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ON UNIFORMLY ALMOST PERIODIC SETS OF FUNCTIONS FOR ALMOST PERIODIC DIFFERENTIAL EQUATIONS

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1. Introduction. Let $f(t, x)$: $R \times R^n \to R^n$ be continuous and for each $x \in R^n$ be almost periodic (a.p. for short) in t. For a definition and a concise discussion of a.p. functions, cf. the book by Besicovitch [1].

The problem of existence of a.p. solutions of the differential equation

$$x' = f(t, x) \qquad (x' = dx/dt) \tag{1}$$

has been studied extensively. It is known that if $x(t)$ is any a.p. function such that $x'(t)$ exists and is uniformly continuous on R, then $x'(t)$ is a.p. It therefore follows that a necessary condition for the existence of such a.p. solutions of (1) is that $f(t, x(t))$ must be a.p. However, in general such composites are not a.p.; cf. the simple example $f(t, x) = \sin tx$, $x = \sin t$, discussed in Fink [2, p. 16]. The question of what additional conditions are required on f so that any such composite is a.p. has led to the concept of f a.p. in t uniformly for x in certain subsets of R^n. More generally, certain concepts of uniformly a.p. (u.a.p. for short) families or sets of functions have been developed. It is the purpose of this paper to examine these concepts and some relationships between them. The author is greatly indebted to his colleague, A.M. Fink, for many fruitful discussions on these topics.

2. Notation, definitions, theorems. For any $x \in R^n$, $|x|$ will denote some fixed norm.

DEFINITION 1. If $f: R \times S \to R^n$, S some nonempty set, and $\varepsilon > 0$, define

$$T(f, S, \varepsilon) = \{\tau \in R: |f(t+\tau, x) - f(t, x)| \leqq \varepsilon \text{ for } (t, x) \in R \times S\}\,.$$

DEFINITION 2. $f: R \times S \to R^n$, where $S \subset X$, X a metric space, is said to be a.p. in t uniformly for $x \in S$ if it is continuous and if for each compact $K \subset S$ and each $\varepsilon > 0$, there exists an $L = L(\varepsilon, K)$ such that for each $a \in R$,

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$$T(f,K,\varepsilon) \cap [a, a+L] \neq \varnothing \ ;$$

here $\varnothing$ denotes the empty set.

This definition follows Yoshizawa's in [3]. In fact, using the results and methods in [3, pp. 6–19], it follows that if S (or X) is separable the following properties hold:

(P_0) If f is a.p. in t uniformly for $x \in S$, then f is uniformly continuous on $R \times K$, K any compact subset of S.

(P_1) A necessary and sufficient condition that $f(t, x)$ is a.p. in t uniformly for $x \in S$ is that given any sequence $\{t_k' \colon k = 1, 2, \cdots\}$, there exists a subsequence $\{t_k \colon k = 1, 2, \cdots\}$ such that $f(t + t_k, x)$ converges uniformly on each set $R \times K$, $K \subset S$, K compact. Also the limit $g(t, x)$ of such a sequence is a.p. in t uniformly for $x \in S$.

(P_2) The set of functions a.p. in t uniformly for $x \in S$, S fixed, is a linear space over the reals.

(P_3) If $x(t)$ is a.p., $x(t) \in K$ compact for $t \in R$, $K \subset S$, and $f(t, x)$ is a.p. in t uniformly for $x \in S$, then $f(t, x(t))$ is a.p.; here the Bohr definition of an X-valued a.p. function is used.

(P_4) If $f(t, x)$ is a.p. in t uniformly for $x \in S$, the set $\Lambda = \Lambda(f)$ defined by

$$\Lambda = \left\{\lambda \in R \colon \lim_{T \to \infty} T^{-1} \int_0^T f(t, x) e^{-i\lambda t} \neq 0 \quad \text{for some} \quad x \in S\right\}$$

is countable. ($\varnothing$ is countable by definition).

A simple example discussed in [3, p. 14], shows that if we replace K by S in Definition 2, and S is not compact, then (P_2) does not hold.

We also note that if $X = R^n$, as is usually assumed in [3], then X is clearly separable.

The properties (P_0)–(P_4) are very useful in applying certain well-known methods for the existence of a.p. solutions for a.p. systems like (1), where f is a.p. in t uniformly for $x \in S \subset R^n$, S open; cf. [2] and [3].

For generalizations of equations like (1) in the direction of including time delays, it becomes profitable in general to consider $f = f(t, \phi)$, where ϕ is an element in a certain function space X, the so-called initial function space, consisting of R^n-valued functions on $[-r, 0]$ or $(-\infty, 0]$. Thus instead of (1) we consider equations like

$$x'(t) = f(t, x_t) \ , \tag{2}$$

where f is as above, and for fixed $t \in R$, x_t is the function $x(t+s) \colon s \in [-r, 0]$ or $(-\infty, 0]$, such that $x_t \in X$.

Introducing a topology in X, we may then use Definition 2 to define

f a.p. in t uniformly for $S \subset X$, and assuming S separable, properties (P_0)–(P_4) will hold.

On the other hand, if S is not separable, a question of interest arises as to whether additional conditions can be added to Definition 2 such that the desirable properties such as (P_0)–(P_4) hold. We show that such additional conditions as given basically by Fink in [2] are adequate to this purpose. These are contained in the following definition.

DEFINITION 3. Let $f: R \times S \to R^n$ where S is some nonempty set. Suppose

(a) For each $\varepsilon > 0$ there exists an $L = L(\varepsilon, S) > 0$ such that for each $a \in R$, $T(f, S, \varepsilon) \cap [a, a + L] \neq \varnothing$,

(b) $0 \in R$ is an interior point of $T(f, S, \varepsilon)$, and

(c) there exists an M such that $|f(t, x)| \leqq M$ for $(t, x) \in R \times S$. Then $\{f(t, x): x \in S\}$ is called a u.a.p. family; cf. [2, p. 17].

Clearly (b) can also be stated as follows:

(b′) f is continuous in t uniformly for $(t, x) \in R \times S$.

We note that a u.a.p. family is in a sense more general than a function f a.p. in t uniformly for $x \in S$, yet also more restrictive, even if $S = R^n$.

We will show that some of the properties (P_0)–(P_4) hold for f such that $\{f(t, x): x \in S\}$ is a u.a.p. family, but the important property (P_3) does not unless an additional continuity condition on f is imposed; in this case S must be a subset of a topological space which for our purposes will be usually a metric space. However, in what follows, unless specified otherwise, S will denote an arbitrary nonempty set.

DEFINITION 4. For any $f: R \times S \to R^n$, we define $H(f)$ to be the set of all functions g such that for some sequence $\{t_k: k = 1, 2, \cdots\}$, $f(t + t_k, x) \to g(t, x)$ as $k \to \infty$ uniformly on $R \times S$. $H(f)$ is called the hull of f.

Since we will be using the following theorems which essentially appear in Fink [2, pp. 22–26], we state these here; the reader may then also more easily compare these results with ours.

PROPOSITION 1 ([2, Theorem 2.7]). *Let* $\{f(t, x): x \in S\}$ *be a* u.a.p. *family. Then given a sequence* $\{t'_k: k = 1, 2, \cdots\}$, *there exists a subsequence* $\{t_k: k = 1, 2, \cdots\}$ *such that* $\lim_{k\to\infty} f(t + t_k, x)$ *exists uniformly on* $R \times S$.

PROPOSITION 2 ([2, Theorem 2.8]). *Let there exist* M *such that* $|f(t, x)| \leqq M$ *for all* $(t, x) \in R \times S$, *and suppose that each sequence* $\{t'_k: k=$

$1, 2, \cdots\}$ *contains a subsequence* $\{t_k: k=1, 2, \cdots\}$ *such that* $\lim_{k\to\infty} f(t+t_k, x)$ *exists uniformly on* $R \times S$. *Then* $\{f(t, x): x \in S\}$ *is a* u.a.p *family.*

PROPOSITION 3 ([2, Theorem 2.10]). *Let S be a compact subset of a metric space X and $f(t, x)$ be continuous on $R \times S$, and* a.p. *in t for each $x \in S$. Then $\{f(t, x): x \in S\}$ is a* u.a.p. *family if and only if f is continuous in x uniformly for $t \in R$.*

We point out that Proposition 3 is stated in [2] only for the case $X = R^n$, but its proof for a general metric space X follows analogously.

THEOREM 1. *Let $\{f(t, x): x \in S\}$ be a* u.a.p. *family. Then for any $g \in H(f)$, $\{g(t, x): x \in S\}$ is a* u.a.p. *family and $f \in H(g)$.*

PROOF. We use Propositions 1 and 2. Let $\{t_k: k = 1, 2, \cdots\}$ be such that $f(t+t_k, x) \to g(t, x)$ as $k \to \infty$ uniformly on $R \times S$. Clearly $|g(t, x)| \leqq M$ for $(t, x) \in R \times S$ since $|f(t, x)| \leqq M$ for $(t, x) \in R \times S$. Let $\{\tau'_k: k = 1, 2, \cdots\}$ be given; by Proposition 1, there exists a subsequence $\{\tau_k: k = 1, 2, \cdots\}$ such that $\{f(t + \tau_k, x): k = 1, 2, \cdots\}$ converges uniformly for $(t, x) \in R \times S$. Then

$$\begin{aligned} &|g(t + \tau_k, x) - g(t + \tau_l, x)| \\ &\quad \leqq |g(t + \tau_k, x) - f(t + \tau_k + t_k, x)| + |f(t + \tau_k + t_k, x) \\ &\quad\quad - f(t + \tau_l + t_k, x)| + |f(t + \tau_l + t_k, x) - g(t + \tau_l, x)| \,. \end{aligned}$$

Using the definition of $\{t_k: k = 1, 2, \cdots\}$ and the Cauchy convergence criterion, we easily see that $\{g(t + t_k, x): k = 1, 2, \cdots\}$ converges uniformly on $R \times S$. By Bochner's normality criterion (cf. [1]), $g(t, x)$ is a.p. in t for each $x \in S$, so by Proposition 2, $\{g(t, x): x \in S\}$ is a u.a.p. family. The fact that $f \in H(g)$ follows easily since $f(t + t_k, x) \to g(t, x)$ as $k \to \infty$ uniformly on $R \times S$ implies $g(t - t_k, x) \to f(t, x)$ as $k \to \infty$ uniformly on $R \times S$. q.e.d.

The fact that the normality condition in (P_1) holds for any f such that $\{f(t, x): x \in S\}$ is a u.a.p. family follows immediately from Proposition 1. However, with a view toward introducing some continuity properties into f, we state

THEOREM 2.1. *Let S be a subset of a metric space and $f: R \times S \to R^n$ be continuous in x uniformly for $(t, x) \in R \times S$. Suppose also that $|f(t, x)| \leqq M$ for $(t, x) \in R \times S$ and that for any sequence $\{t'_k: k = 1, 2 \cdots\}$, there exists a subsequence $\{t_k: k = 1, 2, \cdots\}$ such that $\{f(t + t_k, x): k = 1, 2, \cdots\}$ converges uniformly for $(t, x) \in R \times S$. Then if $g \in H(f)$, $\{g(t, x): x \in S\}$ is a* u.a.p. *family, and g is continuous in x uniformly for $(t, x) \in R \times S$.*

PROOF. Using Proposition 2, we conclude that $\{f(t, x): x \in S\}$ is a u.a.p. family. Thus from Theorem 1, $g \in H(f)$ implies $\{g(t, x): x \in S\}$ is a u.a.p. family. The uniform continuity of g in x follows easily from the corresponding property of f; we omit the details.

THEOREM 2.2. *Let S be a subset of a metric space and $\{f(t, x): x \in S\}$ a* u.a.p. *family where f is continuous in x uniformly for $(t, x) \in R \times S$. Then if $\{t'_k: k = 1, 2, \cdots\}$ is any sequence, there exists a subsequence $\{t_k: k = 1, 2, \cdots\}$ and a $g \in H(f)$ such that $f(t + t_k, x) \to g(t, x)$ as $k \to \infty$ uniformly on $R \times S$; g is also continuous in x uniformly for $(t, x) \in R \times S$.*

PROOF. The existence of $g \in H(f)$ follows immediately from Proposition 1. The continuity of g in x as asserted is a trivial consequence of the uniform convergence of $\{f(t + t_k, x): k = 1, 2, \cdots\}$. q.e.d.

If we assume S is compact, then we can easily get the conclusions of Theorems 2.1 and 2.2 if f is only continuous in x uniformly for $t \in R$. Note that Proposition 3 deals with the case where S is compact. Fink's definition of f a.p. in t uniformly for $x \in S$ is apparently not the same as our Definition 2, or as Yoshizawa's in [3]. For S compact, the following theorem is a trivial consequence of Proposition 3.

THEOREM 3. *Let $\{f(t, x): x \in S\}$ be a* u.a.p. *family, where S is a subset of a metric space. Let f be continuous on $R \times S$. Then $f(t, x)$ is continuous in x uniformly for $t \in R$.*

PROOF. Assume not; then there exist $\bar{x} \in S$, $\varepsilon_1 > 0$, and sequences $\{t_k: k = 1, 2, \cdots\}$, $\{x_k: k = 1, 2, \cdots\}$, $x_k \in S$ such that $x_k \to \bar{x}$ as $k \to \infty$ and

$$|f(t_k, x_k) - f(t_k, \bar{x})| \geqq \varepsilon_1, \quad k = 1, 2, \cdots . \tag{3}$$

Using Definition 3, there exists an $L_1 = L(\varepsilon_1, S) > 0$ such that

$$T(f, S, \varepsilon_1/3) \cap [-t_k, -t_k + L_1] \neq \varnothing , \quad k = 1, 2, \cdots .$$

Hence there exist $\tau_k \in [-t_k, -t_k + L_1]$ such that $\tau_k \in T(f, S, \varepsilon_1/3)$ $k = 1, 2, \cdots$. Since $0 \leqq t_k + \tau_k \leqq L_1$ for $k = 1, 2, \cdots$, and f is continuous on $R \times S$, it follows that

$$|f(t_k + \tau_k, x_k) - f(t_k + \tau_k, \bar{x})| \to 0 \quad \text{as} \quad k \to \infty .$$

Also then

$$\begin{aligned} &|f(t_k, x_k) - f(t_k, \bar{x})| \\ &\quad \leqq |f(t_k, x_k) - f(t_k + \tau_k, x_k)| + |f(t_k + \tau_k, x_k) - f(t_k + \tau_k, \bar{x})| \\ &\qquad\qquad + |f(t_k + \tau_k, \bar{x}) - f(t_k, \bar{x})| \\ &\quad \leqq 2\varepsilon_1/3 + |f(t_k + \tau_k, x_k) - f(t_k + \tau_k, \bar{x})| < \varepsilon_1 \end{aligned}$$

for k sufficiently large. This contradicts (3) and proves the theorem.

Note that in Theorems 2.1 and 2.2 the continuity of f in only x was assumed.

The following is an easy consequence of Theorem 3 and (b′) of Definition 3, but is actually proved in the proof of Proposition 3.

COROLLARY 1. *Let f satisfy the hypotheses of Theorem 3 and S be compact. Then f is uniformly continuous on $R \times S$, and any $g \in H(f)$ is also uniformly continuous on $R \times S$.*

THEOREM 4. *Let $\{f(t, x): x \in S\}$ be a* u.a.p. *family, where S is a subset of a metric space, and suppose f is uniformly continuous on $R \times S$. Let $x(t)$ be* a.p. *and $x(t) \in S$ for $t \in R$. Then $f(t, x(t))$ is* a.p.

PROOF. Note first that this is not quite Theorem 2.11 in [2] but can be proved in a similar manner.

We give a proof using Bochner normality. Given $\{t'_k: k = 1, 2, \cdots\}$ there exists a subsequence $\{t_k: k = 1, 2, \cdots\}$ such that $\{f(t + t_k, x): k = 1, 2 \cdots\}$ converges uniformly for $(t, x) \in R \times S$; this follows by Proposition 1. We may assume $\{t_k: k = 1, 2, \cdots\}$ is such that $\{x(t + t_k): k = 1, 2, \cdots\}$ is a Cauchy sequence uniformly for $t \in R$; this follows as in the proof of in [1, Theorem 3° p. 11]. But

$$\begin{aligned} &|f(t + t_k, x(t + t_k)) - f(t + t_l, x(t + t_l))| \\ &\quad \leqq |f(t + t_k, x(t + t_k)) - f(t + t_l, x(t + t_k))| + |f(t + t_l, x(t + t_k)) \\ &\qquad - f(t + t_l, x(t + t_l))| , \end{aligned}$$

and since f is uniformly continuous on $R \times S$, it follows that $\{f(t + t_k, x(t + t_k)): k = 1, 2, \cdots\}$ is a Cauchy sequence uniformly for $t \in R$. This proves the theorem.

In applications to delay-differential equations with infinite delay, the range of the solution x_t as a function on R to some initial function space X is usually compact. For example, if $X =$ CB, the space of R^n-valued functions continuous and bounded on $(-\infty, 0]$ with norm given by $\|\phi\| = \sup\{|\phi(s)|: s \leqq 0\}$ for $\phi \in$ CB, the compactness of $\{x_t: t \in R\}$ where $x(t)$ is a.p. and $x_t(s) = x(t + s)$, $s \leqq 0$, is an easy consequence of the normality of $x(t)$.

THEOREM 5. *Let S be a compact subset of a metric space. Let f: $R \times S \to R^n$ be uniformly continuous on $R \times S$. Then if f is* a.p. *in t uniformly for $x \in S$, the set $\{f(t, x): x \in S\}$ is a* u.a.p. *family, and conversely.*

PROOF. To show $f(t, x)\colon x \in S$ bounded, suppose it is not. Then there exists $\{(t_k, x_k) \in R \times S\colon k = 1, 2, \cdots\}$ such that $|f(t_k, x_k)| \to \infty$ as $k \to \infty$. We may assume $x_k \to \bar{x} \in S$ as $k \to \infty$. Since as a function of t, $f(t, \bar{x})$ is normal we may assume also that $f(t_k, \bar{x}) \to g(\bar{x})$ as $k \to \infty$. But

$$|f(t_k, x_k) - g(\bar{x})| \leqq |f(t_k, x_k) - f(t_k, \bar{x})| + |f(t_k, \bar{x}) - g(\bar{x})| ,$$

and since f is continuous in x uniformly for $t \in R$ by Theorem 3, it follows that $|f(t_k, x_k)|$ must be bounded by k large, a contradiction. Thus (c) of Definition 3 holds. Since (b′) follows immediately for the uniform continuity of f on $R \times S$, and since (a) holds by Definition 2 we conclude that $\{f(t, x)\colon x \in S\}$ is a u.a.p. family.

Since S is compact, the converse is a simple consequence of Definitions 2 and 3 and our proof is complete. Thus for such S and functions f, the concepts of f a.p. uniformly for $x \in S$ and $\{f(t, x)\colon x \in S\}$ being a u.a.p. family are equivalent.

For any a.p. function $F(t)\colon R \to R^n$, it is well known that the set

$$\Lambda(F) = \left\{\lambda \in R\colon \lim_{T \to \infty} T^{-1}\int_0^T F(t)e^{-i\lambda t}dt \neq 0\right\}$$

is countable; recall that we define the empty set to be countable. The elements λ of $\Lambda(F)$ are called the Fourier exponents of F.

DEFINITION 5. The frequency module mod $F(\cdot)$ of an a.p. function F is the set $\{n_1\lambda_1 + \cdots + n_j\lambda_j\colon n_k$ integers, $\lambda_k \in \Lambda(F)\}$.

THEOREM 6. *Let $\{f(t, x)\colon x \in S\}$ be a* u.a.p. *family, where S is any nonempty set. Then the set*

$$\Lambda(f) = \{\lambda \in R\colon \lim_{T \to \infty} T^{-1}\int_0^T f(t, x)e^{-i\lambda t}dt \neq 0 \quad \textit{for some } x \in S\}$$

is countable.

PROOF. For $x \in S$, define $v_x(\tau) = \sup\{|f(t + \tau, x) - f(t, x)|\colon t \in R\}$ and $v(\tau) = \sup\{v_x(\tau)\colon x \in S\}$. Clearly $v(\tau)$ is well defined and bounded for $\tau \in R$. From Theorem 2.2 in [2, p. 18], $v(\tau)$ is a.p. and for each $x \in S$, $T(f, \{x\}, \varepsilon) \supset T(v, \varepsilon)$, where

$$T(v, \varepsilon) = \{\tau \in R\colon |v(t + \tau) - v(t)| \leqq \varepsilon \quad \text{for} \quad t \in R\} .$$

Using Theorem 4.5 in [2, p. 61], we conclude that for each $x \in S$:

$$\text{mod}\, f(\cdot, x) \subset \text{mod}\, v(\cdot) ,$$

and so $\bigcup_{x \in S} \text{mod}\, f(\cdot, x) \subset \text{mod}\, v(\cdot)$. Since $\text{mod}\, v(\cdot)$ is countable, so is $\bigcup_{x \in S} \text{mod}\, f(\cdot, x)$, and since $\Lambda(f) \subset \bigcup_{x \in S} \text{mod}\, f(\cdot, x)$, our proof is complete.

Thus (P_4) holds if $f(t, x)$ is such that $\{f(t, x): x \in S\}$ is a u.a.p. family and x is in an arbitrary nonempty set.

THEOREM 7. *Let f and g be uniformly continuous on $R \times S$ to R^n, where S is a subset of metric space. Let $\{f(t, x): x \in S\}$ and $\{g(t, x): x \in S\}$ be* u.a.p. *families. Then for any real constants c_1, c_2,*

$$\{c_1 f(t, x) + c_2 g(t, x): x \in S\}$$

is a u.a.p. *family, and each member is uniformly continuous on $R \times S$.*

This theorem is an easy consequence of Propositions 1 and 2 and some obvious continuity arguments; we omit the proof.

We note that by Corollary 1 it is no gain in generality to suppose f only continuous on $R \times S$, if S is compact. We note also that no separability conditions are needed on S in Theorem 7. These results establish, in a sense, the property (P_2) for u.a.p. families.

3. Concluding remarks. From the results in the preceding section it follows that all the properties (P_0)–(P_4) hold for and f on $R \times S$, S a subset of a metric space and f uniformly continuous on $R \times S$, where the condition "f is a.p. in t uniformly for $x \in S$" is replaced by "$\{f(t, x): x \in S\}$ is a u.a.p. family". No separability condition on S is required. Furthermore, some of these properties, in particular, (P_1) and (P_4) hold for f not necessarily continuous on $R \times S$. Theorems 2.1 and 2.2 clearly relate to (P_1), and as has already been observed, (P_4) holds for quite arbitrary u.a.p. families.

In much of the recent work on the existence of a.p. solutions of delay-differential equations with infinite delays and a.p. t-dependence functions on fairly general topological initial function spaces are considered; cf. [3], [4], [5]. In some of this work, these spaces can be assumed metrizable, and the functions are assumed to be a.p. in t uniformly for ϕ in this space. However, in the proofs for the existence of a.p. solutions in, for example [3], [4], and [5], this functions are only considered on $R \times S$, S a compact subset of this function space. Since it is also assumed that f is continuous on $R \times S$, it follows from Corollary 1 that f is in fact uniformly continuous on $R \times S$. So by Theorem 5, $\{f(t, \phi): \phi \in S\}$ is a u.a.p. family and all the properties (P_0)–(P_4) hold. So it seems that these existence results do not require the hypothesis that the initial function space is separable.

Also in the theory of a.p. systems of ordinary and especially functional differential equations of retarded type, the concept of asymptotic almost periodicity is important; cf., for example, [3, pp. 21–29]. To discuss the existence of asymptotic a.p. solutions, it is profitable to define $H^+(f)$ for the $f(t, x)$ in the equation to be the subset of $H(f)$ consisting of functions $g(t, x)$ such that $f(t + t_k, x) \to g(t, x)$ as $k \to \infty$ for some sequence $t_k \to \infty$ as $k \to \infty$. In fact, it is important to know if $f \in H^+(f)$. But this follows easily for either f a.p. in t uniformly for $x \in S$, or $\{f(t, x): x \in S\}$ a u.a.p. family. In the latter case, for example, we use (a) of Definition 3 to get the existence of $t_k \in T(f, S, 1/k) \cap [k, k + L_k]$, $k = 1, 2, \cdots$, where $L_k = L$ in (a) of Definition 3 for $\varepsilon = 1/k$. For such t_k, clearly $|f(t + t_k, x) - f(t, x)| \to 0$ as $k \to \infty$, uniformly on $R \times S$.

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