Tôhoku Math. Journ.
35 (1983), 349-356.

COMPARISON METHOD AND STABILITY PROBLEM IN FUNCTIONAL DIFFERENTIAL EQUATIONS

WANG ZHICHENG

(Received March 11, 1982, revised Octover 18, 1982)

Abstract. In this paper, using the comparison method and borrowing the ideas and terminologies from Kato [1], [2], [3], we discuss the stability in functional differential equations with infinite delay. We also give some extensions of the ideas in [5], [6], [7]. As a corollary to our results, the corresponding stability theorem of Kato [1] is included.

Let X be a linear space of R^n-valued functions on $(-\infty, 0]$ with a semi-norm $\|\cdot\|_X$, and denote by X_τ the space of functions $\phi(s)$ on $(-\infty, 0]$ which are continuous on $[-\tau, 0]$ and satisfying $\phi_{-\tau} \in X$ for $\tau \geqq 0$, where and henceforth ϕ_t denotes the function on $(-\infty, 0]$ defined by $\phi_t(s) = \phi(t+s)$.

The space X is said to be admissible, if the following are satisfied: For any $\tau \geqq 0$ and any $\phi \in X_\tau$

(a) $\phi_t \in X$ for all $t \in [-\tau, 0]$, especially, $\phi_0 = \phi \in X$;

(b) ϕ_t is continuous in $t \in [-\tau, 0]$;

(c) $\mu \|\phi(0)\| \leqq \|\phi\|_X \leqq K(\tau) \sup_{-\tau \leqq s \leqq 0} \|\phi(s)\| + M(\tau) \|\phi_{-\tau}\|_X$,

where $\mu > 0$ is a constant and $K(\tau)$, $M(\tau)$ are continuous.

Consider the functional differential equation

$$\dot{x} = f(t, x_t) \tag{E}$$

and assume that $f(t, 0) \equiv 0$ and that $f(t, \phi)$ is completely continuous on $I \times X$ where X is an admissible space and $I = [0, \infty)$. For the fundamental properties of the solutions of (E), we refer to [4].

Let Y be an admissible space satisfying $X \subset Y$ and

$$\|\phi\|_Y \leqq N \|\phi\|_X \qquad (\phi \in X) ,$$

where $N > 0$ is a constant. Let $x(t)$ be an arbitrary solution of (E).

The definitions of stability in (X, Y) will be given as follows: The zero solution of (E) is said to be

(i) stable in (X, Y), if for any $\varepsilon > 0$ and any $\tau \geqq 0$, there is a $\delta = \delta(\tau, \varepsilon) > 0$ such that $\|x_\tau\|_X < \delta$ implies

$$\|x_t\|_Y < \varepsilon \quad \text{for all} \quad t \geqq \tau ;$$

(ii) uniformly stable in (X, Y), if it is stable in (X, Y) and δ is independent of τ;

(iii) uniformly asymptotically stable in (X, Y), if it is uniformly stable and there is a $\delta_0 > 0$ and a function $T(\varepsilon) > 0$ such that $\|x_\tau\|_X < \delta_0$ implies

$$\|x_t\|_Y < \varepsilon \quad \text{for} \quad t \geqq \tau + T(\varepsilon) .$$

A Liapunov function is a collection $\{v(t, \phi; \tau): \tau \geqq 0\}$ of real-valued, continuous functions $v(t, \phi; \tau)$, defined on $\{(t, \phi): \phi \in X_{t-\tau}, t \geqq \tau\}$ satisfying

$$a(\|\phi\|_Y) \leqq v(t, \phi; \tau)$$

for a continuous nondecreasing positive definite function $a(r)$ and

$$\text{(B)} \qquad v(t, \phi; \tau) \leqq b(t, \tau, \|\phi\|_{X_{t-\tau}})$$

for a function $b(t, \tau, r)$, continuous on I^3, nondecreasing in r and $b(t, \tau, 0) = 0$, where $\|\phi\|_{X_\tau} = \sup_{-\tau \leq t \leq 0} \|\phi_t\|_X$.

Define

$$\dot{v}_{(E)}(t, \phi; \tau) = \sup \limsup_{s \to t+0} [v(s, x_s; \tau) - v(t, \phi; \tau)]/(s - t)$$

for a solution $x(s)$ of (E) satisfying $x_t = \phi$ where the supremum is taken over all such solutions.

Before we state the following theorems concerning the stability in (X, Y), some additional notations are required.

(L): There exist continuous functions $L(t, s, r)$ on I^3, nondecreasing in r with $L(t, s, 0) = 0$, and $\delta_0(t, s)$ on I^2 with $\delta_0(t, s) > 0$ such that any solution $x(t)$ of (E) satisfies

$$\|x_t\|_X \leqq L(t, s, \|x_s\|_X) \quad \text{if} \quad \|x_s\|_X < \delta_0(t, s) , \qquad t \geqq s .$$

Note that, if the zero solution of (E) is unique for the initial value problem, then the condition (L) holds (see [4]).

(UL): In (L), $L(t, s, r)$ and $\delta_0(t, s)$ can be chosen in such a way that $L(t, s, r) = L(t - s, 0, r)$ and $\delta_0(t, s) = \delta_0(t - s, 0)$.

(P): $p(t, r)$ is continuous on $I \times (0, \infty)$, nondecreasing in r and satisfies

$$p(t, r) \leqq t , \qquad p(t, r) \to \infty \quad \text{as} \quad t \to \infty .$$

(UP): In (P) assume that $q(r) = t - p(t, r)$ is positive and independent of t.

It is easy to see that under the condition (P), $\sigma(t, r) = \sup\{s: p(s, r) \leqq t\}$ is continuous on $I \times (0, \infty)$, nonincreasing in r, $\sigma(t, r) \geqq t$ and $p(t, r) \geqq \tau$ if $t \geqq \sigma(\tau, r)$ and $r > 0$.

The following theorem generalizes an analogous theorem of Driver (see [7, Theorem 4]).

THEOREM 1. *Assume that*
(i) *condition* (L) *holds;*
(ii) *there is a Liapunov function* $\{v(t, \phi; \tau): \tau \geqq 0\}$ *which satisfy*

$$\dot{v}_{(E)}(t, \phi; \tau) \leqq w(t, v(t, \phi; \tau)) \tag{1}$$

whenever $v(t, \phi; \tau) > 0$, $p(t, v(t, \phi; \tau)) \geqq \tau$ *and* $v(s, \phi_{s-t}; \tau) \leqq v(t, \phi; \tau)$ *for* $s \in [p(t, v(t, \phi; \tau)), t]$ *where* $w(t, r)$ *is nonnegative, continuous on* I^2, $w(t, 0) = 0$, *and* $p(t, r)$ *is the one in* (P);
(iii) *the zero solution of*

$$\dot{y} = w(t, y) \tag{2}$$

is stable.
Then the zero solution of (E) *is stable in* (X, Y).

PROOF. Let $x(t)$ be a solution of (E) starting at $t = \tau$ for a $\tau \geqq 0$, and let $v(t) = v(t, x_t; \tau)$. For any $\eta > 0$, let $\varepsilon = \min(\eta, a(\eta))$. Since the zero solution of (2) is stable, there is a $\delta_1(\tau, \varepsilon)$, $0 < \delta_1 \leqq \varepsilon$, such that $y_0 = \delta_1$ implies

$$\delta_1 \leqq y(t, \tau, y_0) < \varepsilon \quad \text{for all} \quad t \geqq \tau ,$$

where $y(t) = y(t, \tau, y_0)$ is a maximal solution of (2) starting at $t = \tau$ with the initial value y_0.

For the above $\delta_1 > 0$, there is a $\delta > 0$ such that

$$\sup_{\tau \leqq s \leqq \sigma(\tau, \delta_1)} b(s, \tau, L(s, \tau, \delta)) \leqq \delta_1, \; \delta \leqq \inf_{\tau \leqq s \leqq \sigma(\tau, \delta_1)} \delta_0(s, \tau) .$$

Then by (B) and (L), we have $\|x_\tau\|_X < \delta$ implies

$$v(t) \leqq b(t, \tau, L(t, \tau, \|x_\tau\|_X)) \leqq \delta_1 \quad \text{for} \quad t \in [\tau, \sigma(\tau, \delta_1)] ,$$

so that

$$v(t) \leqq y(t) \quad \text{for} \quad t \in [\tau, \sigma(\tau, \delta_1)] .$$

We now show that

$$v(t) \leqq y(t) \quad \text{for all} \quad t \geqq \tau . \tag{3}$$

Suppose to the contrary, that $v(t_1) > y(t_1)$ for a $t_1 > \sigma(\tau, \delta_1)$. Let $y_m(t)$ be any solution of

$$\dot{y} = w(t, y) + 1/m \quad \text{with} \quad y(\tau) = y_0 , \qquad m = 1, 2, \cdots . \tag{4}$$

It is known that the maximal solution $y(t)$ can be represented as

$$y(t) = \lim_{m\to\infty} y_m(t) .$$

Then there is a number $m > 0$, sufficiently large, such that $v(t_1) > y_m(t_1)$. Since $y_m(t)$ is nondecreasing, for the $t_2 = \inf\{t \in [\tau, t_1]: v(t) > y_m(t)\}$ we see that $v(t_2) = y_m(t_2)$, $v(t_2) \geqq v(t)$ for all $t \in [\tau, t_2]$ and

$$\dot{v}(t_2) \geqq \dot{y}_m(t_2) = w(t_2, y_m(t_2)) + 1/m = w(t_2, v(t_2)) + 1/m .$$

On the other hand, since $v(t_2) > \delta_1$, $t_2 > \sigma(\tau, \delta_1)$, $p(t_2, v(t_2)) \geqq p(t_2, \delta_1) \geqq \tau$ and $v(t) \leqq v(t_2)$ for $t \in [p(t_2, v(t_2)), t_2]$, we have $\dot{v}(t_2) \leqq w(t_2, v(t_2))$, a contradiction.

Therefore, we see that (3) holds and that

$$a(\|x_t\|_Y) \leqq v(t) < a(\eta) \quad \text{for all} \quad t \geqq \tau .$$

Thus

$$\|x_t\|_Y < \eta \quad \text{for all} \quad t \geqq \tau ,$$

and the proof is complete.

THEOREM 2. *In Theorem* 1 *assume that* (L) *is replaced by* (UL), *that in addition to* (UP) *v satisfies* (UB), i.e., $b(t, \tau, r) = b(t - \tau, 0, r)$ *in* (B), *and that the zero solution of* (2) *is uniformly stable. Then the zero solution of* (E) *is uniformly stable in* (X, Y).

PROOF. Note that $\sigma(t, r) = t + q(r)$ and that δ_1 and δ can be chosen as functions of ε alone such that

$$\sup_{0 \leqq \xi \leqq q(\delta_1)} b(\xi, 0, L(\xi, 0, \delta)) < \delta_1, \ \delta \leqq \inf_{0 \leqq \xi \leqq q(\delta_1)} \delta_0(\xi, 0) .$$

The proof is the same as that of Theorem 1.

THEOREM 3. *Assume that*

(i) *condition* (UL) *holds;*

(ii) *there is a Liapunov function* $\{v(t, \phi; \tau): \tau \geqq 0\}$ *which satisfies* (UB) *and*

$$\dot{v}_{(E)}(t, \phi; \tau) = -w(t, v(t, \phi; \tau)) \tag{5}$$

whenever $v(t, \phi; \tau) > 0$, $p(t, v(t, \phi; \tau)) \geqq \tau$ *and* $v(s, \phi_{s-t}; \tau) \leqq F(v(t, \phi; \tau))$ *for* $s \in [p(t, v(t, \phi; \tau)), t]$, *where* $w(t, r)$ *is nonnegative, continuous on* I^2, $w(t, 0) = 0$; $p(t, r)$ *satisfies* (UP), *and* $F(r)$ *is a continuous, nondecreasing function satisfying* $F(r) > r$ *for* $r > 0$.

(iii) *the zero solution of*

$$\dot{z} = -w(t, z) \tag{6}$$

is uniformly asymptotically stable.

Then the zero solution of (E) *is uniformly asymptotically stable in* (X, Y).

PROOF. By (iii), there is a $\delta_0 > 0$ and for any $\eta > 0$, there is a $T_0(\eta) > 0$ such that $0 < z_0 < \delta_0$, $\tau \geqq 0$ imply that

$$(7) \qquad 0 < z(t, \tau, z_0) < \eta \quad \text{for} \quad t \geqq \tau + T_0(\eta) .$$

For the above $\delta_0 > 0$, there is a $\delta_1 > 0$ such that

$$\sup_{0 \leqq \xi \leqq q(\delta_0)} b(\xi, 0, L(\xi, 0, \delta_1)) \leqq \delta_0 , \qquad \delta_1 \leqq \inf_{0 \leqq \xi \leqq q(\delta_0)} \delta_0(\xi, 0) .$$

Then we see that $\|x_\tau\|_X \leqq \delta_1$ implies

$$v(t, x_t, \tau) < \delta_0 \quad \text{for} \quad t \in [\tau, \tau + q(\delta_0)] ,$$

and hence,

$$v(t, x_t; \tau) \leqq \delta_0 \quad \text{for all} \quad t \geqq \tau .$$

In fact, suppose that $v(t_1) > \delta_0$ for a $t_1 > \tau + q(\delta_0)$. Then we can find a $t_2 \in [\tau + q(\delta_0), t_1]$ so that $v(t_2) > \delta_0$, $\dot{v}(t_2) > 0$ and $v(t) \leqq v(t_2)$ for all $t \in [\tau, t_2]$. Since $p(t_2, v(t_2)) \geqq p(t_2, \delta_0) \geqq \tau$ and $v(t) \leqq v(t_2) \leqq F(v(t_2))$ for $t \in [p(t_2, v(t_2)), t_2]$, we have $\dot{v}(t_2) \leqq 0$, a contradiction.

We now show that for any $\eta > 0$ ($\eta < \delta_0$), there is a $T(\eta) > 0$ such that $\|x_\tau\|_X < \delta_1$ implies that

$$(8) \qquad v(t, x_t; \tau) \leqq \eta \quad \text{for} \quad t \geqq \tau + T(\eta) .$$

Let $a = \inf_{\eta \leqq s \leqq \delta_0} [F(s) - s] > 0$, and let m be the first positive integer such that $\eta + ma \geqq \delta_0$. Let $c_n = \eta + na$ ($n = 0, 1, 2, \cdots, m$), $\sigma_i = \sigma(\tau_{i-1}, c_{m-i}) = \tau_{i-1} + q(c_{m-i})$, $\tau_0 = \tau$, $\tau_i = \sigma_i + T_0(\eta)$ and $v(t) = v(t, x_t; \tau)$.

First we show that

$$v(t_1) < c_{m-1} \quad \text{for a} \quad t_1 \in [\sigma_1, \sigma_1 + T_0(\eta)] .$$

Suppose that

$$(9) \qquad v(t) \geqq c_{m-1} \quad \text{for all} \quad t \in [\sigma_1, \sigma_1 + T_0(\eta)] .$$

Then we have

$$F(v(t)) \geqq v(t) + a \geqq c_{m-1} + a = c_m \geqq \delta_0 \geqq v(s)$$

for $s \in [\tau, t]$, Since $t \geqq \sigma_1 = \sigma(\tau, c_{m-1})$ and $p(t, v(t)) \geqq p(t, c_{m-1}) \geqq \tau$, we have

$$F(v(t)) \geqq v(s) \quad \text{for} \quad s \in [p(t, v(t)), t] .$$

By (ii), it follows that

$$\dot{v}(t) \leqq -w(t, v(t)) \quad \text{for} \quad t \in [\sigma_1, \sigma_1 + T_0(\eta)] ,$$

and

$$v(t) \leqq z(t, \sigma_1, z_1) \quad \text{for} \quad t \in [\sigma_1, \sigma_1 + T_0(\eta)] ,$$

where $z_1 = v(\sigma_1, x_{\sigma_1}; \tau) < \delta_0$ and $z(t, \sigma_1, z_1)$ is a maximal solution of (6) starting at $t = \sigma_1$ with the initial value z_1. Since $z_1 < \delta_0$, we have

$$0 < z(t, \sigma_1, z_1) < \eta \quad \text{for} \quad t \geqq \sigma_1 + T_0(\eta) .$$

Thus

$$v(\sigma_1 + T_0(\eta)) < \eta .$$

On the other hand, by (9), we have

$$v(\sigma_1 + T_0(\eta)) \geqq c_{m-1} > \eta ,$$

which is a contradiction.

Next we show that

$$v(t) \leqq c_{m-1} \quad \text{for all} \quad t \geqq t_1 . \tag{10}$$

Suppose it is not the case. Then there is a $t^* > t_1$, such that $v(t^*) > c_{m-1}$ and $\dot{v}(t^*) > 0$. But, since $t^* > \sigma(\tau, c_{m-1})$, $p(t_1, v(t^*)) \geqq p(t^*, c_{m-1}) \geqq \tau$ and $F(v(t^*)) \geqq v(t^*) + a \geqq \delta_0 \geqq v(s)$ for $s \in [\tau, t^*]$, we have $\dot{v}(t^*) \leqq 0$, a contradiction.

With the comparison solution $z(t, \sigma_1, z_1)$ replaced by $z(t, \sigma_k, z_k)$ and by the same type of reasoning as above, we can show that

$$v(t) \leqq c_{m-k} \quad \text{for} \quad t \geqq \sigma_k + T_0(\eta) ,$$

$k = 2, \cdots, m$, where $z_k = v(\sigma_k, x_{\sigma_k}; \tau) < \delta_0$.

Finally, we have

$$v(t) \leqq \eta \quad \text{for} \quad t \geqq \tau + T(\eta) ,$$

where $\tau + T(\eta) = \sigma_m + T_0(\eta)$ and $T(\eta) = q(c_{m-1}) + \cdots + q(c_0) + mT_0(\eta)$. This proves Theorem 3.

REMARK. Driver [7, Theorem 7] and Kato [3, Theorem 4] correspond, respectively, to the cases where $q(r)$ is independent of r and where $w(t, r)$ is independent of t. Therefore, Theorem 3 can be considered as an extension of these theorems.

EXAMPLE. Consider the scalar equation

$$\dot{x}(t) = -ax(t) + bx(t-h) + \int_{-\infty}^{0} g(t, s, x(t+s))ds \tag{11}$$

where a, b and h are constants, $a > 0$, $|b| < a$, $h > 0$. Assume that $g(t, s, x)$ is continuous and satisfies

$$|g(t, s, x)| \leqq m(s)|x| ,$$

where

$$\int_{-\infty}^{0} m(s)ds < a - |b| , \qquad \int_{-\infty}^{0} m(s)e^{-\gamma s}ds < \infty \tag{12}$$

for a $\gamma > 0$. Then the zero solution of (11) is uniformly asymptotically stable in $(C_{\infty}^{\gamma}, R^1)$. Indeed, by (12), we can choose a constant $F > 1$ and a continuous function $q(r)$ on $(0, \infty)$, nondecreasing in r and $q(r) \leqq -h$ for $r > 0$, such that

$$a - |b| - F^{1/2}\int_{-\infty}^{0} m(s)ds = \delta > 0 , \tag{13}$$

and

$$2\int_{-\infty}^{q(r)} m(s)e^{-\gamma s}ds \leqq \delta r^{1/2} . \tag{14}$$

Let $v(t, \phi) = \phi(0)^2$. Then we have

$$\begin{aligned}\dot{v}_{(13)}(t, x_t) \leqq &-2ax^2(t) + 2|b||x(t)||x(t-h)| \\ &+ 2|x(t)|\int_{-\infty}^{0} m(s)|x(t+s)|ds .\end{aligned}$$

Let $v(t) = v(t, x_t)$. Then by (13) and (14), we have

$$\begin{aligned}&2\int_{-\infty}^{q(v(t))} m(s)|x(t+s)|ds \\ &\qquad \leqq 2\,\|x_t\|_{C_{\infty}^{\gamma}} \int_{-\infty}^{q(v(t))} m(s)e^{-\gamma s}ds \\ &\qquad \leqq 2\int_{-\infty}^{q(v(t))} m(s)e^{-\gamma s}ds \\ &\qquad \leqq \delta|x(t)| \quad \text{for} \quad \|x_t\|_{C_{\infty}^{\gamma}} \leqq 1 ,\end{aligned}$$

while

$$\begin{aligned}2\int_{q(v(t))}^{0} m(s)|x(t+s)|ds &\leqq 2F^{1/2}\int_{q(v(t))}^{0} m(s)|x(t)|ds \\ &\leqq 2F^{1/2}|x(t)|\int_{-\infty}^{0} m(s)ds\end{aligned}$$

whenever $v(s) \leqq Fv(t)$ for $s \in [t + q(v(t)), t]$. Then we see that

$$\begin{aligned}\dot{v}_{(13)}(t, \phi) &\leqq -2(a - |b|)\phi(0)^2 + \delta\phi(0)^2 \\ &\quad + 2F^{1/2}\phi(0)^2\int_{-\infty}^{0} m(s)ds \\ &= -\delta\phi(0)^2 = -\delta v(t, \phi) ,\end{aligned}$$

whenever $\|\phi\|_{C^{\gamma}_{\infty}} \leqq 1$, $t + q(v(t, \phi)) \geqq \tau$ and

$$v(s, \phi_{s-t}) \leqq Fv(t, \phi) \quad \text{for} \quad s \in [t + q(v(t, \phi)), t]\,.$$

Namely, the conditions in Theorem 3 are satisfied. Thus, the zero solution of (11) is uniformly asymptotically stable in $(C^{\gamma}_{\infty}, R^1)$.

The author would like to thank Professor J. Kato for his assistance.

References

[1] J. Kato, Liapunov's second method in functional differential equations, Tôhoku Math. J. 32 (1980), 487-497.

[2] J. Kato, Stability in functional differential equations, Lecture Notes in Math. 799, Springer-Verlag, Berlin-Heidelberg-New York, 1980, 252-262.

[3] J. Kato, Stability problem in functional differential equations with infinite delay, Funkcial. Ekvac. 21 (1978), 63-80.

[4] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978), 11-41.

[5] W. Zhicheng and Q. Xiangzheng, The method of Liapunov functionals for functional differential equations, J. of Hunan University, 6(3) (1979), 15-24.

[6] C. Qiang, A method of comparative decision of stability of functional differential equations, J. of Hunan University, 6(4) (1979), 14-21.

[7] R. D. Driver, Existence and stability of solutions of a delay-differential system, Arch. Rat. Mech. Anal. 10 (1962), 401-426.

Department of Mathematics
Hunan University
Changsha, Hunan 1801
People's Republic of China