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# LOWER BOUNDS FOR THE EIGENVALUES OF THE FIXED VIBRATING MEMBRANE PROBLEMS

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**1. Introduction.** Let $\Omega$ be a bounded domain of the Euclidean space $\boldsymbol{R}^n$ with appropriately regular boundary $\partial\Omega$. We consider the classical fixed vibrating membrane problem:

$$\Delta u = \lambda u \text{ on } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega .$$

Here $\Delta$ is the standard Laplacian $-\sum_{i=1}^{n}\partial^2/\partial(x_i)^2$ of the Euclidean space $\boldsymbol{R}^n$. Let $\{\lambda_1 < \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \cdots \uparrow \infty\}$ be the eigenvalues of this problem counted with their multiplicities.

G. Pólya conjectured (cf. [8])

$$\lambda_k \geqq C_n \operatorname{Vol}(\Omega)^{-2/n} k^{2/n} \quad \text{for every} \quad k , \tag{1.1}$$

which was proved by him in case of *space-covering domains* $\Omega$. That is, an infinity of domains congruent to $\Omega$ cover the whole space $\boldsymbol{R}^n$ without gaps and without overlapping except a set of measure zero. Here the positive constant $C_n$ is $4\pi^2\omega_n^{-2/n}$, $\omega_n = \pi^{n/2}/\Gamma((n/2)+1)$ is the volume of the unit ball and $\operatorname{Vol}(\Omega)$ is the volume of $\Omega$. The conjecture of Pólya is closely related to H. Weyl's asymptotic formula (cf. [10])

$$\lambda_k \sim C_n \operatorname{Vol}(\Omega)^{-2/n} k^{2/n} \quad \text{as} \quad k \to \infty , \tag{1.2}$$

which shows the sharpness of Pólya's bounds for higher eigenvalues.

E. H. Lieb [5] has showed that (1.1) is true when $C_n$ is replaced by a smaller constant $D_n^{-2/n}$ where $D_3^{-2/3} = C_3 \times 0.2773$ and $D_3 = 0.1156$. Recently S. Y. Cheng and P. Li (cf. [11, p. 22]) showed

$$\lambda_k \geqq A_n \operatorname{Vol}(\Omega)^{-2/n} k^{2/n} \quad \text{for every} \quad k , \tag{1.3}$$

which is valid for general compact riemannian manifold with smooth boundary. Here the constant $A_n$ is $2c\,n^{-1}e^{-2/n}$, $c = c'^2\ ((n-2)/(2n-2))^2$ and $c'$ is the Sobolev constant $n\omega_n^{1/n}$ which satisfies the inequality $\operatorname{Vol}(\partial\Omega)^n \geqq c'^n \operatorname{Vol}(\Omega)^{n-1}$. It should be noted that the constant $A_n$ is asymptotically $e^2 2^{-1} n^{-1}$ as $n \to \infty$.

In this paper, we show the following:

THEOREM 1. *For every eigenvalue $\lambda_k$ of the fixed vibrating membrane* *problem for a bounded domain* $\Omega$ *in the Euclidean space* $\boldsymbol{R}^n$, *we have*

$$\lambda_k \geqq C_n \operatorname{Vol}(\Omega)^{-2/n} k^{2/n} \delta_L(\Omega)^{2/n} , \tag{1.4}$$

*where the constant* $\delta_L(\Omega)$ *is the lattice packing density of* $\Omega$ (cf. [9, p. 22] *or* §2).

Here we note some remarks for the constant $\delta_L(\Omega)$ of the inequality (1.4).

REMARK 1. For space-covering domains $\Omega$, $\delta_L(\Omega) = 1$. Theorem 1 can be regarded as a natural generalization of Pólya's result.

REMARK 2. For convex bounded domains $\Omega$ in $\boldsymbol{R}^n$, it is known (cf. [9, p. 10]) that

$$\delta_L(\Omega) \geqq 2(n!)^2/(2n)! . \tag{1.5}$$

In particular, when $n = 2$,

$$\delta_L(\Omega) \geqq 3/4 = 0.75 \quad \text{(cf. [12])} . \tag{1.6}$$

Since the right hand side of (1.5) is asymptotically $2(\pi n)^{1/2}4^{-n}$ (cf. [9, p. 10]) as $n \to \infty$, we have

$$\delta_L(\Omega)^{2/n} \geqq (2(n!)^2/(2n)!)^{2/n} \sim 1/16 = 0.0625 \quad \text{as} \quad n \to \infty , \tag{1.7}$$

which shows the sharpness of (1.4) for large $n$.

REMARK 3. For a symmetrical (i.e., $-x \in \Omega$ whenever $x \in \Omega$) convex bounded domain $\Omega$,

$$\delta_L(\Omega) \geqq \zeta(n)/2^{n-1} , \qquad \zeta(n) = \sum_{k=1}^{\infty} k^{-n} . \tag{1.8}$$

When $n = 3$, for all symmetrical convex bounded domains $\Omega$ in $\boldsymbol{R}^3$,

$$\delta_L(\Omega)^{2/3} \geqq 0.4486 , \tag{1.9}$$

which is sharper than the constant of Lieb in this case.

## 2. Lattice packing of bounded domain.

**2.1.** Following Rogers [9], we explain the lattice packing density $\delta_L(\Omega)$ for a bounded domain $\Omega$ in the Euclidean space $\boldsymbol{R}^n$. If $\{a_1, \cdots, a_n\}$ is a basis of $\boldsymbol{R}^n$, the set $\Lambda = \Lambda(a_1, \cdots, a_n)$ of all vectors of the form $\sum_{i=1}^n m_i a_i (m_i \in \boldsymbol{Z},\ i = 1, \cdots, n)$ is called a lattice. Let $\{a_1, a_2, \cdots, a_n, a_{n+1}, \cdots\}$ be an enumeration of the points of $\Lambda$. A system $Z = Z_{\Lambda,\Omega}$ consisting of the translates $\Omega + a_i = \{x + a_i;\ x \in \Omega\}$ of a given bounded domain $\Omega$ is called a *lattice packing* of $\Omega$ with lattice $\Delta$ when $\Omega + a_i \cap \Omega + a_j = \varnothing$ $(i \neq j)$. For such a lattice packing $Z$, put

$$\rho(Z, C) = \mathrm{Vol}(C)^{-1} \sum_{(\Omega+a_i)\cap C \neq \emptyset} \mathrm{Vol}(\Omega + a_i) ,$$

where $C$ is a cube in $\boldsymbol{R}^n$ with the edge length $s(C)$. Define

$$\rho(Z) = \limsup_{s(C)\to\infty} \rho(Z, C) \leqq 1 .$$

The *lattice packing density* $\delta_L(\Omega)$ (cf. [9, p. 24]) of $\Omega$ is defined by

$$\delta_L(\Omega) = \sup_Z \rho(Z) ,$$

the supremum being taken over all lattice packings $Z$ of the set $\Omega$.

**2.2.** Translating a bounded domain $\Omega$ in $\boldsymbol{R}^n$, we may assume the origin $o$ of $\boldsymbol{R}^n$ belongs to $\Omega$. For a small positive constant $h$, put $\Omega_h = \{hx; x \in \Omega\}$. Then

$$\mathrm{Vol}(\Omega_h) = h^n \mathrm{Vol}(\Omega) . \tag{2.1}$$

Let $\boldsymbol{K}$ be the open unit cube $\{x \in \boldsymbol{R}^n; |x_i| < 1/2 \ (i = 1, \cdots, n\}$ in $\boldsymbol{R}^n$. For a lattice packing $Z_{\Lambda,h}$ of $\Omega_h$ with lattice $\Lambda = \Lambda(a_1, \cdots, a_n)$, let $\Omega(h, \Lambda)$ be the union of $\Omega_h + a_i$ $(i = 1, 2, \cdots)$ which are included in $\boldsymbol{K}$ (see Figure 1).

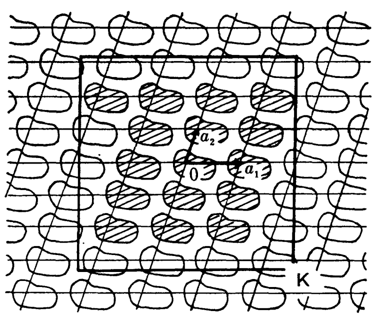


FIGURE 1. Lattice packing $Z_{\Lambda,h}$ of $\Omega_h$ and $\Omega(h, \Lambda)$.

Let $m(h, \Lambda)$ be the number of $\Omega_h + a_i$ $(i = 1, 2, \cdots)$ being included in $\boldsymbol{K}$. For a small positive number $h$, define

$$m(h) = \sup_{Z_{\Lambda,h}} m(h, \Lambda) ,$$

where the supremum is taken over all lattice packings $Z_{\Lambda,h}$ of $\Omega_h$. Then it is clear that

$$\lim_{h\to 0} m(h) = \infty . \tag{2.2}$$

Moreover we have:

$$\limsup_{h\to 0} \sup_{Z_{\Lambda,h}} \mathrm{Vol}(\Omega(h, \Lambda)) \geqq \delta_L(\Omega) , \tag{2.3}$$

where the supremum is taken over all lattice packings $Z_{\Lambda,h}$ of $\Omega_h$.

REMARK. It seems that the above inequality is in fact the equality.

PROOF OF (2.3). By (2.1), the left hand side of (2.3) coincides with

$$\lim_{h\to 0}\sup \sup_{Z} \operatorname{Vol}\left(\frac{1}{h}\boldsymbol{K}\right)^{-1} \sum_{\Omega+a_i\subset(1/h)\boldsymbol{K}} \operatorname{Vol}(\Omega + a_i)\,,$$

where $Z$ varies over all lattice packings of $\Omega$ and $(1/h)\boldsymbol{K} = \{(1/h)x;\ x \in \boldsymbol{K}\}$. Then we have

$$\sup_{Z} \operatorname{Vol}\left(\frac{1}{h}\boldsymbol{K}\right)^{-1} \sum_{\Omega+a_i\subset(1/h)\boldsymbol{K}} \operatorname{Vol}(\Omega + a_i) \geqq \operatorname{Vol}\left(\frac{1}{h}\boldsymbol{K}\right)^{-1} \sum_{\Omega+b_i\subset(1/h)\boldsymbol{K}} \operatorname{Vol}(\Omega + b_i)$$
$$\geqq \operatorname{Vol}\left(\frac{1}{h}\boldsymbol{K}\right)^{-1} \sum_{\Omega+b_i\cap(1/h)\boldsymbol{K}\neq\emptyset} \operatorname{Vol}(\Omega + b_i) - h^n\left\{2ns(\Omega)\left(\frac{1}{h} + 2s(\Omega)\right)^{n-1}\right\}\,,$$

for any lattice packing $Z_{\Lambda'}$ of $\Omega$ with lattice $\Lambda' = \Lambda(b_1, \cdots, b_n)$. Here $s(\Omega)$ is the length of the edge of any fixed cube including $\Omega$. Therefore the left hand side of (2.3) is not less than

$$\lim_{h\to 0}\sup \operatorname{Vol}\left(\frac{1}{h}\boldsymbol{K}\right)^{-1} \sum_{\Omega+b_i\cap(1/h)\boldsymbol{K}\neq\emptyset} \operatorname{Vol}(\Omega + b_i) = \rho(Z_{\Lambda'})\,,$$

for any lattice packing $Z_{\Lambda'}$ of $\Omega$ with lattice $\Lambda' = \Lambda(b_1, \cdots, b_n)$. Thus we have (2.3).

Combining (2.1) and (2.3), we have immediately

$$\lim_{h\to 0} m(h)h^n \geqq \delta_L(\Omega)\operatorname{Vol}(\Omega)^{-1}\,. \tag{2.4}$$

**3. Proof of Theorem 1.** Let $\Omega$ be any bounded domain in $\boldsymbol{R}^n$. We preserve the notations and situations in §2.

For the $k$-th eigenvalue $\lambda_k(\Omega)$ of the fixed vibrating membrane problem for $\Omega$, it is well-known that

$$\lim_{k\to\infty} \lambda_k(\boldsymbol{K})k^{-2/n} = C_n\,, \quad \text{and} \tag{3.1}$$

$$\lambda_k(\Omega_h) = h^{-2}\lambda_k(\Omega)\,, \quad k = 1, 2, \cdots\,, \tag{3.2}$$

for every positive number $h$. Moreover for every lattice packing $Z_{\Lambda,h}$ of $\Omega_h$ with lattice $\Lambda$, we have

$$\lambda_{km(h,\Lambda)}(\boldsymbol{K}) \leqq \lambda_k(\Omega_h) \quad \text{for every} \quad k = 1, 2, \cdots\,, \tag{3.3}$$

because of the inequalities

$$\lambda_{km(h,\Lambda)}(\boldsymbol{K}) \leqq \lambda_{km(h,\Lambda)}(\Omega(h, \Lambda)) \leqq \lambda_k(\Omega_h)$$

by [3, p. 408, Theorem 2]. Therefore we have

(3.3′) $$\lambda_{km(h)}(\boldsymbol{K}) \leqq \lambda_k(\Omega_h) .$$

Then we obtain

$$\begin{aligned}\lambda_k(\Omega) &= h^2\lambda_k(\Omega_h) \quad (\text{by } (3.2)) \\ &\geqq h^2\lambda_{km(h)}(\boldsymbol{K}) \quad (\text{by } (3.3')) \\ &= \lambda_{km(h)}(\boldsymbol{K})(km(h))^{-2/n}(km(h))^{2/n}h^2\end{aligned}$$

for all $k = 1, 2, \cdots$ and $h > 0$. Letting $h \to 0$ on the right hand side of the above inequality, we have

$$\begin{aligned}&\lim_{h\to 0} \lambda_{km(h)}(\boldsymbol{K})(km(h))^{-2/n}(km(h))^{2/n}h^2 \\ &\quad = \left\{\lim_{h\to 0} \lambda_{km(h)}(\boldsymbol{K})(km(h))^{-2/n}\right\}\left\{\lim_{h\to 0} m(h)^{2/n}h^2\right\}k^{2/n} \\ &\quad \geqq C_n \operatorname{Vol}(\Omega)^{-2/n}\delta_L(\Omega)^{2/n}k^{2/n}\end{aligned}$$

by (2.2), (3.1) and (2.4). Thus we have Theorem 1.

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