscr{D}_1 \times \mathscr{D}_2$, where $\mathscr{D}_1$ is a Siegel domain of the second kind in $C^n \times C^{m_1}$ and $\mathscr{D}_2$ is a homogeneous bounded circular domain in C^{m_2} containing the origin.*

THEOREM 1.2. *Let $\mathscr{D}$ be a generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2}$ with exponent* (1/2, 0). *Suppose that $\mathscr{D}$ admits a discrete subgroup Γ of* Aut($\mathscr{D}$) *with compact quotient $\mathscr{D}/\Gamma$. Then $\mathscr{D}$ is symmetric.*

After long series of lemmas, we first clarify the structure of the Lie algebra $\mathfrak{g}(\mathscr{D})$ in Theorem 1.22. And, using this result, we give the proofs of the above theorems at the end of this section.

The proof of the following proposition is similar to that of Vey [5, Proposition 1.1] and hence is left to the reader:

PROPOSITION 1.3. *Let $\mathscr{D}$ be a holomorphically convex generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2}$ with exponent (c_1, c_2). We put*

$$\mathscr{D}_0 = \mathscr{D} \cap (C^n \times \{0\} \times \{0\}) .$$

Then we have the following:

(1) *$\mathscr{D}_0$ is a Siegel domain of the first kind in C^n, and hence it is expressed as*

$$\mathscr{D}_0 = \{z \in C^n \mid \operatorname{Im} z \in \Omega\} ,$$

where Ω is an open convex cone in R^n containing no straight line.

(2) *Let $(z, w_1, w_2) \in \mathscr{D}$. Then $(z, \lambda w_1, \mu w_2) \in \mathscr{D}$ for any $\lambda, \mu \in C$ with $|\lambda| \leqq 1$ and $|\mu| \leqq 1$.*

Throughout the rest of this section we denote by $\mathscr{D}$ a sweepable generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2}$ with exponent (1/2, 0). We also use the following notations:

$$\mathfrak{Z}_{\mu\nu_1\nu_2} = \{Z_{\mu\nu_1\nu_2}\} ;$$
$$\mathfrak{W}^{\lambda}_{\mu\nu_1\nu_2} = \{W^{\lambda}_{\mu\nu_1\nu_2}\} \qquad \text{for} \quad \lambda = 1, 2 ,$$

where $Z_{\mu\nu_1\nu_2}$ and $W^{\lambda}_{\mu\nu_1\nu_2}$ are polynomial vector fields defined in [3, Section 1]. We put $\mathfrak{Z}_{\mu\nu_1\nu_2} = \{0\}$ and $\mathfrak{W}^{\lambda}_{\mu\nu_1\nu_2} = \{0\}$ if μ, ν_1 or ν_2 are negative.

In the case where $\mathscr{D}$ is a generalized Siegel domain with exponent (1/2, 0) we have from [3, (1.1)] that

$$(1.1) \qquad \begin{cases} [\partial, Z_{\mu\nu_1\nu_2}] = (\mu - 1 + \nu_1/2) Z_{\mu\nu_1\nu_2} ; \\ [\partial, W^1_{\mu\nu_1\nu_2}] = (\mu + \nu_1/2 - 1/2) W^1_{\mu\nu_1\nu_2} ; \\ [\partial, W^2_{\mu\nu_1\nu_2}] = (\mu + \nu_1/2) W^2_{\mu\nu_1\nu_2} . \end{cases}$$

Now, let X be an arbitrary holomorphic vector field on $\mathscr{D}$ belonging to $\mathfrak{g}(\mathscr{D})$. By [3, Theorem B] X can be written in the form

$$X=\sum_{\mu\geqq 0}\{Z_{\mu 00}+Z_{\mu 10}+Z_{\mu 01}+W^1_{\mu 20}+W^1_{\mu 11}+W^1_{\mu 02}+W^1_{\mu 10}+W^1_{\mu 01}+W^1_{\mu 00}$$
$$+W^2_{\mu 20}+W^2_{\mu 11}+W^2_{\mu 02}+W^2_{\mu 10}+W^2_{\mu 01}+W^2_{\mu 00}\}\ .$$

Using the bracket relation (1.1), we then have

$$\begin{aligned}\operatorname{ad}\partial\cdot X=\sum_{\mu\geqq 0}\{&(\mu-1)(Z_{\mu 00}+Z_{\mu 01})+(\mu-1/2)(Z_{\mu 10}+W^1_{\mu 02}+W^1_{\mu 01}+W^1_{\mu 00})\\&+\mu(W^1_{\mu 11}+W^1_{\mu 10}+W^2_{\mu 02}+W^2_{\mu 01}+W^2_{\mu 00})\\&+(\mu+1/2)(W^1_{\mu 20}+W^2_{\mu 11}+W^2_{\mu 10})+(\mu+1)W^2_{\mu 20}\}\ .\end{aligned}$$

Thus, putting

(1.2) $X_\mu=Z_{(\mu+1)00}+Z_{(\mu+1)01}+W^1_{\mu 11}+W^1_{\mu 10}+W^2_{\mu 02}+W^2_{\mu 01}+W^2_{\mu 00}+W^2_{(\mu-1)20}$;

(1.3) $X_{\mu+1/2}=Z_{(\mu+1)10}+W^1_{(\mu+1)02}+W^1_{(\mu+1)01}+W^1_{(\mu+1)00}+W^1_{\mu 20}+W^2_{\mu 11}+W^2_{\mu 10}$

for $\mu=-1, 0, 1, 2, \cdots$, we have

$$X=\sum_{\lambda\in\Lambda}X_\lambda$$

and

$$\phi(\operatorname{ad}\partial)\cdot X=\sum_{\lambda\in\Lambda}\phi(\lambda)\cdot X_\lambda$$

for every polynomial $\phi(x)\in\boldsymbol{R}[x]$, where

$$\Lambda=\{\lambda\in\boldsymbol{R}\mid 2\lambda\in\boldsymbol{Z},\ \lambda\geqq -1\}\ .$$

Therefore we obtain the following theorem as in the case where $\mathscr{D}$ is a generalized Siegel domain with exponent 1/2 due to Kaup, Matsushima and Ochiai [2, Theorem 2]:

THEOREM 1.4. *Let $\mathscr{D}$ be a generalized Siegel domain with exponent* (1/2, 0). *For each $\lambda\geqq -1$, let $\mathfrak{g}_\lambda$ be the subspace of $\mathfrak{g}(\mathscr{D})$ consisting of all vector fields in $\mathfrak{g}(\mathscr{D})$ of the form* (1.2) *or* (1.3) *according as λ is an integer or a half-integer. Then we have*

(1) *$\mathfrak{g}_\lambda$ is the eigen space of* ad ∂ *for the eigen-value λ;*

(2) $\mathfrak{g}(\mathscr{D})=\sum_{\lambda\in\Lambda}\mathfrak{g}_\lambda$;

(3) $[\mathfrak{g}_\lambda, \mathfrak{g}_\sigma]\subset\mathfrak{g}_{\lambda+\sigma}$.

LEMMA 1.5. *For $\mu=-1, 0, 1, 2, \cdots$, we have $\mathfrak{g}_\mu=\mathfrak{g}'_\mu\oplus\mathfrak{g}''_\mu$, where*

$$\mathfrak{g}'_\mu=\mathfrak{g}(\mathscr{D})\cap(\mathfrak{Z}_{(\mu+1)01}\oplus\mathfrak{W}^1_{\mu 11}\oplus\mathfrak{W}^2_{\mu 02}\oplus\mathfrak{W}^2_{\mu 00})\ ;$$
$$\mathfrak{g}''_\mu=\mathfrak{g}(\mathscr{D})\cap(\mathfrak{Z}_{(\mu+1)00}\oplus\mathfrak{W}^1_{\mu 10}\oplus\mathfrak{W}^2_{\mu 01})\ .$$

PROOF. Consider an arbitrary vector field X in $\mathfrak{g}_\mu$. By (1.2) X is then expressed as

$$X = Z_{(\mu+1)00} + Z_{(\mu+1)01} + W^1_{\mu 11} + W^1_{\mu 10} + W^2_{\mu 02} + W^2_{\mu 01} + W^2_{\mu 00} + W^2_{(\mu-1)20} \,.$$

By routine calculations we obtain

$$\begin{aligned} &\mathrm{ad}\,\partial^1\cdot X = 2\sqrt{-1}\,W^2_{(\mu-1)20}\,; \\ &(\mathrm{ad}\,\partial^1)^2\cdot X = -4W^2_{(\mu-1)20}\,; \\ &(\mathrm{ad}\,\partial^2)^2\cdot X = -\{Z_{(\mu+1)01} + W^1_{\mu 11} + W^2_{\mu 02} + W^2_{\mu 00} + W^2_{(\mu-1)20}\}\,. \end{aligned}$$

The first two equalities mean that $W^2_{(\mu-1)20} = 0$, since $\mathfrak{g}(\mathscr{D}) \cap \sqrt{-1}\mathfrak{g}(\mathscr{D}) = \{0\}$ by H. Cartan's principle. Therefore we have $X = X_1 + X_2$, where

$$\begin{cases} X_1 = Z_{(\mu+1)00} + W^1_{\mu 10} + W^2_{\mu 01} \in \mathfrak{g}(\mathscr{D})\,; \\ X_2 = Z_{(\mu+1)01} + W^1_{\mu 11} + W^2_{\mu 02} + W^2_{\mu 00} \in \mathfrak{g}(\mathscr{D})\,, \end{cases}$$

which implies our assertion. q.e.d.

LEMMA 1.6. *We have the following*:

(1) $\mathfrak{g}_{-1} = \{\sum_{k=1}^n a_k\partial/\partial z_k \mid (a_1, a_2, \cdots, a_n) \in \boldsymbol{R}^n\}$;

(2) $\mathfrak{g}_{-1/2} = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{Z}_{010} \oplus \mathfrak{W}^1_{000})$;

(3) $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus \mathfrak{g}_0''$, *where*

$$\begin{cases} \mathfrak{g}_0' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}^1_{011} \oplus \mathfrak{W}^2_{002} \oplus \mathfrak{W}^2_{000})\,, \\ \mathfrak{g}_0'' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{Z}_{100} \oplus \mathfrak{W}^1_{010} \oplus \mathfrak{W}^2_{010})\,; \end{cases}$$

(4) $\mathfrak{g}_{1/2} = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{Z}_{110} \oplus \mathfrak{W}^1_{100} \oplus \mathfrak{W}^1_{020} \oplus \mathfrak{W}^2_{011})$;

(5) $\mathfrak{g}_1 = \mathfrak{g}_1' \oplus \mathfrak{g}_1''$, *where*

$$\begin{cases} \mathfrak{g}_1' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}^1_{111} \oplus \mathfrak{W}^2_{102} \oplus \mathfrak{W}^2_{100})\,, \\ \mathfrak{g}_1'' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{Z}_{200} \oplus \mathfrak{W}^1_{110} \oplus \mathfrak{W}^2_{101})\,. \end{cases}$$

PROOF. It is clear that $\partial/\partial z_k$ belongs to $\mathfrak{g}_{-1}$ for $k = 1, 2, \cdots, n$. Now, let $X = Z_{000} + Z_{001}$ be an arbitrary vector field belonging to $\mathfrak{g}_{-1}$. Then

$$\begin{cases} \mathrm{ad}\,\partial^2\cdot X = \sqrt{-1}Z_{001}\,; \\ (\mathrm{ad}\,\partial^2)^2\cdot X = -Z_{001}\,, \end{cases}$$

and hence $Z_{001} = 0$. It remains to show that the coefficients of $X = Z_{000}$ are real. But this follows from the proof of [2, Theorem 3].

Let $X = Z_{010} + W^1_{002} + W^1_{001} + W^1_{000} \in \mathfrak{g}_{-1/2}$. Then

$$\begin{cases} \mathrm{ad}\,\partial^2\cdot X = 2\sqrt{-1}\,W^1_{002} + \sqrt{-1}\,W^1_{001}\,; \\ (\mathrm{ad}\,\partial^2)^3\cdot X = -8\sqrt{-1}\,W^1_{002} - \sqrt{-1}\,W^1_{001}\,. \end{cases}$$

From this we have $\sqrt{-1}\,W^1_{001}$, $\sqrt{-1}\,W^1_{002} \in \mathfrak{g}(\mathscr{D})$, and hence

$$\begin{cases} W^1_{002} = -(1/2)\cdot \mathrm{ad}\, \partial^2 \cdot (\sqrt{-1}\, W^1_{002}) \in \mathfrak{g}(\mathscr{D})\,; \\ W^1_{001} = -\mathrm{ad}\, \partial^2 \cdot (\sqrt{-1}\, W^1_{001}) \in \mathfrak{g}(\mathscr{D})\,. \end{cases}$$

Noting the fact $\mathfrak{g}(\mathscr{D}) \cap \sqrt{-1}\mathfrak{g}(\mathscr{D}) = \{0\}$, we have

$$W^1_{002} = W^1_{001} = 0\,,$$

which implies our assertion (2).

Let $X = Z_{101} + W^1_{011} + W^2_{002} + W^2_{000}$ be an arbitrary vector field in $\mathfrak{g}'_0$. By virtue of Lemma 1.5 it is enough to show that $Z_{101} = 0$. By (1) we have

$$[\partial/\partial z_k, Z_{101}] = [\partial/\partial z_k, X] \in \mathfrak{g}_{-1} \cap \mathfrak{Z}_{001} = \{0\}$$

for every $k = 1, 2, \cdots, n$, which means $Z_{101} = 0$, as desired.

To prove (4), we take an arbitrary vector field X in $\mathfrak{g}_{1/2}$. By (1.3) X can be written in the form

$$X = Z_{110} + W^1_{102} + W^1_{101} + W^1_{100} + W^1_{020} + W^2_{011} + W^2_{010}\,.$$

Simple computations give the following equalities

$$\begin{cases} \mathrm{ad}\, \partial^2 \cdot X = 2\sqrt{-1}\, W^1_{102} + \sqrt{-1}\, W^1_{101} - \sqrt{-1}\, W^2_{010}\,; \\ (\mathrm{ad}\, \partial^2)^3 \cdot X = -8\sqrt{-1}\, W^1_{102} - \sqrt{-1}\, W^1_{101} + \sqrt{-1}\, W^2_{010}\,, \end{cases}$$

from which we have

$$\sqrt{-1}\, W^1_{102},\ \sqrt{-1}\, W^1_{101} - \sqrt{-1}\, W^2_{010} \in \mathfrak{g}(\mathscr{D})\,. \tag{1.4}$$

Then, since the vector field $W^1_{102} = -(1/2)\cdot \mathrm{ad}\, \partial^2 \cdot (\sqrt{-1}\, W^1_{102})$ also belongs to $\mathfrak{g}(\mathscr{D})$, we see that $W^1_{102} = 0$. Moreover, putting

$$X_1 = -\mathrm{ad}\, \partial^2 \cdot (\sqrt{-1}\, W^1_{101} - \sqrt{-1}\, W^2_{010}) = W^1_{101} + W^2_{010}\,, \tag{1.5}$$

we have from (2) that

$$[\partial/\partial z_k, W^1_{101}] = [\partial/\partial z_k, X_1] \in \mathfrak{g}_{-1/2} \cap \mathfrak{W}^1_{001} = \{0\}$$

for every $k = 1, 2, \cdots, n$. This means $W^1_{101} = 0$. Hence, from (1.4) and (1.5) we obtain

$$\sqrt{-1}\, W^2_{010},\ W^2_{010} \in \mathfrak{g}(\mathscr{D})\,,$$

which shows $W^2_{010} = 0$. We have thus shown that $\mathfrak{g}_{1/2}$ is contained in $\mathfrak{g}(\mathscr{D}) \cap (\mathfrak{Z}_{110} \oplus \mathfrak{W}^1_{100} \oplus \mathfrak{W}^1_{020} \oplus \mathfrak{W}^2_{011})$. The reverse inclusion is trivial, completing the proof of (4).

Finally let $X = Z_{201} + W^1_{111} + W^2_{102} + W^1_{100}$ be an arbitrary vector field belonging to $\mathfrak{g}'_1$. Then, since

$$[\partial/\partial z_k, X] \in \mathfrak{g}_0\,, \quad [\partial/\partial z_k, Z_{201}] \in \mathfrak{Z}_{101}\,, \quad [\partial/\partial z_k, W^1_{111}] \in \mathfrak{W}^1_{011}\,,$$
$$[\partial/\partial z_k, W^2_{102}] \in \mathfrak{W}^2_{002}\,, \quad [\partial/\partial z_k, W^2_{100}] \in \mathfrak{W}^2_{000}$$

and the $\mathfrak{Z}_{101}$-component of any vector field belonging to $\mathfrak{g}_0$ does not appear by (3), we conclude that

$$[\partial/\partial z_k, Z_{201}] = 0 \qquad \text{for} \quad k = 1, 2, \cdots, n ,$$

which implies $Z_{201} = 0$. Our assertion (5) is now an immediate consequence of Lemma 1.5. q.e.d.

REMARK 1. We see later in Lemma 1.11 that

$$\mathfrak{g}_\lambda = \{0\} \qquad \text{for} \quad \lambda \geqq 3/2 .$$

LEMMA 1.7. *For* $\mu = 0, 1, 2, \cdots$, *we have*

(1) $\mathfrak{g}'_\mu = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}^1_{\mu 11} \oplus \mathfrak{W}^2_{\mu 02} \oplus \mathfrak{W}^2_{\mu 00})$;

(2) $\mathfrak{g}_{\mu+1/2} = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{Z}_{(\mu+1)10} \oplus \mathfrak{W}^1_{(\mu+1)00} \oplus \mathfrak{W}^1_{\mu 20} \oplus \mathfrak{W}^2_{\mu 11})$.

PROOF. We prove these by induction on μ. It is already proved in Lemma 1.6 that (1) holds for $\mu = 0$ or 1. Supposing that (1) is true for $\mu = s - 1$ $(s \geqq 2)$, we now consider an arbitrary vector field X belonging to $\mathfrak{g}'_s$. By Lemma 1.5 X may be expressed as

$$X = Z_{(s+1)01} + W^1_{s11} + W^2_{s02} + W^2_{s00} .$$

It is sufficient to show that $Z_{(s+1)01} = 0$. Now, noting that $[\partial/\partial z_k, X] \in \mathfrak{g}_{s-1}$ and that the $\mathfrak{Z}_{s01}$-component of any vector field belonging to $\mathfrak{g}_{s-1}$ is zero by induction assumption, a reasoning similar to the one in the proof of Lemma 1.6, (5) yields also the equality

$$[\partial/\partial z_k, Z_{(s+1)01}] = 0 \qquad \text{for} \quad k = 1, 2, \cdots, n ,$$

which implies $Z_{(s+1)01} = 0$, as desired.

By induction on μ, we can easily verify the second assertion in the same way as in the proof of (4) of Lemma 1.6. q.e.d.

LEMMA 1.8. *Let* $\mathfrak{r}$ *be the radical of* $\mathfrak{g}(\mathscr{D})$. *Then we have*

$$\mathfrak{r} = \sum_{\lambda \in \Lambda} \mathfrak{r}_\lambda , \quad \mathfrak{r}_\lambda = \mathfrak{r} \cap \mathfrak{g}_\lambda .$$

Moreover, $\mathfrak{r}_\lambda = \mathfrak{g}_\lambda$ *for* $\lambda \geqq 3/2$.

PROOF. This can be proved in exactly the same way as Kaup, Matsushima and Ochiai [2, Lemma 4.1]. q.e.d.

Now, let $A = \sum_{k=1}^n a_k \partial/\partial z_k$ $(a_k \in \boldsymbol{R})$ be an element of $\mathfrak{g}_{-1}$. According to Kaup, Matsushima and Ochiai [2], we define the linear mappings $\Psi_A: \mathfrak{g}_{1/2} \to \mathfrak{g}_{-1/2}$ and $\Phi_A: \mathfrak{g}_1 \to \mathfrak{g}_{-1}$ as follows:

$$(1.6) \qquad \begin{cases} \Psi_A(X) = \operatorname{ad} \partial^1 \cdot \operatorname{ad} A \cdot X & \text{for} \quad X \in \mathfrak{g}_{1/2} ; \\ \Phi_A(X) = (1/2) \cdot (\operatorname{ad} A)^2 \cdot X & \text{for} \quad X \in \mathfrak{g}_1 . \end{cases}$$

Then, by a straightforward computation we can show that

$$(1.7)\qquad \begin{cases} X(\sqrt{-1}a, 0, 0) = -\Psi_A(X)(\sqrt{-1}a, 0, 0) & \text{for } X \in \mathfrak{g}_{1/2}\,; \\ X(\sqrt{-1}a, 0, 0) = -\Phi_A(X)(\sqrt{-1}a, 0, 0) & \text{for } X \in \mathfrak{g}_1''\,, \end{cases}$$

where $a = (a_1, a_2, \cdots, a_n)$. Using these equalities, we can prove the following lemma with the same arguments as those in the proof of Kaup, Matsushima and Ochiai [2, Lemma 4.2], and hence the proof is omitted:

LEMMA 1.9. *$\mathfrak{r} \cap \mathfrak{g}_{1/2} = \{0\}$ and $\mathfrak{r} \cap \mathfrak{g}_1'' = \{0\}$.*

LEMMA 1.10. *For $\mu = 1, 2, \cdots$, we have*

$$\mathfrak{g}_{\mu+1/2} = \{0\} \quad \textit{and} \quad \mathfrak{g}_{\mu+1}'' = \{0\}\,.$$

PROOF. We notice by a simple calculation that $[\mathfrak{g}_{-1}, \mathfrak{g}_\mu''] \subset \mathfrak{g}_{\mu-1}''$ for every μ. From this fact, using Lemmas 1.5, 1.7, 1.8 and 1.9, our proof can be carried out with exactly the same arguments as [2, Lemma 4.3]. q.e.d.

For a given generalized Siegel domain $\mathscr{D}$ in $C^n \times C^{m_1} \times C^{m_2}$ we put

$$\begin{cases} \mathscr{D}_0 = \mathscr{D} \cap (C^n \times \{0\} \times \{0\})\,; \\ \mathscr{D}_1 = \mathscr{D} \cap (C^n \times C^{m_1} \times \{0\}) \end{cases}$$

and also

$$\begin{cases} \mathscr{D}(z) = \{(z, w_1, w_2) \in \mathscr{D}\} \subset \{z\} \times C^{m_1} \times C^{m_2} & \text{for } z \in \mathscr{D}_0\,; \\ \mathscr{D}(z, w_1) = \{(z, w_1, w_2) \in \mathscr{D}\} \subset \{z\} \times \{w_1\} \times C^{m_2} & \text{for } (z, w_1) \in \mathscr{D}_1\,. \end{cases}$$

LEMMA 1.11. *$\mathfrak{g}_\mu' = \{0\}$ for $\mu = 2, 3, \cdots$.*
In particular, by Lemmas 1.5 and 1.10 we have

$$\mathfrak{g}_\lambda = \{0\} \quad \textit{for} \quad \lambda \geqq 3/2\,.$$

PROOF. We first observe that $\mathfrak{g}_2' = \{0\}$. For this we consider an arbitrary vector field X belonging to $\mathfrak{g}_2'$. By Lemma 1.7 X may be expresed as

$$(1.8)\qquad X = W_{211}^1 + W_{2C2}^2 + W_{200}^2\,.$$

Take a point $a = (a_1, a_2, \cdots, a_n) \in R^n$ in such a way that $(\sqrt{-1}a, 0, 0) \in \mathscr{D}$. Then, putting

$$\begin{cases} A = \sum_{k=1}^n a_k \partial/\partial z_k \in \mathfrak{g}_{-1}\,; \\ X' = (1/2)\cdot(\operatorname{ad} A)^2 \cdot X \in \mathfrak{g}_0'\,, \end{cases}$$

we can show by a routine calculation that

$$X'(z, w_1, w_2) = X(a, w_1, w_2) = -X(\sqrt{-1}a, w_1, w_2)\,,$$

since X is homogeneous of degree two in z_k $(1 \leqq k \leqq n)$ by (1.8). Thus the vector field $X + X'$ vanishes on the fiber $\mathscr{D}(\sqrt{-1}a)$. Noting that the C^n-component of $X + X'$ is zero, the same reasoning as that in the proof of Vey [5, Lemma 7.4] yields that $X + X' = 0$, and hence

$$X = 0 \quad \text{and} \quad X' = 0 ,$$

which implies that $\mathfrak{g}_2' = \{0\}$. The verification of the lemma is now straightforward by induction on μ. q.e.d.

LEMMA 1.12. *Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}(\mathscr{D})$ generated by ∂^2 and $\mathfrak{g}_0'$. Then we have*

$$\mathfrak{h} = \mathfrak{g}_0' \oplus \mathfrak{h} \cap (\mathfrak{W}_{010}^1 \oplus \mathfrak{W}_{001}^2) .$$

PROOF. Clearly $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}_0$ and any vector field X belonging to $\mathfrak{h}$ is independent of z. Thus, by Lemma 1.6 X can be expressed as

$$X = W_{010}^1 + W_{001}^2 + W_{011}^1 + W_{002}^2 + W_{000}^2$$

with $W_{011}^1 + W_{002}^2 + W_{000}^2 \in \mathfrak{g}_0' \subset \mathfrak{h}$, and hence $W_{010}^1 + W_{001}^2 \in \mathfrak{h} \cap (\mathfrak{W}_{010}^1 \oplus \mathfrak{W}_{001}^2)$.
q.e.d.

LEMMA 1.13 (Vey [5, Lemma 8.2]). *Let $C\colon \mathfrak{h} \to C^{m_2}$ be the linear mapping defined by $C(X) = X(0, 0, 0)$ for $X \in \mathfrak{h}$. Then we have*

(1) *C is injective on $\mathfrak{g}_0'$;*

(2) $\operatorname{Ker} C = \mathfrak{h} \cap (\mathfrak{W}_{010}^1 \oplus \mathfrak{W}_{001}^2)$, *where* $\operatorname{Ker} C$ *denotes the kernel of the linear mapping C;*

(3) *$\mathfrak{h}(0, 0, 0)$ is a C-subspace of C^{m_2}, that is, $\mathfrak{h}^c(0, 0, 0) = \mathfrak{h}(0, 0, 0)$.*

PROOF. To prove (1), we consider an arbitrary vector field $X = W_{011}^1 + W_{002}^2 + W_{000}^2$ belonging to $\mathfrak{g}_0'$. Suppose that $C(X) = 0$. By the definition of C, this means that $W_{000}^2 = 0$. Then we have $X = W_{011}^1 + W_{002}^2$, and hence the vector field $\sqrt{-1}(W_{011}^1 + W_{002}^2) = \operatorname{ad} \partial^2 \cdot X$ also belongs to $\mathfrak{g}(\mathscr{D})$. Recalling the fact $\mathfrak{g}(\mathscr{D}) \cap \sqrt{-1}\mathfrak{g}(\mathscr{D}) = \{0\}$, we conclude that $X = W_{011}^1 + W_{002}^2 = 0$.

The verification of (2) is straightforward. Next, to prove the assertion (3), take an arbitrary vector field $X = W_{011}^1 + W_{002}^2 + W_{000}^2$ belonging to $\mathfrak{g}_0'$. Then we have

$$\operatorname{ad} \partial^2 \cdot X = \sqrt{-1}\, W_{011}^1 + \sqrt{-1}\, W_{002}^2 - \sqrt{-1}\, W_{000}^2 ,$$

and hence

$$\sqrt{-1} X(0, 0, 0) = -(\operatorname{ad} \partial^2 \cdot X)(0, 0, 0) \in \mathfrak{h}(0, 0, 0)$$

which is our last assertion. q.e.d.

LEMMA 1.14. $\dim_R \mathfrak{g}_0' = 2h$ *for some* h, $0 \leqq h \leqq m_2$. *Moreover, there exists an* $\boldsymbol{R}$*-basis* $\{Y_1, \cdots, Y_h, \tilde{Y}_1, \cdots, \tilde{Y}_h\}$ *for* $\mathfrak{g}_0'$ *such that*

(1) $\tilde{Y}_j = [\partial^2, Y_j]$ *for* $j = 1, 2, \cdots, h$;

(2) $\{C(Y_1), \cdots, C(Y_h)\}$ *forms a* $\boldsymbol{C}$*-basis for* $\mathfrak{h}^c(0, 0, 0) = \mathfrak{h}(0, 0, 0)$.

PROOF. This is an immediate consequence of Lemma 1.13. q.e.d.

The proofs of the following two lemmas are similar to those of Lemmas 1.13 and 1.14, and hence are left to the reader (cf. Vey [5, Lemme 4.1]):

LEMMA 1.15. *Let* $C: \mathfrak{g}_{-1/2} \to \boldsymbol{C}^{m_1}$ *be the linear mapping defined by* $C(X) = X(0, 0, 0)$ *for* $X \in \mathfrak{g}_{-1/2}$. *Then we have*

(1) C *is injective*;

(2) $\mathfrak{g}_{-1/2}(0, 0, 0)$ *is a* $\boldsymbol{C}$*-subspace of* $\boldsymbol{C}^{m_1}$, *that is,* $\mathfrak{g}^c_{-1/2}(0, 0, 0) = \mathfrak{g}_{-1/2}(0, 0, 0)$.

LEMMA 1.16. $\dim_R \mathfrak{g}_{-1/2} = 2k$ *for some* k, $0 \leqq k \leqq m_1$. *Moreover, there exists an* $\boldsymbol{R}$*-basis* $\{X_1, \cdots, X_k, \tilde{X}_1, \cdots, \tilde{X}_k\}$ *for* $\mathfrak{g}_{-1/2}$ *such that*

(1) $\tilde{X}_j = [\partial^1, X_j]$ *for* $j = 1, 2, \cdots, k$;

(2) $\{C(X_1), \cdots, C(X_k)\}$ *forms a C-basis for* $\mathfrak{g}^c_{-1/2}(0, 0, 0) = \mathfrak{g}_{-1/2}(0, 0, 0)$.

LEMMA 1.17. *Let* X_j $(1 \leqq j \leqq k)$ *and* Y_j $(1 \leqq j \leqq h)$ *be the vector fields as in Lemmas* 1.16 *and* 1.14. *Then*

$$\{(\partial/\partial z_1)(p_a), \cdots, (\partial/\partial z_n)(p_a), X_1(p_a), \cdots, X_k(p_a), Y_1(p_a), \cdots, Y_h(p_a)\}$$

is a $\boldsymbol{C}$*-basis for* $\mathfrak{g}(\mathscr{D})^c(p_a)$, *where* $p_a = (\sqrt{-1}a, 0, 0) \in \mathscr{D}$ *and* $a \in \boldsymbol{R}^n$.

PROOF. Since $\mathfrak{g}(\mathscr{D}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1$ by Theorem 1.4 and Lemma 1.11, it is sufficient to show that, for any vector field X belonging to each $\mathfrak{g}_\lambda$ $(-1 \leqq \lambda \leqq 1)$, $X(p_a)$ can be expressed as a linear combination of the $n + k + h$ vectors as given in the lemma. If X belongs to $\mathfrak{g}_{-1}$, $\mathfrak{g}_{-1/2}$ or $\mathfrak{g}_0'$, this is obvious. So, consider an arbitrary vector field $X = Z_{100} + W^1_{010} + W^2_{001}$ in $\mathfrak{g}_0''$. Then $X(p_a) = Z_{100}(p_a) \in \boldsymbol{C}^n$, and hence is a linear combination of $(\partial/\partial z_1)(p_a), \cdots, (\partial/\partial z_n)(p_a)$.

Next, taking a vector field X belonging to $\mathfrak{g}_{1/2}$ (resp. $\mathfrak{g}_1''$), we have

$$X(p_a) = -\Psi_A(X)(p_a) \in \mathfrak{g}_{-1/2}(p_a) \quad (\text{resp. } X(p_a) = -\Phi_A(X)(p_a) \in \mathfrak{g}_{-1}(p_a))$$

by (1.7), and hence $X(p_a)$ is a linear combination of $X_1(p_a), \cdots, X_k(p_a)$ (resp. $(\partial/\partial z_1)(p_a), \cdots, (\partial/\partial z_n)(p_a)$). Finally we consider a vector field

$$X = W^1_{111} + W^2_{102} + W^2_{100} \tag{1.9}$$

belonging to $\mathfrak{g}_1'$. Noting that $X = \sum_{k=1}^n z_k[\partial/\partial z_k, X]$ by (1.9) and $[\partial/\partial z_k, X] \in \mathfrak{g}_0'$ for every $k = 1, 2, \cdots, n$, we have

$$X(p_a) = \sum_{k=1}^{n} (\sqrt{-1}a_k)[\partial/\partial z_k, X](p_a) \in \mathfrak{g}_0'^c(p_a) ,$$

and hence $X(p_a)$ is a linear combination of $Y_1(p_a), \cdots, Y_h(p_a)$. q.e.d.

LEMMA 1.18 (Vey [5, Proposition 4.1 and Lemme 8.5]).

$$\dim_R \mathfrak{g}_{-1/2} = 2m_1 \quad \textit{and} \quad \dim_R \mathfrak{g}_0' = 2m_2 .$$

PROOF. We have to show that $k = m_1$ and $h = m_2$ in Lemma 1.17. Now, since $\mathscr{D}$ is sweepable by assumption, Vey [5, Proposition 2.3] says that

$$\dim_C \mathfrak{g}(\mathscr{D})^c(p) = \text{constant} ,$$

where the constant is independent of the point p of $\mathscr{D}$. Hence, by Lemma 1.17 we have

$$\dim_C \mathfrak{g}(\mathscr{D})^c(p) = n + k + h$$

for any point p of $\mathscr{D}$, and therefore, putting $p_a = (\sqrt{-1}a, 0, 0) \in \mathscr{D}$, $a \in \boldsymbol{R}^n$ as before, we can take an open neighborhood V of the point p_a such that

$$\{(\partial/\partial z_1)(p), \cdots, (\partial/\partial z_n)(p), X_1(p), \cdots, X_k(p), Y_1(p), \cdots, Y_h(p)\}$$

forms a $\boldsymbol{C}$-basis for $\mathfrak{g}(\mathscr{D})^c(p)$ whenever p is contained in V. Then, considering the vector field

$$\partial = \sum_{k=1}^{n} z_k \partial/\partial z_k + (1/2) \sum_{\alpha=1}^{m_1} w_\alpha^1 \partial/\partial w_\alpha^1 \in \mathfrak{g}_0''$$

and the points in V of the form $(z, w_1, 0)$, we can choose the complex numbers λ^j, μ^j, which may be dependent on (z, w_1), in such a way that

$$z + w_1/2 = \sum_{j=1}^{k} \lambda^j X_j(z, w_1, 0) + \sum_{j=1}^{n} \mu^j (\partial/\partial z_j)(z, w_1, 0)$$

(note that $Y_j(z, w_1, 0) \in \boldsymbol{C}^{m_2}$ for every j, $1 \leqq j \leqq h$, by (3) of Lemma 1.6), which implies that

$$w_1/2 = \sum_{j=1}^{k} \lambda^j C(X_j) \tag{1.10}$$

for any $(z, w_1, 0) \in V$, where $C: \mathfrak{g}_{-1/2} \to \boldsymbol{C}^{m_1}$ is the linear mapping defined in Lemma 1.15. Recalling that $C(X_1), C(X_2), \cdots, C(X_k)$ are linearly independent in $\boldsymbol{C}^{m_1}$ by (2) of Lemma 1.16, we have the first assertion $k = m_1$ by (1.10).

To prove the second assertion, we take a vector field X in $\mathfrak{g}_0'$. By Lemma 1.6 X can be written in the form

$$X = W^1_{011} + W^2_{002} + W^2_{000}$$

and hence

$$X(z, 0, w_2) = W^2_{002}(z, 0, w_2) + W^2_{000}(z, 0, w_2)$$

for any $(z, w_2) \in C^n \times C^{m_2}$. Therefore, applying the arguments in the proof of Vey [5, Lemme 8.5] to our case, we can prove that

$$(1.11) \qquad Y_j(z, 0, w_2) \in \mathfrak{h}(0, 0, 0)\ , \quad j = 1, 2, \cdots, h$$

for any $(z, w_2) \in C^n \times C^{m_2}$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}(\mathscr{D})$ defined in Lemma 1.12. Now, considering the vector field

$$\partial^2 = \sqrt{-1} \sum_{\alpha=1}^{m_2} w^2_\alpha \partial/\partial w^2_\alpha \in \mathfrak{g}''_0$$

and the points in V of the form $(z, 0, w_2)$, we can choose the complex numbers μ^j in such a way that

$$\sqrt{-1} w_2 = \sum_{j=1}^{h} \mu^i Y_j(z, 0, w_2)$$

for any $(z, 0, w_2) \in V$. Note that $(\partial/\partial z_i)(z, 0, w_2)$, $X_j(z, 0, w_2) \in C^n \oplus C^{m_1}$ for every i and j, $1 \leqq i \leqq n$, $1 \leqq j \leqq k = m_1$. By (1.11) we then have

$$\sqrt{-1} w_2 \in \mathfrak{h}^c(0, 0, 0) = \mathfrak{h}(0, 0, 0)$$

for any $(z, 0, w_2) \in V$. Obviously this shows that

$$h = \dim_C \mathfrak{g}'_0(0, 0, 0) = \dim_C \mathfrak{h}(0, 0, 0) = m_2\ . \qquad \text{q.e.d.}$$

LEMMA 1.19. *The domain $\mathscr{D}_1 = \mathscr{D} \cap (C^n \times C^{m_1} \times \{0\})$ is a Siegel domain of the second kind in $C^n \times C^{m_1}$. More precisely, $\mathscr{D}_1$ can be expressed as*

$$\mathscr{D}_1 = \{(z, w_1) \in C^n \times C^{m_1} \mid \operatorname{Im} z - F(w_1, w_1) \in \Omega\}$$

where Ω is the open convex cone in R^n appearing in Proposition 1.3 and $F: C^{m_1} \times C^{m_1} \to C^n$ is a suitable Ω-Hermitian form.

PROOF. Our proof is almost identical to that of Vey [5, Proposition 5.1]. First of all, since $\dim_R \mathfrak{g}_{-1/2} = 2m_1$ by Lemma 1.18, it follows from the proof of Vey [5, Lemme 5.1] that

$$\mathfrak{g}_{-1/2} = \left\{\sum_{k=1}^{n}\left(\sum_{\alpha,\beta=1}^{m_1} f^k_{\alpha\beta} w^1_\alpha \bar{c}_\beta\right)\partial/\partial z_k + \sum_{\alpha=1}^{m_1} c_\alpha \partial/\partial w^1_\alpha \,\middle|\, (c_1, c_2, \cdots, c_{m_1}) \in C^{m_1}\right\}\ ,$$

where $f^k_{\alpha\beta}$ are complex constants depending only on $\mathfrak{g}_{-1/2}$. For $k = 1, 2, \cdots, n$, we put

$$F^k(u, v) = (1/2\sqrt{-1}) \sum_{\alpha,\beta=1}^{m_1} f^k_{\alpha\beta} u_\alpha \bar{v}_\beta$$

for all $u = (u_\alpha)$, $v = (v_\alpha) \in \boldsymbol{C}^{m_1}$ and define the mapping $F: \boldsymbol{C}^{m_1} \times \boldsymbol{C}^{m_1} \to \boldsymbol{C}^n$ by

$$F(u, v) = (F^1(u, v), \cdots, F^n(u, v)) \quad \text{for} \quad u, v \in \boldsymbol{C}^{m_1} .$$

Then we have

(1.12) $\quad F(\lambda u, \mu v) = \lambda\bar{\mu}F(u, v) \quad$ for $\quad u, v \in \boldsymbol{C}^{m_1} \quad$ and $\quad \lambda, \mu \in \boldsymbol{C}$;

(1.13) Every vector field X belonging to $\mathfrak{g}_{-1/2}$ can be written in the form

$$X = 2\sqrt{-1}\sum_{k=1}^{n} F^k(w_1, c)\partial/\partial z_k + \sum_{\alpha=1}^{m_1} c_\alpha \partial/\partial w_\alpha^1$$

for some $c = (c_1, c_2, \cdots, c_{m1}) \in \boldsymbol{C}_{m_1}$.

Moreover, we assert that

$$F(v, u) = F(u, v)^- \quad \text{for} \quad u, v \in \boldsymbol{C}^{m_1} ,$$

where the right hand side is the complex conjugate of $F(u, v)$. To this end, it is sufficient to show that the matrix A_k defined by

$$A_k = (f^k_{\alpha\beta})_{1 \leqq \alpha, \beta \leqq m_1}$$

is skew-Hermitian, that is, ${}^t\bar{A}_k = -A_k$ for each k, $1 \leqq k \leqq n$. Now, to simplify the notation, we put

$$\begin{aligned} X_c &= \sum_{k=1}^{n}\Big(\sum_{\alpha,\beta=1}^{m_1} f^k_{\alpha\beta} w^1_\alpha \bar{c}_\beta\Big)\partial/\partial z_k + \sum_{\alpha=1}^{m_1} c_\alpha \partial/\partial w^1_\alpha \\ &= 2\sqrt{-1}\sum_{k=1}^{n} F^k(w_1, c)\partial/\partial z_k + \sum_{\alpha=1}^{m_1} c_\alpha \partial/\partial w^1_\alpha \end{aligned}$$

for all $c = (c_1, c_2, \cdots, c_{m_1}) \in \boldsymbol{C}^{m_1}$. Then, by a routine calculation we have

$$[X_c, [\partial^1, X_c]] = 2\sqrt{-1}\sum_{k=1}^{n}\Big(\sum_{\alpha,\beta=1}^{m_1} f^k_{\alpha\beta} c_\alpha \bar{c}_\beta\Big)\partial/\partial z_k . \tag{1.14}$$

Noting that the vector field $[X_c, [\partial^1, X_c]]$ belongs to $\mathfrak{g}_{-1}$ and the coefficients of any vector field in $\mathfrak{g}_{-1}$ are real by Lemma 1.6, we see from (1.14) that $\sum_{\alpha,\beta=1}^{m_1} f^k_{\alpha\beta} c_\alpha \bar{c}_\beta$ is pure imaginary for each k, $1 \leqq k \leqq n$. Therefore

$$\sum_{\alpha,\beta=1}^{m_1} (f^k_{\alpha\beta} + \bar{f}^k_{\beta\alpha}) c_\alpha \bar{c}_\beta = 0$$

for all $c = (c_1, c_2, \cdots, c_{m_1}) \in \boldsymbol{C}^{m_1}$. Obviously, this implies that

$$f^k_{\alpha\beta} + \bar{f}^k_{\beta\alpha} = 0 \quad \text{for} \quad 1 \leqq k \leqq n , \quad 1 \leqq \alpha, \beta \leqq m_1$$

and so

$${}^t\bar{A}_k = -A_k \quad \text{for} \quad k = 1, 2, \cdots, n ,$$

as desired.

Next, by a simple computation we have

(1.15) $\exp X_c \cdot (z, w_1, w_2) = (z + 2\sqrt{-1}F(w_1, c) + \sqrt{-1}F(c, c), w_1 + c, w_2)$

for all $(z, w_1, w_2) \in C^n \times C^{m_1} \times C^{m_2}$ and all $c \in C^{m_1}$. In particular,

$$\begin{cases} \exp X_{-w_1} \cdot (z, w_1, 0) = (z - \sqrt{-1}F(w_1, w_1), 0, 0) \,; \\ \exp X_{w_1} \cdot (z - \sqrt{-1}F(w_1, w_1), 0, 0) = (z, w_1, 0) \end{cases}$$

for all $(z, w_1) \in \mathscr{D}_1$. Recalling Proposition 1.3, we conclude from this that

(1.16) $$\mathscr{D}_1 = \{(z, w_1) \in C^n \times C^{m_1} \mid \operatorname{Im} z - F(w_1, w_1) \in \Omega\} \,.$$

Finally we claim that

(1.17) $F(u, u) \in \bar{\Omega}$ for any $u \in C^{m_1}$ and $F(u, u) = 0$ only if $u = 0$,

where $\bar{\Omega}$ denotes the topological closure of Ω. Indeed, let $u \in C^{m_1}$ and $(z, 0, 0) \in \mathscr{D}$. We have from (1.15) that

$$\exp tX_u \cdot (z, 0, 0) = (z + \sqrt{-1}t^2F(u, u), tu, 0) \in \mathscr{D}$$

for all $t \in R$, and hence $z + \sqrt{-1}t^2F(u, u) \in \mathscr{D}_0$, or equivalently,

$$\operatorname{Im} z + t^2F(u, u) \in \Omega$$

for all $t \in R$ by Proposition 1.3, which implies that $F(u, u) \in \bar{\Omega}$. Next, suppose that there exists a point $u_0 \in C^{m_1}$ such that $u_0 \neq 0$ and $F(u_0, u_0) = 0$. Then, taking a point $(z_0, 0, 0) \in \mathscr{D}$, we see from (1.16) that $\mathscr{D}_1$, and hence $\mathscr{D}$, contains the complex affine line $\{(z_0, \lambda u_0, 0) \mid \lambda \in C\}$. But this is impossible, since $\mathscr{D}$ is holomorphically equivalent to a bounded domain in $C^{n+m_1+m_2}$. We have thus shown that $\mathscr{D}_1$ is the Siegel domain of the second kind defined by the convex cone Ω and the Ω-Hermitian form F. q.e.d.

REMARK 2. From (1.13) in the proof, we have

$$\mathfrak{g}_{-1/2} = \left\{2\sqrt{-1}\sum_{k=1}^{n} F^k(w_1, c)\partial/\partial z_k + \sum_{\alpha=1}^{m_1} c_\alpha \partial/\partial w_\alpha^1 \mid c = (c_1, \cdots, c_{m_1}) \in C^{m_1}\right\} \,.$$

LEMMA 1.20. *We have*

(1) $\mathfrak{g}_0' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}_{002}^2 \oplus \mathfrak{W}_{000}^2)$:

(2) $\mathfrak{g}_1' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}_{102}^2 \oplus \mathfrak{W}_{100}^2)$.

PROOF. To prove (1), consider an arbitrary vector field X belonging to $\mathfrak{g}_0'$. By Lemma 1.6 X can be expressed as $X = W_{011}^1 + W_{002}^2 + W_{000}^2$. We have only to show that $W_{011}^1 = 0$. Now, taking a vector field $Y = Z_{010} + W_{000}^1$ in $\mathfrak{g}_{-1/2}$ arbitrarily, we have

$$[Z_{010}, W_{011}^1] + [W_{000}^1, W_{011}^1] = [Y, X] \in \mathfrak{g}_{-1/2} \,.$$

On the other hand, since

$$[Z_{010}, W^1_{011}] \in \mathfrak{Z}_{011}\,, \quad [W^1_{000}, W^1_{011}] \in \mathfrak{W}^1_{001}$$

and the $\mathfrak{Z}_{011}$(resp. $\mathfrak{W}^1_{001}$)-component of any vector field in $\mathfrak{g}_{-1/2}$ is zero by Lemma 1.6, we conclude that

$$(1.18) \qquad [Z_{010}, W^1_{011}] = 0 \quad \text{and} \quad [W^1_{000}, W^1_{011}] = 0\,.$$

Let $W^1_{000} = \sum_{\alpha=1}^{m_1} c_\alpha \partial/\partial w^1_\alpha$ and $W^1_{011} = \sum_{\beta=1}^{m_1} P^\beta_{011}\partial/\partial w^1_\beta$. It follows then from the second equality of (1.18) that

$$\sum_{\beta=1}^{m_1}\left(\sum_{\alpha=1}^{m_1} c_\alpha \partial P^\beta_{011}/\partial w^1_\alpha\right)\partial/\partial w^1_\beta = 0\,,$$

and hence

$$\sum_{\alpha=1}^{m_1} c_\alpha \partial P^\beta_{011}/\partial w^1_\alpha = 0 \qquad \text{for} \quad \beta = 1, 2, \cdots, m_1\,.$$

Recalling that $\dim_C \{C(X) \mid X \in \mathfrak{g}_{-1/2}\} = m_1$ by Lemma 1.18, this means that

$$\partial P^\beta_{011}/\partial w^1_\alpha = 0 \qquad \text{for} \quad 1 \leqq \alpha, \beta \leqq m_1\,.$$

Clearly this implies that $W^1_{011} = 0$, as desired.

To verify the assertion (2), we next consider a vector field X belonging to $\mathfrak{g}'_1$. By Lemma 1.6 X can be written in the form $X = W^1_{111} + W^2_{102} + W^2_{100}$. From the assertion (1) we have

$$[\partial/\partial z_k, X] \in \mathfrak{g}'_0 = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}^2_{002} \oplus \mathfrak{W}^2_{000})$$

for every k, $1 \leqq k \leqq n$. Therefore, the $\mathfrak{W}^1_{011}$-component of $[\partial/\partial z_k, X]$ satisfies

$$[\partial/\partial z_k, W^1_{111}] = 0 \qquad \text{for} \quad k = 1, 2, \cdots, n\,,$$

from which we have $W^1_{111} = 0$. This proves the second assertion. q.e.d.

LEMMA 1.21. $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{W}^2_{001} \oplus \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}^2_{002} \oplus \mathfrak{W}^2_{000})$.

PROOF. Recall that $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}(\mathscr{D})$ generated by $\{\partial^2, \mathfrak{g}'_0\}$. Since $\mathfrak{g}'_0 = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}^2_{002} \oplus \mathfrak{W}^2_{000})$ by Lemma 1.20, it is obvious that any vector field in $\mathfrak{h}$ is independent of (z, w_1). Thus, our assertion follows immediately from Lemma 1.12. q.e.d.

Summarizing our results, we have the following

THEOREM 1.22. *Let $\mathscr{D}$ be a sweepable generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2}$ with exponent* (1/2, 0). *We put $\mathscr{D}_1 = \mathscr{D} \cap (C^n \times C^{m_1} \times \{0\})$. Then we have from Lemma* 1.19 *that*

(1) *$\mathscr{D}_1 = \mathscr{D}_1(\Omega, F)$ is the Siegel domain of the second kind defined*

by the convex cone Ω and the Ω-Hermitian form F.

For each $\lambda \geqq -1$, let $\mathfrak{g}_\lambda$ be the subspace of $\mathfrak{g}(\mathscr{D})$ as defined in Theorem 1.4. Then we have the following:

(2) $\mathfrak{g}(\mathscr{D}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1$, $[\mathfrak{g}_\lambda, \mathfrak{g}_\sigma] \subset \mathfrak{g}_{\lambda+\sigma}$, *where*

(i) $\mathfrak{g}_{-1} = \{\sum_{k=1}^n a_k \partial/\partial z_k \mid (a_1, a_2, \cdots, a_n) \in \boldsymbol{R}^n\}$;

(ii) *Putting* $F(u, v) = (F^1(u, v), \cdots, F^n(u, v))$,

$$\mathfrak{g}_{-1/2} = \left\{2\sqrt{-1} \sum_{k=1}^n F^k(w_1, c)\partial/\partial z_k + \sum_{\alpha=1}^{m_1} c_\alpha \partial/\partial w_\alpha^1 \,\middle|\, c = (c_1, c_2, \cdots, c_{m_1}) \in \boldsymbol{C}^{m_1}\right\} ;$$

(iii) $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus \mathfrak{g}_0''$, *where*

$$\begin{cases} \mathfrak{g}_0' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}^2_{002} \oplus \mathfrak{W}^2_{000}) , \\ \mathfrak{g}_0'' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{Z}_{100} \oplus \mathfrak{W}^1_{010} \oplus \mathfrak{W}^2_{001}) ; \end{cases}$$

(iv) $\mathfrak{g}_{1/2} = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{Z}_{110} \oplus \mathfrak{W}^1_{100} \oplus \mathfrak{W}^1_{020} \oplus \mathfrak{W}^2_{011})$;

(v) $\mathfrak{g}_1 = \mathfrak{g}_1' \oplus \mathfrak{g}_1''$, *where*

$$\begin{cases} \mathfrak{g}_1' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}^2_{102} \oplus \mathfrak{W}^2_{100}) , \\ \mathfrak{g}_1'' = \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{Z}_{200} \oplus \mathfrak{W}^1_{110} \oplus \mathfrak{W}^2_{101}) . \end{cases}$$

In particular, denoting by $\pi: \boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \boldsymbol{C}^{m_2} \to \boldsymbol{C}^n \times \boldsymbol{C}^{m_1}$ the canonical projection $\pi(z, w_1, w_2) = (z, w_1)$, we have:

(3) *Every vector field in $\mathfrak{g}(\mathscr{D})$ is π-projectable.*

Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}(\mathscr{D})$ generated by ∂^2 and $\mathfrak{g}_0'$. Then we have

(4) $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{W}^2_{001} \oplus \mathfrak{g}(\mathscr{D}) \cap (\mathfrak{W}^2_{002} \oplus \mathfrak{W}^2_{000})$;

(5) $\mathfrak{h}(0, 0, 0) = \boldsymbol{C}^{m_2}$.

We are now in a position to prove Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1. Put $\mathscr{D}_1 = \mathscr{D} \cap (\boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \{0\})$. Then, by Theorem 1.22 $\mathscr{D}_1$ is a Siegel domain of the second kind in $\boldsymbol{C}^n \times \boldsymbol{C}^{m_1}$. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}(\mathscr{D})$ defined in Theorem 1.22 and H the analytic subgroup of $\mathrm{Aut}(\mathscr{D})$ corresponding to $\mathfrak{h}$. We put $\mathscr{D}(z, w_1) = \{(z, w_1, w_2) \in \mathscr{D}\} \subset \{z\} \times \{w_1\} \times \boldsymbol{C}^{m_2}$ for any $(z, w_1) \in \mathscr{D}_1$ as before. Since we know from (4) and (5) of Theorem 1.22 that $\mathfrak{h}(z, w_1, 0) = \mathfrak{h}(0, 0, 0) = \boldsymbol{C}^{m_2}$, the following two assertions can be verified as in the proof of Vey [5, Lemme 8.5]:

(1.19) H acts transitively on $\mathscr{D}(z, w_1)$;

(1.20) $\mathscr{D}(z, w_1)$ is a bounded domain in $\{z\} \times \{w_1\} \times \boldsymbol{C}^{m_2}$.

On the other hand, since $[\mathfrak{Z}_{000} \oplus \mathfrak{W}^1_{000}, \mathfrak{h}] = \{0\}$ by (4) of Theorem 1.22, every element of H commutes with any parallel translation

$$P_{(\alpha,\beta)}: (z, w_1, w_2) \to (z + \alpha, w_1 + \beta, w_2)$$

of $C^n \times C^{m_1} \times C^{m_2}$ onto itself, where $\alpha \in C^n$ and $\beta \in C^{m_1}$. Then, denoting by $\mathscr{D}_2 = H\cdot(0, 0, 0) \subset \{0\} \times \{0\} \times C^{m_2}$ the orbit of H passing through the origin, it follows that $\mathscr{D}(z, w_1) = P_{(z,w_1)}(\mathscr{D}_2)$ for any $(z, w_1) \in \mathscr{D}_1$. Clearly this implies that $\mathscr{D}$ is the direct product $\mathscr{D}_1 \times \mathscr{D}_2$. Moreover, $\mathscr{D}_2$ is a homogeneous bounded domain in $C^{m_2} = \{0\} \times \{0\} \times C^{m_2}$ containing the origin, since so is any fiber $\mathscr{D}(z, w_1)$. q.e.d.

LEMMA 1.23. *Under the same situation as in Theorem* 1.1, *we have*

$$\mathfrak{g}(\mathscr{D}) = \mathfrak{g}(\mathscr{D}_1) \oplus \mathfrak{g}(\mathscr{D}_2) , \quad [\mathfrak{g}(\mathscr{D}_1), \mathfrak{g}(\mathscr{D}_2)] = \{0\} .$$

PROOF. Since $\mathscr{D} = \mathscr{D}_1 \times \mathscr{D}_2$ (direct product) by Theorem 1.1, this is a classical result of H. Cartan [1]. q.e.d.

PROOF OF THEOREM 1.2. Suppose that there exists a discrete subgroup Γ of Aut$(\mathscr{D})$ such that $\mathscr{D}/\Gamma$ is compact. Since $\mathscr{D}$ is a sweepable domain, it follows immediately from Theorem 1.1 that $\mathscr{D}$ is the direct product $\mathscr{D}_1 \times \mathscr{D}_2$, where $\mathscr{D}_1$ is a Siegel domain of the second kind in $C^n \times C^{m_1}$ and $\mathscr{D}_2$ is a homogeneous bounded circular domain in C^{m_2} containing the origin. Moreover, by Lemma 1.23 we have $\mathfrak{g}(\mathscr{D}) = \mathfrak{g}(\mathscr{D}_1) \oplus \mathfrak{g}(\mathscr{D}_2)$ (direct sum of ideals). On the other hand, we know from Vey [5, Proposition 6.1] that $\mathfrak{g}(\mathscr{D})$ is unimodular, hence so is $\mathfrak{g}(\mathscr{D}_1)$. It then follows from another result of Vey [5] that $\mathscr{D}_1$ is symmetric. Since $\mathscr{D}_2$ is homogeneous and circular, it is obvious that $\mathscr{D}_2$ is symmetric. Finally, being the direct product of two symmetric domains, $\mathscr{D}$ is also symmetric. q.e.d.

2. Vector fields which are independent of $z_1, z_2, \cdots, z_n$. In this section we study holomorphic vector fields in $\mathfrak{g}(\mathscr{D})$ which are independent of $z_1, z_2, \cdots, z_n$, where $\mathscr{D}$ is a sweepable generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2} \times \cdots \times C^{m_s}$ with exponent $(c_1, c_2, \cdots, c_{s-1}, 0)$.

With the same notations as in [3], we shall first prove the following:

LEMMA 2.1. *Let $\mathscr{D}$ be a sweepable generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2} \times \cdots \times C^{m_s}$ with exponent $(c_1, c_2, \cdots, c_{s-1}, 0)$. Let X be a holomorphic vector field in $\mathfrak{g}(\mathscr{D})$ which is independent of $z_1, z_2, \cdots, z_n$. Assume that*

(*) $c_1 \neq 1/2$;

(**) *the exponents c_α $(1 \leqq \alpha \leqq s-1)$ are all mutually distinct and $c_\alpha \neq 0$ $(1 \leqq \alpha \leqq s-1)$.*

Then X can be written in the form

$$X = Z_0 + \sum_{1<\lambda<s} Z_\lambda + W^s_{2,s} + \sum_{1<\lambda<s} W^\lambda_{2,s} + \sum_{\substack{1<\alpha<\beta<s \\ \lambda\neq 1,\alpha,\beta}} W^\lambda_{\alpha\beta} + \sum_{\alpha<s} W^\alpha_{\alpha s} + W^1_1$$
$$+ \sum_{1<\alpha,\lambda} W^\lambda_\alpha + \sum_{1<\lambda} W^\lambda_0 \, .$$

PROOF. As in [3, Section 2], X can be written as

$$X = Z_0 + \sum_\lambda Z_\lambda + \sum_\lambda \left(\sum_\alpha W^\lambda_{2,\alpha} + \sum_{\alpha<\beta} W^\lambda_{\alpha\beta} \right) + \sum_{\alpha,\lambda} W^\lambda_\alpha + \sum_\lambda W^\lambda_0 \, .$$

Moreover, from [3, (2.5), (2.7)] we have

$$\begin{aligned} 0 &= \mathrm{ad}\,\partial\cdot(\mathrm{ad}(\partial^1 + \cdots + \partial^s))^2\cdot X - \sum_\nu c_\nu (\mathrm{ad}(\partial^1 + \cdots + \partial^s))^2\cdot(\mathrm{ad}\partial^\nu)^2\cdot X \\ &= \sum_\lambda (1-2c_\lambda) Z_\lambda - 2\sum_\lambda c_\lambda W^\lambda_{2,\lambda} - 6\sum_{\lambda\neq\alpha} c_\alpha W^\lambda_{2,\alpha} - 2\sum_{\substack{\lambda\neq\alpha,\beta \\ \alpha<\beta}} (c_\alpha + c_\beta) W^\lambda_{\alpha\beta} \\ &\qquad - 2\sum_{\alpha<\beta} c_\alpha W^\beta_{\alpha\beta} - 2\sum_{\alpha<\beta} c_\beta W^\alpha_{\alpha\beta} \\ &= \sum_{\lambda<s} (1-2c_\lambda) Z_\lambda + Z_s - 2\sum_{\lambda<s} c_\lambda W^\lambda_{2,\lambda} - 6\sum_{\substack{\lambda\neq\alpha \\ \alpha\neq s}} c_\alpha W^\lambda_{2,\alpha} - 2\sum_{\substack{\lambda\neq\alpha,\beta \\ \alpha<\beta<s}} (c_\alpha + c_\beta) W^\lambda_{\alpha\beta} \\ &\qquad - 2\sum_{\substack{\alpha<s \\ \lambda\neq\alpha,s}} c_\alpha W^\lambda_{\alpha s} - 2\sum_{\alpha<\beta} c_\alpha W^\beta_{\alpha\beta} - 2\sum_{\alpha<\beta<s} c_\beta W^\alpha_{\alpha\beta} \, , \end{aligned}$$

and hence we conclude by our assumptions (*) and (**) that

$$\begin{cases} Z_1,\ Z_s,\ W^\lambda_{2,\lambda}\ (1 \leqq \lambda \leqq s-1)\,, \quad W^\lambda_{2,\alpha}\ (\lambda\neq\alpha,\ 1\leqq\alpha\leqq s-1)\,, \\ W^\lambda_{\alpha s}\ (\lambda\neq\alpha, s,\ 1\leqq\alpha\leqq s-1)\,, \quad W^\beta_{\alpha\beta}\ (1\leqq\alpha<\beta\leqq s) \\ \text{and } W^\alpha_{\alpha\beta}\ (1\leqq\alpha<\beta\leqq s-1) \end{cases}$$

are all equal to zero. Then the vector field X is of the form

$$X = Z_0 + \sum_{1<\lambda<s} Z_\lambda + W^s_{2,s} + \sum_{\lambda<s} W^\lambda_{2,s} + \sum_{\substack{\lambda\neq\alpha,\beta \\ \alpha<\beta<s}} W^\lambda_{\alpha,\beta} + \sum_{\alpha<s} W^\alpha_{\alpha s} + \sum_{\alpha,\lambda} W^\lambda_\alpha + \sum_\lambda W^\lambda_0 \, .$$

For each $\nu < s$, we have, by direct computations, the equalities

$$(\mathrm{ad}(\partial^1 + \cdots + \partial^s))^2\cdot(\mathrm{ad}\,\partial^\nu)^2\cdot X = Z_\nu + W^\nu_{2,s} + \sum_{\substack{\alpha<\beta<s \\ \alpha,\beta\neq\nu}} W^\nu_{\alpha\beta} + \sum_{\substack{\alpha<\nu \\ \lambda\neq\nu}} W^\lambda_{\alpha\nu} + \sum_{\substack{\nu<\alpha \\ \lambda\neq\nu}} W^\lambda_{\nu\alpha} + W^\nu_0 \, ;$$

$$\mathrm{ad}\,\partial^\nu\cdot\mathrm{ad}(\partial^1 + \cdots + \partial^s)\cdot(\mathrm{ad}\,\partial^\nu)^2\cdot X = Z_\nu - W^\nu_{2,s} - \sum_{\substack{\alpha<\beta<s \\ \alpha,\beta\neq\nu}} W^\nu_{\alpha\beta} + \sum_{\substack{\alpha<\nu \\ \lambda\neq\nu}} W^\lambda_{\alpha\nu} + \sum_{\substack{\nu<\alpha \\ \lambda\neq\nu}} W^\lambda_{\nu\alpha} + W^\nu_0 \, ,$$

and hence

$$\text{(2.1)} \qquad \begin{cases} W^\nu_{2,s} + \sum_{\substack{\alpha<\beta<s \\ \alpha,\beta\neq\nu}} W^\nu_{\alpha\beta} \in \mathfrak{g}(\mathscr{D})\,; \\ Z_\nu + \sum_{\substack{\alpha<\nu \\ \lambda\neq\nu}} W^\lambda_{\alpha\nu} + \sum_{\substack{\nu<\alpha \\ \lambda\neq\nu}} W^\lambda_{\nu\alpha} + W^\nu_0 \in \mathfrak{g}(\mathscr{D})\,. \end{cases}$$

In the particular case where $\nu = 1$ in (2.1), we can show in the same way as in the proof of [3, (2.14)] that the two vector fields

$$X_1 = W^1_{2,s} + \sum_{1<\alpha<\beta<s} W^1_{\alpha\beta}$$

and

$$X_2 = \sum_{1<\alpha,\lambda} W^{\lambda}_{1\alpha} + W^1_0$$

are equal to zero. Thus X may be expressed as

$$(2.2)\quad X = Z_0 + \sum_{1<\lambda<s} Z_\lambda + W^s_{2,s} + \sum_{1<\lambda<s} W^{\lambda}_{2,s} + \sum_{\substack{1<\alpha<\beta<s \\ \lambda\neq 1,\alpha,\beta}} W^{\lambda}_{\alpha\beta} + \sum_{\alpha<s} W^{\alpha}_{\alpha s} + \sum_{\alpha,\lambda} W^{\lambda}_{\alpha} + \sum_{1<\lambda} W^{\lambda}_0 .$$

From this, by a straightforward computation we have

$$\begin{cases} \operatorname{ad}\partial\cdot(\operatorname{ad}\partial^1)^2\cdot X = \sum_{1<\alpha}(c_1 - c_\alpha)W^1_\alpha - \sum_{1<\alpha}(c_1 - c_\alpha)W^\alpha_1 ; \\ \operatorname{ad}\partial^\alpha\cdot\operatorname{ad}\partial^1\cdot X = W^1_\alpha + W^\alpha_1 \quad \text{for} \quad \alpha = 2, 3, \cdots, s . \end{cases}$$

Then, with the same arguments as in the proof of [3, Lemma 2.1] we can prove that

$$(2.3)\qquad W^1_\alpha = 0 \quad \text{and} \quad W^\alpha_1 = 0 \quad \text{for} \quad \alpha = 2, 3, \cdots, s .$$

As a result, from (2.2) and (2.3) we conclude that the vector field X has the desired form as in Lemma. q.e.d.

LEMMA 2.2. *Under the same situation as in Lemma 2.1, we have the following:*

(1) *The veceor field ∂^1 belongs to the center $\mathfrak{z}$ of $\mathfrak{g}(\mathscr{D})$;*

(2) *Let V be the set of common zeros of vector fields belonging to $\mathfrak{z}$. Then*

$$\mathscr{D} \supset \mathscr{D} \cap (\boldsymbol{C}^n \times \{0\} \times \boldsymbol{C}^{m_2} \times \cdots \times \boldsymbol{C}^{m_s}) \supset V \supset \mathscr{D} \cap (\boldsymbol{C}^n \times \{0\} \times \cdots \times \{0\}) .$$

PROOF. By Lemma 2.1, the assertion (1) can be proved in the same way as Vey [5, Lemme 3.2].

Our proof of (2) is similar to that of Vey [5, Lemme 3.3]. Consider the vector field $\partial^1 = \sqrt{-1}\sum_{\alpha=1}^{m_1} w^1_\alpha \partial/\partial w^1_\alpha$. By the first assertion (1) ∂^1 belongs to $\mathfrak{z}$, and moreover it is obvious that

$$\partial^1 = 0 \quad \text{on} \quad \mathscr{D} \cap (\boldsymbol{C}^n \times \{0\} \times \boldsymbol{C}^{m_2} \times \cdots \times \boldsymbol{C}^{m_s})$$

and hence

$$\mathscr{D} \cap (\boldsymbol{C}^n \times \{0\} \times \boldsymbol{C}^{m_2} \times \cdots \times \boldsymbol{C}^{m_s}) \supset V .$$

Next, taking an arbitrary vector field X belonging to $\mathfrak{z}$, we have $[\partial/\partial z_k, X] = 0$ for every $k = 1, 2, \cdots, n$. This implies that X is independent of $z_1, z_2, \cdots, z_n$. Thus we may assume that X has the explicit form as in Lemma 2.1. Then, by a straightforward computation we have

$$0 = [\partial, X] = -Z_0 + \operatorname{ad}\partial\cdot\Big\{\sum_{1<\lambda<s} Z_\lambda + W^s_{2,s} + \sum_{1<\lambda<s} W^{\lambda}_{2,s} + \sum_{\substack{1<\alpha<\beta<s \\ \lambda\neq 1,\alpha,\beta}} W^{\lambda}_{\alpha\beta}$$

$$+\sum_{\alpha<s} W^{\alpha}_{\alpha s} + W^1_1 + \sum_{1<\alpha,\lambda} W^{\lambda}_{\alpha}\Big\} - \sum_{1<\lambda<s} c_\lambda W^{\lambda}_0$$

and also

$$0 = [\partial^s, X] = \mathrm{ad}\,\partial^s\cdot\Big\{Z_0 + \sum_{1<\lambda<s} Z_\lambda + W^s_{2,s} + \sum_{1<\lambda<s} W^{\lambda}_{2,s} + \sum_{\substack{1<\alpha<\beta<s\\ \lambda\neq 1,\alpha,\beta}} W^{\lambda}_{\alpha\beta}$$
$$+\sum_{\alpha<s} W^{\alpha}_{\alpha s} + W^1_1 + \sum_{1<\alpha,\lambda} W^{\lambda}_{\alpha}\Big\} - \sqrt{-1}\,W^s_0 \,.$$

From these equalities, it is now easy to see that

$$Z_0 = 0 \quad \text{and} \quad W^{\lambda}_0 = 0 \ (1 < \lambda \leqq s)\,,$$

and hence X can be expressed as

$$X = \sum_{1<\lambda<s} Z_\lambda + W^s_{2,s} + \sum_{1<\lambda<s} W^{\lambda}_{2,s} + \sum_{\substack{1<\alpha<\beta<s\\ \lambda\neq 1,\alpha,\beta}} W^{\lambda}_{\alpha\beta} + \sum_{\alpha<s} W^{\alpha}_{\alpha s} + W^1_1 + \sum_{1<\alpha,\lambda} W^{\lambda}_{\alpha}\,,$$

which says that $X = 0$ on $\mathscr{D} \cap (\boldsymbol{C}^n \times \{0\} \times \cdots \times \{0\})$. Since X is an arbitrary vector field belonging to the center $\mathfrak{z}$ of $\mathfrak{g}(\mathscr{D})$, we conclude that $V \supset \mathscr{D} \cap (\boldsymbol{C}^n \times \{0\} \times \cdots \times \{0\})$, proving our assertion (2). q.e.d.

The proofs of the following lemmas are almost identical to those of Lemmas 2.1 and 2.2 above. Therefore we omit the proofs.

LEMMA 2.3. *Let $\mathscr{D}$ be a sweepable generalized Siegel domain in $\boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \boldsymbol{C}^{m_2} \times \cdots \times \boldsymbol{C}^{m_s}$ $(s \geqq 3)$ with exponent $(c_1, c_2, \cdots, c_{s-1}, 0)$. Let X be a holomorphic vector field in $\mathfrak{g}(\mathscr{D})$ which is independent of $z_1, z_2, \cdots, z_n$. Assume that*

$(*)'$ $c_1 = 1/2$;

$(**)$ *the exponents c_α $(1 \leqq \alpha \leqq s-1)$ are all mutually distinct and $c_\alpha \neq 0$ $(1 \leqq \alpha \leqq s-1)$.*

Then X can be written in the form

$$X = Z_0 + Z_1 + W^s_{2,s} + \sum_{\lambda\neq 2,s} W^{\lambda}_{2,s} + \sum_{\alpha<s} W^{\alpha}_{\alpha s} + \sum_{2<\alpha<\beta<s} W^1_{\alpha\beta} + \sum_{\substack{2<\alpha<s\\ \lambda\neq\alpha\\ 2<\lambda}} W^{\lambda}_{1\alpha}$$
$$+ \sum_{\substack{2<\alpha<\beta<s\\ \lambda\neq\alpha,\beta\\ 2<\lambda}} W^{\lambda}_{\alpha\beta} + W^2_2 + \sum_{\alpha,\lambda\neq 2} W^{\lambda}_{\alpha} + \sum_{\lambda\neq 2} W^{\lambda}_0 \,.$$

LEMMA 2.4. *Under the same situation as in Lemma 2.3, we have the following*:

(1) *The vector field ∂^2 belongs to the center $\mathfrak{z}$ of $\mathfrak{g}(\mathscr{D})$*;

(2) *Let V be the set of common zeros of vector fields belonging to $\mathfrak{z}$. Then*

$$\mathscr{D} \supset \mathscr{D} \cap (\boldsymbol{C}^n \times \boldsymbol{C}^{m_1} \times \{0\} \times \boldsymbol{C}^{m_3} \times \cdots \times \boldsymbol{C}^{m_s})$$
$$\supset V \supset \mathscr{D} \cap (\boldsymbol{C}^n \times \{0\} \times \cdots \times \{0\})\,.$$

REMARK 3. In Lemma 2.3, the case when $s = 2$ has been already considered in Section 1.

3. Proofs of theorems. To begin with, we shall prove the following lemma, from which our problem will be reduced to the special case where $\mathscr{D}$ is a generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2}$ with exponent $(1/2, 0)$.

LEMMA 3.1. *Let $\mathscr{D}$ be a sweepable generalized Siegel domain in $C^n \times C^{m_1} \times C^{m_2} \times \cdots \times C^{m_s}$ $(s \geqq 2)$ with exponent $(c_1, c_2, \cdots, c_{s-1}, 0)$. Suppose that $c_\alpha \neq 0$ for $1 \leqq \alpha \leqq s-1$. Then we have $c_\alpha = 1/2$ for all $\alpha = 1, 2, \cdots, s-1$.*

PROOF. We first claim that all c_α's are identical. Indeed, using Lemma 2.4, this can be verified with exactly the same arguments as in the proof of [3, Theorem].

Secondly we shall show that $c_1 = 1/2$. Since $c_1 = c_2 = \cdots = c_{s-1}$ as above, our domain $\mathscr{D}$ is now a sweepable generalized Siegel domain in $C^n \times C^{\tilde{m}_1} \times C^{\tilde{m}_2}$ with exponent $(c_1, 0)$, where $\tilde{m}_1 = m_1 + m_2 + \cdots + m_{s-1}$ and $\tilde{m}_2 = m_s$. Let V be the set of common zeros of vector fields belonging to the center $\mathfrak{z}$ of $\mathfrak{g}(\mathscr{D})$. Now, suppose that $c_1 \neq 1/2$. Then, by (2) of Lemma 2.2 we have

$$V \neq \emptyset \tag{3.1}$$

and

$$\begin{cases} \dim \mathfrak{z}^c(p) = 0 & \text{for} \quad p \in V\,; \\ \dim \mathfrak{z}^c(p) \neq 0 & \text{for} \quad p \in \mathscr{D} - \mathscr{D} \cap (C^n \times \{0\} \times C^{\tilde{m}_2})\,. \end{cases} \tag{3.2}$$

On the other hand, since $\mathscr{D}$ is sweepable and the center $\mathfrak{z}$ is obviously stable under the adjoint action of any subgroup of $\mathrm{Aut}(\mathscr{D})$, it follows from a result of Vey [5] that $\dim \mathfrak{z}^c(p) =$ constant, where the constant is independent of the point p of $\mathscr{D}$. This is a contradiction. Therefore we have shown that $c_1 = c_2 = \cdots = c_{s-1} = 1/2$. q.e.d.

PROOF OF THEOREM II. *Case* 1. $\tilde{m}_1 = 0$: In this case D is a generalized Siegel domain in $C^n \times C^{\tilde{m}_2}$ with exponent $c = 0$. Hence, this fact follows from (A-2) of Vey's theorem in the Introduction.

Case 2. $\tilde{m}_1 > 0$: First we note that $\tilde{m}_2 > 0$. By considering a suitable linear transformation $\mathscr{L}_\sigma$: $C^{n+m_1+\cdots+m_s} \to C^{n+m_1+\cdots+m_s}$ as defined in the Introduction if necessary, we may assume without loss of generality that

$$c_\alpha \neq 0 \ (1 \leqq \alpha \leqq s-k) \quad \text{and} \quad c_\beta = 0 \ (s-k+1 \leqq \beta \leqq s)\,. \tag{3.3}$$

Thus $\mathscr{D}$ may be regarded as a sweepable generalized Siegel domain in $C^n \times C^{m_1} \times \cdots \times C^{m_{s-k}} \times C^{\tilde{m}_{s-k+1}}$ with exponent $(c_1, c_2, \cdots, c_{s-k}, 0)$ such that $c_\alpha \neq 0$ $(1 \leqq \alpha \leqq s-k)$, where $\tilde{m}_{s-k+1} = m_{s-k+1} + \cdots + m_s$. By Lemma 3.1 we then have $c_1 = c_2 = \cdots = c_{s-k} = 1/2$. Hence $\mathscr{D}$ is a sweepable generalized Siegel domain in $C^n \times C^{\tilde{m}_1} \times C^{\tilde{m}_2}$ with exponent $(1/2, 0)$, where $\tilde{m}_1 = m_1 + \cdots + m_{s-k}$ and $\tilde{m}_2 = \tilde{m}_{s-k+1} = m_{s-k+1} + \cdots + m_s$. Our theorem is now an immediate consequence of Theorem 1.1 in Section 1. q.e.d.

PROOF OF THEOREM III. Suppose that there exists a discrete subgroup Γ of Aut$(\mathscr{D})$ such that $\mathscr{D}/\Gamma$ is compact. We now have the following three cases to consider.

Case 1. $c_\alpha = 0$ *for all* $\alpha = 1, 2, \cdots, s$: In this case $\mathscr{D}$ is a generalized Siegel domain in $C^n \times C^{m_1+m_2+\cdots+m_s}$ with exponent $c = 0$. Hence this theorem follows from (B) of Vey's theorem in the Introduction.

Case 2. $c_\alpha \neq 0$ *for all* $\alpha = 1, 2, \cdots, s$: In this case $\mathscr{D}$ is a Siegel domain of the second kind in $C^n \times C^{m_1+m_2+\cdots+m_s}$ by Theorem I in the Introduction. Therefore, Theorem III again follows from (B) of Vey's theorem (see [3, Corollary]).

Case 3. $c_\alpha \neq 0$ *and* $c_\beta = 0$ *for two exponents* c_α *and* c_β: Since $\mathscr{D}$ is a sweepable domain, it follows immediately from Theorem II that, in this case, $\mathscr{D}$ may be regarded as the direct product $\mathscr{D}_1 \times \mathscr{D}_2$, where $\mathscr{D}_1$ is a Siegel domain of the second kind in $C^n \times C^{\tilde{m}_1}$ and $\mathscr{D}_2$ is a homogeneous bounded circular domain in $C^{\tilde{m}_2}$ containing the origin. Thus, Theorem III follows now from Theorem 1.2 in Section 1. q.e.d.

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