Tôhoku Math. Journ.
36 (1984), 159-174.

ON TOPOLOGICAL BLASCHKE CONJECTURE II

NONEXISTENCE OF BLASCHKE STRUCTURE ON FAKE QUATERNION PROJECTIVE PLANES

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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(Received July 9, 1982)

Let (M, g) be a compact Riemannian manifold. We say that (M, g) is a Blaschke manifold if, for each point $m \in M$, the tangential cut locus is a sphere of a constant radius (see Besse [2] for details).

In a previous paper [5], one of the authors has shown that a Blaschke manifold whose integral cohomology ring is equal to that of the complex projective space $\boldsymbol{C}P^n$ is homeomorphic to $\boldsymbol{C}P^n$ for $n > 0$.

Let $\boldsymbol{H}P^2$ denote the quaternion projective plane, and let N be a closed simply connected smooth manifold with the same integral cohomology ring as that of $\boldsymbol{H}P^2$. We say that N is a fake quaternion projective plane if N is not homeomorphic to $\boldsymbol{H}P^2$. It was proved by Eells-Kuiper [3] and Tamura [9] that there are infinitely many non-homeomorphic fake quaternion projective planes.

In this paper, we prove the following theorem.

THEOREM. *Let (M, g) be a Blaschke manifold whose integral cohomology ring is equal to that of $\boldsymbol{H}P^2$. Then M is* PL*-homeomorphic to $\boldsymbol{H}P^2$. Consequently, there are no Blaschke structures on fake quaternion projective planes.*

For the proof, we show that the first Pontrjagin class of M is equal to that of $\boldsymbol{H}P^2$. Then by a result of differential topology (Proposition 17), we know that M is PL-homeomorphic to $\boldsymbol{H}P^2$. For this purpose we study the homotopy class of a map from the cut locus of a point in M to a Grassman manifold, which is naturally associated to the sphere bundle in Proposition 1 (see, §1 below).

After the first draft of this paper was written, the mimeographed notes of Gluck-Warner-Yang came to the authors' attention. They treat the same problem by a somewhat different method.

1. Allamigeon's results. In this section, we recall results of

Allamigeon, and state some other related results which we use in later sections.

Let S^{n-1} denote the unit sphere in the Euclidean space $\boldsymbol{R}^n$. By a great sphere, we mean the intersection of a vector subspace of $\boldsymbol{R}^n$ with S^{n-1}.

Let (M, g) be a Blaschke manifold of dimension n. For a point $m \in M$, we denote by C_m the cut locus of m. By a fibration, we mean a smooth fiber bundle. The following results are due to Allamigeon [1]. See also Besse [2].

PROPOSITION 1. *The cut locus C_m has a natural structure of a smooth manifold, and is diffeomorphic to the base space of a fibration of S^{n-1} such that each fiber is a great sphere with structure group in the orthogonal group.*

Let $p: S^{n-1} \to B$ be a fibration by great spheres, with structure group in the orthogonal group, and let $k-1$ be the dimension of the fiber so that the dimension of B is equal to $n-k$. Let $V(n, k)$ and $G(n, k)$, respectively, be the Stiefel and the Grassmann manifolds consisting of k-frames and k-planes in $\boldsymbol{R}^n$, respectively. The natural projection $q: V(n, k) \to G(n, k)$ defines a principal $O(k)$-bundle. By taking the k-plane determined by the fiber, we obtain a continuous map $g: B \to G(n, k)$. Let $F(n, k)$ be the S^{k-1}-bundle associated with q. Then $F(n, k)$ is equal to the set of points (P, x) in $G(n, k) \times S^{n-1}$ such that x is contained in the k-plane P. Let $q_1: F(n, k) \to G(n, k)$ and $q_2: F(n, k) \to S^{n-1}$ be the projections to the first and the second factor. Then the bundle $p: S^{n-1} \to B$ is naturally isomorphic to the induced bundle $g^*(F(n, k))$. The bundle map $\tilde{g}: S^{n-1} \to F(n, k)$ covering g is given by $\tilde{g}(x) = (gp(x), x)$ for $x \in S^{n-1}$. We have the following commutative diagram:

$$\begin{array}{ccccc} & & G(n,k) \times S^{n-1} & & \\ & & \cup & \searrow^{q_2} & \\ S^{n-1} & \xrightarrow{\tilde{g}} & F(n,k) & \xrightarrow{q_2} & S^{n-1} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q_1} & & \\ B & \xrightarrow{g} & G(n,k) & & \end{array}$$

The following is a direct consequence of the above diagram.

LEMMA 2. *The composition $q_2 \cdot \tilde{g}$ is equal to the identity map.*

Since $\tilde{g}$ is a bundle map and is an embedding, the smooth map g is of maximal rank. Since S^{n-1} is foliated by great spheres, g is injective.

Consequently we obtain the following.

PROPOSITION 3. *The map* $g: B \to G(n, k)$ *is a smooth embedding satisfying the following property*:

$$(*) \qquad g(b) \cap g(b') = \{0\} \quad \text{for any } b \neq b' \text{ in } B .$$

Conversely the following holds.

PROPOSITION 4. *Let* B *be a closed connected smooth manifold of dimension* $n-k$, *and let* $g: B \to G(n, k)$ *be an embedding satisfying the condition* $(*)$. *Then* B *is the quotient space of a foliation of* S^{n-1} *by great spheres.*

PROOF. Consider the subset $A = q_1^{-1}(g(B))$ in $F(n, k)$. Then A is a compact connected smooth manifold of dimension $n-1$. By the condition $(*)$, $q_2|A$ is a homeomorphism into S^{n-1}. By the invariance theorem of domains, $q_2|A$ is surjective. Thus we obtain a foliation of S^{n-1} by great $(k-1)$-spheres.

Let E be the disc bundle associated with the sphere bundle $p: S^{n-1} \to B$ which is obtained from (M, g) by Proposition 1. Then S^{n-1} is the boundary of the manifold E. We have the following (cf. [2, Theorem 5.43]).

PROPOSITION 5. *A Blaschke manifold* M *is diffeomorphic to the union of the unit disc* D *and the disc bundle* E *glued by a diffeomorphism along their boundaries.*

2. Fibrations of S^7 by great 3-spheres. Let $p: S^7 \to B$ be a fibration of S^7 with structure group in $O(4)$ such that each fiber is a great 3-sphere. Then B is homotopy equivalent to S^4. As is stated in §1, we obtain the map $g: B \to G(8, 4)$.

Let $\mathrm{BO}(4)$ denote the classifying space of $O(4)$. Note that, since $\mathrm{BO}(4) = \lim_{N\to\infty} G(N, 4)$, we have the natural inclusion of $G(8, 4)$ in $\mathrm{BO}(4)$. To know the isomorphism class of the bundle $p: S^7 \to B$, we study the homotopy class $\{g\}$ of g.

Let x_0 denote the 4-plane in $\boldsymbol{R}^8$ defined by the natural embedding of $\boldsymbol{R}^4$; $\boldsymbol{R}^4 = \boldsymbol{R}^4 \times \{0\} \subset \boldsymbol{R}^4 \times \boldsymbol{R}^4 = \boldsymbol{R}^8$. We can regard x_0 as an element in $G(8, 4)$. Define K to be the closed subset in $X = G(8, 4)$ consisting of 4-planes which fail to be transverse to x_0. Suppose that there exists a point b_0 in B such that $g(b_0) = x_0$. Since g satisfies the condition $(*)$ of §1, we have the relation

$$g(B - b_0) \subset X - K .$$

The 16-dimensional manifold X has the canonical cell decomposition by the Schubert cells (see e.g., [4]).

By the definition of Schubert cells, we easily obtain the following.

PROPOSITION 6. *The subset K is equal to the union of open Schubert cells of dimension smaller than* 16.

PROOF. A 4-plane x in $\boldsymbol{R}^8$ transverse to x_0 if and only if $\dim(x \cap x_0) = 0$. Thus the Schubert symbol of an open cell containing x is equal to (5, 6, 7, 8) and the dimension of the cell is 16. But we have only one cell of dimension 16 in X.

We have an obvious corollary.

COROLLARY 7. *The open manifold $X - K$ is contractible.*

Let D^4 be a closed 4-disc in B centered at the base point b_0.

PROPOSITION 8. *Let g_0 and g_1 be two embeddings of B in X which satisfy the following*:

(1) $g_i(b_0) = x_0$, $g_i(B - b_0) \subset X - K$ *for* $i = 0, 1$.

(2) $g_0 | D^4$ *and* $g_1 | D^4$ *are homotopic by a homotopy* $H = \{h_t (0 \leqq t \leqq 1)\}$: $D^4 \times I \to X$ *with* $h_0 = g_0 | D^4$ *and* $h_1 = g_1 | D^4$ *such that*

$$H(\{b_0\} \times I) = x_0$$
$$H((D^4 - \{b_0\}) \times I) \subset X - K .$$

Then the homotopy classes $\{g_0\}$ and $\{g_1\}$ in $\pi_4(X) = \pi_4(X, x_0)$ are equal.

PROOF. Denote by $-(B - \operatorname{Int} D^4)$ the manifold $B - \operatorname{Int} D^4$ with the orientation reversed. Let DB denote the union

$$-(B - \operatorname{Int} D^4) \cup (\partial D^4) \times I \cup (B - \operatorname{Int} D^4) ,$$

with boundaries glued by the identity maps. We have the map

$$\begin{aligned} h \equiv (g_0 | -(B - \operatorname{Int} D^4) \cup (H | (\partial D^4) \times I) \cup (g_1 | (B - \operatorname{Int} D^4)) \\ : DB \to X . \end{aligned}$$

Then $h(DB) \subset X - K$. Note that DB is homotopy equivalent to S^4. By Corollary 7, the map h is homotopic to the constant map. Thus we can construct a homotopy connecting g_0 and g_1 keeping b_0 fixed.

In the following, the condition in Proposition 8 on $g_0 \circ e$ and $g_1 \circ e$ is simply stated as follows: The embeddings $g_0 \circ e$ and $g_1 \circ e$ are homotopic in $X - K$ keeping the center fixed.

3. Transition function. Let $M(n, k)$ denote the set of real $(n \times k)$-matrices. We write simply $M(n)$ for $M(n, n)$. Let $\mathrm{GL}(n)$ denote the group of real non-singular $(n \times n)$-matrices. For $k < n$, define a subgroup $P(n, k)$ of $\mathrm{GL}(n)$ by

$$P(n, k) = \left\{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix};\ A \in \mathrm{GL}\,(k),\ C \in \mathrm{GL}\,(n-k),\ B \in M(k, n-k)\right\}.$$

Let $\mathfrak{gl}(n)$ be the Lie algebra of $\mathrm{GL}\,(n)$. The Lie algebra $\mathfrak{p}(n, k)$ of $P(n, k)$ is given by

$$\mathfrak{p}(n, k) = \left\{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix};\ A \in M(k),\ C \in M(n-k),\ B \in M(k, n-k)\right\}.$$

The quotient space $\mathrm{GL}\,(n)/P(n, k)$ is diffeomorphic to $G(n, k)$.

Define a 16-dimensional vector subspace $\mathfrak{m}$ of $\mathfrak{gl}(8)$ by

$$\mathfrak{m} = \left\{\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix};\ A \in M(4)\right\}.$$

Then we have the vector space direct sum decomposition

$$\mathfrak{gl}(8) = \mathfrak{p}(8, 4) + \mathfrak{m}.$$

We often identify $\mathfrak{m}$ with $M(4)$, and $\mathfrak{m}$ is naturally identified with the tangent space $T_{x_0}X$, $X = G(8, 4)$.

Define the map

$$\mathrm{Exp}\colon \mathfrak{m} \to G(8, 4) = \mathrm{GL}\,(8)/P(8, 4)$$

by $\mathrm{Exp}\,(A) = \{\exp\,(A)\}$, the class in $\mathrm{GL}\,(8, 4)/P(8, 4)$ represented by $\exp\,(A)$, where $A \in \mathfrak{m}$ and $\exp\colon \mathfrak{gl}(8) \to \mathrm{GL}\,(8)$ is the exponential mapping of the Lie group $\mathrm{GL}\,(8)$. We express a 4-frame in $\boldsymbol{R}^8$ by an (8×4)-matrice. Then $\mathrm{Exp}\,(A)$ is represented by the (8×4)-matrix

$$\begin{pmatrix} I_4 \\ A \end{pmatrix},$$

where I_4 is the identity matrix of $\mathrm{GL}\,(4)$.

Let $x_0^\perp \in G(8, 4)$ denote the 4-plane orthogonal to the base point $x_0 \in G(8, 4)$. Then the map Exp is a diffeomorphism such that the image $\mathrm{Exp}\,(\mathfrak{m})$ is equal to the set of 4-planes transverse to $x_0^\perp$, for $x_0^\perp$ is represented by

$$\begin{pmatrix} 0 \\ I_4 \end{pmatrix}.$$

Let us define a subset K' of $\mathfrak{m} = M(4)$ by $K' = \mathrm{Exp}^{-1}\,(K)$.

LEMMA 9. *The space K' is equal to the set of matrices $A \in M(4)$ such that* $\det A = 0$. *Consequently, K' is a linear cone centered at* 0.

PROOF. An element $A \in \mathfrak{m}$ belongs to K' if and only if $\mathrm{Exp}\,(A)$ is not transverse to x_0. But the 4-plane $\mathrm{Exp}\,(A)$ is transverse to x_0 if and only if $\det A \neq 0$.

We have the Stiefel manifold $V(8, 4)$ and the GL (4)-principal bundle $q\colon V(8, 4) \to G(8, 4) = X$. An element in $V(8, 4)$ is represented by an (8×4)-matrix of rank 4. We give a trivialization h of the bundle q restricted over $\operatorname{Exp}(\mathfrak{m})$,

$$h\colon \operatorname{Exp}(\mathfrak{m}) \times \mathrm{GL}(4) \to q^{-1}(\operatorname{Exp}(\mathfrak{m}))$$

by

$$h(a, g) = \begin{pmatrix} g \\ Ag \end{pmatrix}, \quad \text{where} \quad a = \operatorname{Exp} A,\ A \in \mathfrak{m} .$$

Similarly we define the map $\operatorname{Exp}^{\perp}\colon \mathfrak{m} \to G(8, 4)$ by

$$\operatorname{Exp}^{\perp}(A) = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \exp(A) \quad \text{in} \quad \mathrm{GL}(8)/P(8, 4) .$$

Then $\operatorname{Exp}^{\perp}(0) = x_0^{\perp}$ and $\operatorname{Exp}(A)$ is represented by

$$\begin{pmatrix} A \\ I_4 \end{pmatrix} .$$

The image $\operatorname{Exp}^{\perp}(\mathfrak{m})$ is nothing but the Schubert top cell of $G(8, 4)$. As before we have a trivialization $h^{\perp}$ of the bundle q over $\operatorname{Exp}^{\perp}(\mathfrak{m})$,

$$h^{\perp}\colon \operatorname{Exp}(\mathfrak{m}) \times \mathrm{GL}(4) \to q^{-1}(\operatorname{Exp}^{\perp}(\mathfrak{m}))$$

defined by

$$h^{\perp}(a, g) = \begin{pmatrix} Ag \\ g \end{pmatrix}, \quad \text{where} \quad a = \operatorname{Exp}^{\perp}(A),\ A \in \mathfrak{m} .$$

Note that By Lemma 9, the intersection $\operatorname{Exp}(\mathfrak{m}) \cap \operatorname{Exp}^{\perp}(\mathfrak{m})$ is equal to the set $\operatorname{Exp}(\mathrm{GL}(4))$.

PROPOSITION 10. *On* $\operatorname{Exp}(\mathfrak{m}) \cap \operatorname{Exp}^{\perp}(\mathfrak{m}) = \operatorname{Exp}(\mathrm{GL}(4))$, *the transition function* $k\colon \operatorname{Exp}(\mathrm{GL}(4)) \to \mathrm{GL}(4)$ *of two trivializations* h *and* $h^{\perp}$ *is given by*

$$k(\operatorname{Exp}(A)) = A , \qquad \textit{for} \quad A \in \mathrm{GL}(4) ,$$

that is, $h(a, g) = h^{\perp}(a, Ag)$, *for* $(a, g) \in \operatorname{Exp}(\mathfrak{m}) \times \mathrm{GL}(4)$.

PROOF. If $a = \operatorname{Exp}(A)$ for $A \in \mathrm{GL}(4)$, then $a = \operatorname{Exp}^{\perp}(A^{-1})$. Solving the equation

$$\begin{pmatrix} I_4 \\ A \end{pmatrix} = \begin{pmatrix} A^{-1}h \\ h \end{pmatrix},$$

we obtain that $h = A$ at a.

4. Non-singularity of the matrices in $g_*(T_{b_0}B)$. Let B be the base manifold of the fibration of S^7 by great 3-spheres and $g: B \to G(8, 4)$ be the smooth embedding of §2. By homogeneity of $G(8, 4)$, we may suppose that $g(b_0) = x_0$, where b_0 is a base point of B.

Identify $T_0\mathfrak{m}$ with $\mathfrak{m}$ itself and identify $T_{x_0}G(8, 4)$ with $\mathfrak{m} = M(4)$ by $(\mathrm{Exp})_*$, the differential of the map Exp at x_0.

PROPOSITION 11. *For any non-zero vector X in the tangent space $T_{b_0}B$, $g_*(X) \in M(4)$ is a non-singular matrix.*

Before the proof, we study the tangent spaces at the base points of the manifolds $F(8, 4)$ and S^7 treated in §1. We define a subgroup $P^+(n, n-1)$ in $\mathrm{GL}\,(n)$ by

$$P^+(n, n-1) = \left\{\begin{pmatrix} a & b \\ 0 & C \end{pmatrix} ;\ 0 < a \in R,\ b \in M(1, n-1),\ C \in \mathrm{GL}\,(n-1)\right\} ,$$

and define H in $\mathrm{GL}\,(8)$ by

$$H = \left\{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} ;\ A \in P^+(4, 3),\ B \in M(4),\ C \in \mathrm{GL}\,(4)\right\} .$$

Then $F(8, 4)$ and S^7 are diffeomorphic to homogeneous spaces $\mathrm{GL}\,(8)/H$ and $\mathrm{GL}\,(8)/P^+(8, 7)$, respectively.

Let $\mathfrak{m}^F$ and $\mathfrak{m}^S$ be vector subspaces of $\mathfrak{gl}(8)$ defined by

$$\mathfrak{m}^F = \left\{\begin{pmatrix} 0 & 0 & \\ a & 0 & 0 \\ & B & 0 \end{pmatrix} ;\ a \in M(3, 1),\ B \in M(4, 4)\right\} ,$$

$$\mathfrak{m}^S = \left\{\begin{pmatrix} 0 & 0 & \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} ;\ a \in M(3, 1),\ b \in M(4, 1)\right\} .$$

Then the exponential mapping exp of $\mathrm{GL}\,(8)$ defines the smooth maps

$$\mathrm{Exp}^F: \mathfrak{m}^F \to F(8, 4)$$
$$\mathrm{Exp}^S: \mathfrak{m}^S \to S^7 .$$

We denote the base point of $F(8, 4)$ and S^7 by the same letter 0. Then

$$(\mathrm{Exp}^F)_*: \mathfrak{m}^F \to T_0F(8, 4)$$
$$(\mathrm{Exp}^S)_*: \mathfrak{m}^S \to T_0S^7$$

are isomorphisms. We identify $T_{x_0}G(8, 4)$, $T_0F(8, 4)$ and T_0S^7 with $\mathfrak{m}$, $\mathfrak{m}^F$

and $\mathfrak{m}^S$ by $(\mathrm{Exp})_*$, $(\mathrm{Exp}^F)_*$, and $(\mathrm{Exp}^S)_*$, respectively. Note that the differential $(q_2)_*$: $T_0F(8, 4) \to T_0S^7$ is equal to the natural projection of $\mathfrak{m}^F$ onto $\mathfrak{m}^S$.

PROOF OF PROPOSITION 11. Put $V = g_*(X) \in M(4)$. There exist P_1 and P_2 in $O(4)$ such that $P_2VP_1^{-1}$ is the diagonal matrix

$$\begin{pmatrix} v_1 & & & \\ & v_2 & & \\ & & v_3 & \\ & & & v_4 \end{pmatrix}$$

with $0 \leqq |v_1| \leqq |v_2| \leqq |v_3| \leqq |v_4|$. Let $Q \in \mathrm{GL}\,(8)$ be the matrix defined by

$$Q = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

Denote by Q^S, Q^G and Q^F the left multiplication of Q on S^7, $G(8, 4)$ and $F(8, 4)$, respectively. Then Q^G fixes x_0 in $G(8, 4)$. The differential $(Q^G)_*$: $\mathfrak{m} \to \mathfrak{m}$ is given by $(Q^G)_*(A) = P_2AP_1^{-1}$. Consequently, we have

$$(Q^G)_*(V) = \begin{pmatrix} v_1 & & & \\ & v_2 & & \\ & & v_3 & \\ & & & v_4 \end{pmatrix}.$$

Let $i\colon \mathfrak{m} \hookrightarrow \mathfrak{m}^F$ be the natural inclusion. Let y be the point in S^7 defined by $y = \tilde{g}^{-1}(Q^F)^{-1}(0)$. Take $Y \in T_y(S^7)$ such that $X = p_*(Y)$. Then $(Q^F)_*\tilde{g}_*(Y) \in T_0F(8, 4)$. Since $q_1 \circ Q^F = Q^G \circ q_1$,

$$\begin{aligned}(q_1)_*(Q^F)_*\tilde{g}_*(Y) &= (Q^G)_*(q_1)_*\tilde{g}_*(Y) \\ &= (Q^G)_*(V) = (q_1)_*i((Q^G)_*(V)) .\end{aligned}$$

Consequently, the difference $(Q^F)_*\tilde{g}_*(Y) - i((Q^G)_*(V))$ is a vector tangent to the fiber $q_1^{-1}(x_0)$. Since $q_1^{-1}(x_0) = \tilde{g}(p^{-1}(b_0))$, there exists $Z \in T_y(S^7)$ with $g_*p_*(Z) = V$ such that $(Q^F)_*\tilde{g}_*(Z) = i((Q^G)_*(V))$. We can naturally identify $\mathfrak{m}^S$ with the set of column 7-vectors. We express an element in $\mathfrak{m}^S$ by the transpose of a row 7-vector. We have

$$(q_2)_*(Q^F)_*\tilde{g}_*(Z) = (q_2)_*i((Q^G)_*(V)) = {}^t(0, 0, 0, v_1, 0, 0, 0) .$$

Note that $q_2 \circ Q^F = Q^S \circ q_2$ and $q_2 \circ \tilde{g} =$ identity. Thus

$${}^t(0, 0, 0, v_1, 0, 0, 0) = (Q^S)_*(q_2)_*\tilde{g}_*(Z) = (Q^S)_*(Z) .$$

Since g_* is injective, $V \neq 0$ and $Z \neq 0$. Consequently, it follows that

$v_1 \neq 0$. Then $v_i \neq 0$ for $1 \leqq i \leqq 4$ and $\det V \neq 0$. The proof is complete.

5. Linear map. We have identified $M(4)$ with $T_{x_0}G(8, 4)$. By a linear isomorphism, we also identify $\boldsymbol{R}^4$ with $T_{b_0}B$. By Proposition 11, we have the linear map $g_*: \boldsymbol{R}^4 \to M(4)$ such that $g_*(V) \in \mathrm{GL}(4)$ for $V \neq 0$. Let S^3 be the unit sphere in $\boldsymbol{R}^4$. If $g_*(S^3)$ is contained in the connected component of the identity of $\mathrm{GL}(4)$, which we denote by $\mathrm{GL}^+(4)$, then we define σ to be the homotopy class of $g_* | S^3: S^3 \to \mathrm{GL}^+(4)$. For all base points x of $\mathrm{GL}^+(4)$, $\pi_3(\mathrm{GL}^+(4), x)$ are canonically isomorphic, and we simply denote it by $\pi_3(\mathrm{GL}^+(4))$. Thus σ is an element of $\pi_3(\mathrm{GL}^+(4))$.

Consider the case where $g_*(S^3)$ is contained in the different connected component of $\mathrm{GL}^+(4)$. Let R denote the element in $\mathrm{GL}(8)$ defined by

$$R = \begin{pmatrix} I_7 & 0 \\ 0 & -1 \end{pmatrix},$$

where I_k denote the identity matrix of $\mathrm{GL}(k)$. We define an automorphism J of the Stiefel manifold $V(8, 4)$ by $J(a) = \{RA\}$, where $A \in \mathrm{GL}(8)$ represents $a \in V(8, 4)$. The automorphism J induces an automorphism J' of $G(8, 4)$. Thus J is a bundle isomorphism and J' fixes the base point x_0. The differential $(J')_*$ is equal to the multiplication of

$$\begin{pmatrix} I_3 & 0 \\ 0 & -1 \end{pmatrix}$$

on $M(4)$. In particular J' maps the subset K onto itself. The composition $(J')_* g_*$ maps S^3 into $\mathrm{GL}^+(4)$. We define $\sigma \in \pi_3(\mathrm{GL}^+(4))$ to be the homotopy class of $(J')_* g_* | S^3: S^3 \to \mathrm{GL}^+(4)$.

Remark that the composition

$$(J')_* g_*: \boldsymbol{R}^4 \to M(4)$$

is a linear map of vector spaces. The class σ will be shown to coincide with the homotopy class of the characteristic map of the bundle $p: S^7 \to B$.

6. Homotopy class of linear map. In this section, we study the homotopy class of $f | S^3$, where $f: \boldsymbol{R}^4 \to M(4)$ is a linear map such that $f(\boldsymbol{R}^4 - \{0\}) \subset \mathrm{GL}^+(4)$.

The Lie group $S^3 = \mathrm{Sp}(1)$ is naturally considered as the subgroup of $\mathrm{GL}^+(4)$. Let $\mathrm{GL}^+(4)/S^3$ be the coset space and let $\beta: \mathrm{GL}^+(4) \to \mathrm{GL}^+(4)/S^3$ be the projection. Since $\mathrm{GL}^+(4)/S^3$ is diffeomorphic to $\mathrm{SO}(3) \times \boldsymbol{R}^{10}$, we have $\pi_3(\mathrm{GL}^+(4)/S^3) \cong \boldsymbol{Z}$. The homotopy class of the composition $\beta \circ f | S^3$ defines an element in $\pi_3(\mathrm{GL}^+(4)/S^3)$, the isomorphism class of which is independent of the choice of the base point. Let λ be a generator of $\pi_3(\mathrm{GL}^+(4)/S^3)$.

The aim of this section is to prove the following.

PROPOSITION 12. *Let $f\colon \boldsymbol{R}^4 \to M(4)$ be a linear map such that $f(\boldsymbol{R}^4 - \{0\}) \subset \mathrm{GL}^+(4)$. Write*

$$\{\beta \circ f \mid S^3\} = m\lambda \in \pi_3(\mathrm{GL}^+(4)/S^3),$$

where $m \in \boldsymbol{Z}$. Then we have

$$|m| \leqq 2 .$$

For the proof, we need several lemmas. Firstly, we define a map $r\colon \mathrm{GL}(4) \to \mathrm{GL}(3)$ as follows. Denote by $\boldsymbol{H}$ the field of quaternions and put $\boldsymbol{H}^* = \boldsymbol{H} - \{0\}$. We naturally identify $\boldsymbol{H}$ with $\boldsymbol{R}^4$. To each element $x \in \boldsymbol{H}$, we associate a matrix $m(x) \in M(4)$ defined by $m(x)y = xy$ for $y \in \boldsymbol{H} \cong \boldsymbol{R}^4$, where xy denotes the product of x and y in $\boldsymbol{H}$. Thus $m(x) \in \mathrm{GL}^+(4)$ if $x \neq 0$. Represent an element $X \in \mathrm{GL}(4)$ by (X_1, X_2, X_3, X_4), where X_i are column vectors and considered as elements in $\boldsymbol{H}^*$. For any $x \in H$, we denote by $\mathrm{Im}\, x$ the imaginary part of x. We regard $\mathrm{Im}\, x$ as a 3-dimensional column vector. Define $r(X)$ by

$$\begin{aligned} r(X) &= (\mathrm{Im}\, X_1^{-1}X_2, \mathrm{Im}\, X_1^{-1}X_3, \mathrm{Im}\, X_1^{-1}X_4), \\ &= (\mathrm{Im}\, m(X_1^{-1})X_2, \mathrm{Im}\, m(X_1^{-1})X_3, \mathrm{Im}\, m(X_1^{-1})X_4). \end{aligned}$$

LEMMA 13. *For any $X \in \mathrm{GL}(4)$, the (3×3)-matrix $r(X)$ is non-singular.*

PROOF. We have $\det(1, X_1^{-1}X_2, X_1^{-1}X_3, X_1^{-1}X_4) = \det(m(X_1)^{-1}X)$. If $X \in \mathrm{GL}(4)$, then $X_1 \neq 0$ and $\det m(X_1) \neq 0$. Thus $\det r(X) = \det(\mathrm{Im}\, X_1^{-1}X_2, \mathrm{Im}\, X_1^{-1}X_3, \mathrm{Im}\, X_1^{-1}X_4) = \det(m(X_1^{-1})) \det X \neq 0$.

Note that if $X \in \mathrm{GL}^+(4)$, then $r(X) \in \mathrm{GL}^+(3)$.

Secondly, we want to write down the composition $r \circ f\colon \boldsymbol{R}^4 \to \mathrm{GL}^+(3)$. Define vectors e^1, e^2, e^3, e^4 in $\boldsymbol{R}^4 - \{0\}$ by

$$e^1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e^2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad e^3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad e^4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

For $i = 1, 2, 3, 4$, put $F^i = f(e^i)$. Then $F^i \in \mathrm{GL}^+(4)$. Write

$$F^i = (a^i, b^i, c^i, d^i),$$

where $a^i, b^i, c^i, d^i \in \boldsymbol{R}^4 - \{0\} = H^*$ are column vectors. Define a (4×4)-

matrix A by $A = (a^1, a^2, a^3, a^4)$. Similarly we define (4×4)-matrices B, C and D. Then for a vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

in $\boldsymbol{R}^4$, we have

$$\begin{aligned} f(x) &= \sum x_i F^i \\ &= (\sum x_i a^i, \sum x_i b^i, \sum x_i c^i, \sum x_i d^i) \\ &= (Ax, Bx, Cx, Dx) . \end{aligned}$$

Since $f(x) \in \mathrm{GL}^+(4)$ for $x \neq 0$, A, B, C and D are non-singular (4×4)-matrices.

If $A \in \mathrm{GL}^+(4)$, the homotopy class of $f \mid S^3 \colon S^3 \to \mathrm{GL}^+(4)$ coincides with the homotopy class of $A^{-1}f \mid S^3 \colon S^3 \to \mathrm{GL}^+(4)$. If $A \in \mathrm{GL}^-(4)$, Image $(A^- f \mid S^3)$ is contained in $\mathrm{GL}^-(4)$. But $\mathrm{GL}^-(4)$ is diffeomorphic to $\mathrm{GL}^+(4)$ and all arguments work as in the case where $A \in \mathrm{GL}^+(4)$. Thus in any case, we assume that A is the identity matrix I_4 and $f(x) = (x, Bx, Cx, Dx)$.

Define skew-symmetric (4×4)-matrices P_1, P_2 and P_3 by

$$P_1 = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \quad P_2 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}, \quad P_3 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

LEMMA 14. *Assuming that $A = I_4$, for $x \in \boldsymbol{R}^4 - \{0\} = \boldsymbol{H}^*$, we have*

$$r \circ f(x) = \frac{1}{|x|^2} \begin{pmatrix} {}^t x(P_1 B)x, & {}^t x(P_1 C)x, & {}^t x(P_1 D)x \\ {}^t x(P_2 B)x, & {}^t x(P_2 C)x, & {}^t x(P_2 D)x \\ {}^t x(P_3 B)x, & {}^t x(P_3 C)x, & {}^t x(P_3 D)x \end{pmatrix} .$$

PROOF. By our definition, $r \circ f(x) = (\operatorname{Im} x^{-1}Bx, \operatorname{Im} x^{-1}Cx, \operatorname{Im} x^{-1}Dx)$. Then the proof is a direct calculation using

$$m(x^{-1}) = \frac{1}{|x|^2} m(\bar{x}) = \frac{1}{|x|^2} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & x_4 & -x_3 \\ -x_3 & -x_4 & x_1 & x_2 \\ -x_4 & x_3 & -x_2 & x_1 \end{pmatrix} .$$

By this lemma, we know that each entry of the matrix $r \circ f(x)$ is a homogeneous polynomial of degree 2 in four variables x_1, x_2, x_3, x_4.

Thirdly, let N be the subgroup of $\mathrm{GL}^+(3)$ defined by $N=\{g=(g_{ij})\in \mathrm{GL}^+(3);\, g_{ij}=0(i>j),\, g_{ii}>0\}$. Then N is diffeomorphic to $\boldsymbol{R}^6$ and $\mathrm{GL}^+(3)$ is diffeomorphic to $\mathrm{SO}(3)\times N$, by the orthonormalization of Gramm-Schmidt. Let ω: $\mathrm{GL}^+(3)\to \mathrm{SO}(3)$ denote the canonical projection.

The following is easy to see.

LEMMA 15. *The homotopy class* $\{\beta\circ f|S^3\}$ *is equal to the homotopy class* $\{\omega\circ r\circ f|S^3\}\in \pi_3(\mathrm{GL}^+(4)/S^3)$, *where we identify* $\pi_3(\mathrm{SO}(3))$ *with* $\pi_3(\mathrm{GL}^+(4)/S^3)$ *by the inclusion.*

Fourthly, in order to know the homotopy class of the composition

$$\omega\circ r\circ f|S^3\colon S^3\to \mathrm{GL}^+(4)\to \mathrm{GL}^+(3)\to \mathrm{SO}(3)\,,$$

we count the degree. For $x\in S^3$, represent the (3×3)-matrix $(r\circ f|S^3)(x)$ by $(h_{ij}(x))$. By Lemma 14, $h_{ij}(x)$ is a homogeneous polynomial of degree 2. By the definition of ω, it follows that the inverse image $(\omega\circ r\circ f|S^3)^{-1}(I)$, I being the identity of $\mathrm{SO}(3)$, is contained in the set of points x in S^3 such that $h_{12}(x)=h_{23}(x)=h_{13}(x)=0$. We consider the solutions of real homogeneous polynomial equations of degree 2 in $\boldsymbol{C}P^3$.

LEMMA 16. *We can choose a map* $H'\colon S^3\to \mathrm{GL}^+(3)$ *with* $H'(x)=(h'_{ij}(x))$ *which satisfies the following conditions*:

(i) H' *is homotopic to* $r\circ f|S^3$.

(ii) $h'_{ij}(x)$ *is a real homogeneous polynomial of degree* 2 *for* $1\leqq i,\, j\leqq 3$.

(iii) *The number of points* y *in* $\boldsymbol{C}P^3$ *such that* $h'_{12}(y)=h'_{23}(y)=h'_{13}(y)=0$ *is finite.*

PROOF. A complex homogeneous polynomial of degree 2 in $\boldsymbol{C}^4$ is written as $\sum_{1\leqq i\leqq j\leqq 4} a_{ij}x_ix_j$ with $a_{ij}\in \boldsymbol{C}$. Thus it corresponds to the point (a_{ij}) in $\boldsymbol{C}^{10}$. We put $h'_{ij}(x)=h_{ij}(x)$ for $i\geqq j$ and choose $h'_{ij}(x)$ sufficiently near to $h_{ij}(x)$ for $i<j$, so that $\det(h_{ij}(x)+t(h'_{ij}(x)-h_{ij}(x)))\neq 0$ for $0\leqq t\leqq 1$ and for any $x\in S^3$ as follows. In the product space $\boldsymbol{C}P^3\times(\boldsymbol{C}^{10})^3$, we have the algebraic manifold $V=\{(x;h^1,h^2,h^3);\, h^j(x)=0$ for $j=1,2,3$, h^j are complex homogeneous polynomials of degree $2\}$. The codimension of V in $\boldsymbol{C}P^3\times(\boldsymbol{C}^{10})^3$ is equal to 3. Let $p\colon V\to(\boldsymbol{C}^{10})^3$ denote the projection to the second factor. The set $W=\{v\in V;\, \dim p^{-1}(p(v))\geqq 1\}$ is a Zariski closed set. The closure $\overline{p(W)}$ of $p(W)$ in the usual topology of $(\boldsymbol{C}^{10})^3$ is Zariski closed algebraic set. The codimension of $\overline{p(W)}$ in $(\boldsymbol{C}^{10})^3$ is greater than 0. Let U be an open set in $(\boldsymbol{R}^{10})^3$. Then the Zariski closure U^c is equal to $(\boldsymbol{C}^{10})^3$. Consequently, the intersection $\overline{p(W)}\cap(\boldsymbol{R}^{10})^3$ does not contain any open set in $(\boldsymbol{R}^{10})^3$. Thus, for any point $k\in(\boldsymbol{R}^{10})^3$, we can choose k' in $(\boldsymbol{R}^{10})^3$ near to k', such that k' is not contained in $\overline{P(W)}$. The

point k' defines real homogeneous polynomials h'_{12}, h'_{23}, h'_{13} of degree 2 with the desired properties.

Now we are in a position to prove Proposition 12.

PROOF OF PROPOSITION 12. Let P^3 denote the real projectives 3-space and let $\xi: S^3 \to P^3$ be the covering map. Since $H'(x) = H'(-x)$ for $x \in S^3$, there exists a map $f': P^3 \to \mathrm{GL}^+(3)$ such that $H' = f' \circ \xi$: $S^3 \to \mathrm{GL}^+(3)$. For $y \in P^3$, write $f'(y) = (f'_{ij}(y))$. Define a subset Z in P^3 by $Z = \{y \in P^3, f'_{12}(y) = f'_{23}(y) = f'_{13}(y) = 0\}$ and Z^C in CP^3 by $Z^C = \{y \in CP^3, f'_{12}(y) = f'_{23}(y) = f'_{13}(y) = 0\}$. Then $Z \subset Z^C$ and Z^C is a finite set by our definition of H'. By Bezout's theorem in CP^3 (see e.g., [6, Chapter IV]), the set Z^C consists of $2^3 = 8$ points and Z consists of at most 8 points. Denote by D the dihedral group of order 4 in $\mathrm{SO}(3)$. The group D is isomorphic to $\boldsymbol{Z}_2 + \boldsymbol{Z}_2$ and generated by

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

Then the inverse image $(\omega \circ f')^{-1}(D)$ is contained in Z. If $\omega \circ f': P^3 \to \mathrm{SO}(3)$ is not surjective, then $\omega \circ f'$ is homotopic to the trivial map. So assume that $\omega \circ f'$ is surjective. Then there exists a point v in D such that $(\omega \circ f')^{-1}(v)$ consists of at most two points. Since the homotopy class in $\boldsymbol{Z} \cong \pi_3(\mathrm{GL}^+(4)/S^3)$ is equal to the degree of the map $f': P^3 \to \mathrm{SO}(3)$, the proof is completed.

7. Proof of Theorem. The following is known (see [3], [7], [9]).

PROPOSITION 17. *Let N_1 and N_2 be two closed simply connected smooth manifolds with the same integral cohomology ring as that of $\boldsymbol{H}P^2$. Denote by $p_1(N_1)$ and $p_1(N_2)$ their first Pontrjagin classes. Then N_1 and N_2 are* PL*-homeomorphic if and only if*

$$p_1(N_1) = \pm p_1(N_2) .$$

The proof is given as follows. Embed S^4 smoothly in N_i $(i = 1, 2)$, so that S^4 is a generator of $H_4(N_i; \boldsymbol{Z})$. Let T_i be a tubular neighborhood of S^4. Then N_i is PL-homeomorphic to the union $T_i \cup D^8$. If $p_1(N_1) = \pm p_1(N_2)$, then T_1 and T_2 are diffeomorphic and N_1 and N_2 are PL-homeomorphic. Conversely, if N_1 and N_2 are PL-homeomorphic, then T_1 and T_2 are bundle isomorphic and $p_1(N_1) = \pm p_1(N_2)$.

Suppose that N is a Blaschke manifold with the same cohomology ring as that of $\boldsymbol{H}P^2$. A Blaschke manifold is known to be simply connected unless the cohomology ring is equal to that of a real projective space ([2, 7.23]).

By Proposition 5, N is diffeomorphic to the union $E \cup D^8$, where E is the 4-disc bundle over B associated with the sphere bundle $p: S^7 \to B$. To know the isomorphism class of the bundle E, it is sufficient to know the homotopy class of $g: B \to X = G(8, 4)$ with $g(b_0) = x_0$. The differential g_* is the map from $\boldsymbol{R}^4 \cong T_{b_0}B$ to $T_{x_0}X = \mathfrak{m}$.

Let $D^4(r)$ be a closed 4-disc of radius $r > 0$ in $T_{b_0}B$ and $e: T_{b_0}B \to B$ be the exponential map for some Riemannian metric of B. Since $K' = \mathrm{Exp}^{-1}(K)$ is a linear cone in $\mathfrak{m} = T_{x_0}X$ by Lemma 9, for small $r > 0$, we can choose a map $g': B \to X$ such that

(i) $g' \circ e = \mathrm{Exp} \circ g_*$ on $D^4(r/2)$,

(ii) $g' = g$ outside $e(D^4(r))$,

(iii) $g' \circ e | D^4(r)$ and $g \circ e | D^4(r)$ are homotopic in $X - K$ keeping the center fixed (see the remark after Proposition 8). By Proposition 8, the homotopy classes of g and g' are equal in $\pi_4(X)$. Note that g' satisfies the following relations:

$$g'(B - e(D^4(r))) \subset \mathrm{Exp}^{\perp}(\mathfrak{m}) ,$$

$$g'(e(D^4(r))) \subset \mathrm{Exp}(\mathfrak{m}) .$$

LEMMA 18. *The homotopy class of the characteristic map of the bundle E is equal to $\sigma \in \pi_3(\mathrm{GL}^+(4))$ defined in* §5.

PROOF. Since g has the above property, $p^{-1}e(D^4(r))$ and $p^{-1}(B - e(D^4(r)))$ have trivializations induced from those of $q^{-1}(\mathrm{Exp}(\mathfrak{m}))$ and $q^{-1}(\mathrm{Exp}^{\perp}(\mathfrak{m}))$. Write S^3 for $\partial D^4(r/2)$. Then by the definition of g' and Proposition 10, we have

$$k \circ g' \circ e | S^3 = k \circ \mathrm{Exp} \circ g_* | S^3 = g_* | S^3 .$$

Thus the characteristic map of E is given by the map

$$g_* | S^3 : S^3 \to \mathrm{GL}(4) .$$

If the image of $g_* | S^3$ is not contained in $\mathrm{GL}^+(4)$, changing the trivialization of $q^{-1}(\mathrm{Exp}(\mathfrak{m}))$ by J (see §5), we can assume that $g_*(S^3)$ is in $\mathrm{GL}^+(4)$. The proof is completed.

Let f and g be maps from S^3 to $\mathrm{SO}(4)$ defined by $f(x)y = xyx^{-1}$, $g(x)y = xy$ where x and y are quaternions with norm 1. Denote their homotopy classes by λ and μ. Then λ and μ generate $\pi_3(\mathrm{SO}(4)) \cong \pi_3(\mathrm{GL}^+(4)) \cong \boldsymbol{Z} + \boldsymbol{Z}$. Thus we can write $\sigma = m\lambda + n\mu$, where $m, n \in \boldsymbol{Z}$.

Let α be a generator of $H^4(B; \boldsymbol{Z})$ and let $p_1(E)$ denote the first Pontrjagin class of the bundle E. Then the following holds.

LEMMA 19. *If* $\sigma = m\lambda + n\mu$, *then*

$$p_1(E) = \pm 2(2m + n)\alpha \, .$$

PROOF. In the case where B is diffeomorphic to S^4, this lemma is proved, e.g., in Tamura [8]. Since the proof uses only the obstruction theory, this holds for any closed base manifold B homotopy equivalent to S^4.

By Proposition 12, we have $|m| \leqq 2$. Since the boundary E is homeomorphic to S^7, we have $n = \pm 1$. Choosing an orientation of E, we may assume that $n = 1$.

The following holds.

LEMMA 20. *Suppose that* $\sigma = m\lambda + \mu$. *Then* E *is diffeomorphic to* S^7 *if and only if* $m(m+1) \equiv 0 \bmod 56$.

PROOF. This is proved in [9], [10] when B is diffeomorphic to S^4. Since the proof uses only the Pontrjagin classes, the result is true for any closed smooth base manifold homotopy equivalent to S^4.

PROOF OF THEOREM. The first Pontrjagin class $p_1(N)$ of the manifold N is equal to $p_1(E)$. The integer m with $|m| \leqq 2$ which satisfies the relation $m(m+1) \equiv 0 \bmod 56$ is equal to either 0 or -1. In these cases, $p_1(N) = \pm 2\alpha$ by Lemma 19. Since $p_1(\boldsymbol{H}P^2) = \pm 2$, it follows from Proposition 17 that N is PL-homeomorphic to $\boldsymbol{H}P^2$. The proof of Theorem is completed.

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