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ON EXTREMAL QUASICONFORMAL MAPPINGS COMPATIBLE WITH A FUCHSIAN GROUP WITH A DILATATION BOUND

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1. Introduction. Let U be the upper half-plane and let Γ be a Fuchsian group. That is, Γ is a discrete subgroup of the real Möbius group $PSL(2, \boldsymbol{R})$, possibly consisting of only the identity transformation of $PSL(2, \boldsymbol{R})$. Let $L_\infty(\Gamma)$ be the closed linear subspace of $L_\infty(U)$ consisting of those $\nu \in L_\infty(U)$ which satisfy

$$(1.1) \qquad \nu(\gamma(w))\overline{\gamma'(w)}/\gamma'(w) = \nu(w) \quad \text{for every} \quad \gamma \in \Gamma .$$

We denote by $M(\Gamma)$ the open unit ball of $L_\infty(\Gamma)$. For ν in $M(\Gamma)$, we denote by $z = F_\nu(w)$ the uniquely determined automorphism of U which is a generalized solution in U of the Beltrami equation $F_{\bar{w}} = \nu F_w$ and which leaves 0, 1, ∞ fixed. The mapping F_ν is called the normalized quasiconformal automorphism of U with complex dilatation $\nu = \nu(w)$ (see Lehto and Virtanen [12, p. 185 and p. 194]). As is known, F_ν is extensible to a homeomorphism of the closure $\bar{U} = U \cup \hat{R}$ of U in the extended complex plane $\hat{C}$, which we denote by the same letter F_ν.

Let σ be a Γ-invariant closed subset of the extended real line $\hat{R}$, which contains 0, 1 and ∞. Let E be a Γ-invariant measurable, possibly empty, subset of U such that the closure of E/Γ in $\{U \cup (\hat{R} \setminus \sigma)\}/\Gamma$ is a compact proper subset of $\{U \cup (\hat{R} \setminus \sigma)\}/\Gamma$. Let $b(w)$ be a non-negative bounded measurable function on E, being automorphic for Γ and satisfying

$$(1.2) \qquad 0 \leqq c_1 = \operatorname*{ess\,sup}_{w \in E} b(w) < 1 .$$

Let $D = U \setminus E$. By the above property of E, we easily see that the set D/Γ has a positive measure. For ν in $M(\Gamma)$, we put

$$(1.3) \qquad K(\nu|_D) = (1 + \|\nu|_D\|_\infty)/(1 - \|\nu|_D\|_\infty) ,$$

where $\|\nu|_D\|_\infty$ means the L_∞ norm of the restriction $\nu|_D$ of ν to D. In the case E is empty, we use the notation $K(\nu)$ instead of $K(\nu|_U)$.

Suppose that μ is a prescribed element in $M(\Gamma)$ satisfying $|\mu(w)| \leqq b(w)$ a.e. in E. We consider the class $M_\mu \equiv M_\mu(\Gamma, \sigma, E, b)$ consisting of those $\nu \in M(\Gamma)$ which satisfy the conditions $F_\nu|_\sigma = F_\mu|_\sigma$ and $|\nu(w)| \leqq b(w)$ a.e. in E. We put

$$(1.4) \qquad k(M_\mu) = \inf \|\nu|_D\|_\infty , \quad K(M_\mu) = (1 + k(M_\mu))/(1 - k(M_\mu)) ,$$

where the infimum is taken over all $\nu \in M_\mu$. An element ν in M_μ which satisfies $\|\nu|_D\|_\infty = k(M_\mu)$ is said to be extremal within the class M_μ. By means of a normal family argument of quasiconformal mappings in [12, pp. 71–74] and Strebel [18, Satz on page 469], we see that there exists at least one extremal element within M_μ. We note that $M_\nu = M_\mu$ for every $\nu \in M_\mu$. We put

$$(1.5) \qquad c_0 = \operatorname*{ess\,inf}_{w \in E} b(w) .$$

Reich gave characterizations of extremal elements in [16, Theorems 3, 4 and 5] in the case $\Gamma = 1$ and $0 < c_0 \leqq c_1 < 1$. Later, in the case $c_1 = 0$, Gardiner proved in [11] two theorems which can be viewed, to a certain extent, as analogous to those presented in [16]. Our objective is to investigate to what extent their results can be generalized.

Main results in this note are Theorems 1, 2, 3 and Corollaries 1, 2 to Theorem 2. In Section 2, we state them with the proof of Theorem 3. Theorem 1 gives sufficient conditions for extremality, and the others characterize extremal elements under certain conditions. In particular, Theorem 2, Corollaries 1 and 2 can be viewed as generalized forms of the corresponding results in [11] and [16] (see Remark 2). In Section 4, we give the proofs of Theorems 1 and 2. In their proofs, some results from Teichmüller space theory and modified arguments in [11] and [16] shall play important roles. In Section 3, we give a property of extremal elements with a substantial boundary point (see Theorems 4 and 5).

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2. Main theorems. In order to formulate our main theorems, first we collect necessary definitions and notations. For a prescribed class $M_\mu = M_\mu(\Gamma, \sigma, E, b)$, we fix the notations under the following relations. We put

$$(2.1) \qquad \begin{aligned} &F = F_\mu , \quad f = F^{-1} , \quad \kappa(z) = f_{\bar z}(z)/f_z(z) , \\ &G = F\Gamma F^{-1} \quad \text{and} \quad \delta = F(\sigma) . \end{aligned}$$

Let E_0 be the subset of E consisting of those $w \in E$ with $b(w) = 0$. We put

$$(2.2) \qquad c_0' = \operatorname*{ess\,inf}_{w \in E \setminus E_0} b(w) .$$

We denote by $\hat\kappa$ the extension of $\kappa|_{F(D \cup E_0)}$ to U, which satisfies

$$\hat{\kappa}(z) = \|\mu|_D\|_\infty \kappa(z)/b(f(z)) \quad \text{for} \quad z \in F(E \setminus E_0) . \tag{2.3}$$

As is known, the property of (1.1) of μ implies that G is a Fuchsian group. A holomorphic function ϕ on U is called a quadratic differential for G on U if $\phi(g(z))g'(z)^2 = \phi(z)$ for every $g \in G$. We denote by $A(G, \delta)$ the space consisting of all the quadratic differentials ϕ for G on U, which are continuously extensible to $\hat{R} \setminus \delta$ and real on $\hat{R} \setminus \delta$, and satisfy

$$\|\phi\|_G \equiv \iint_{U/G} |\phi(z)| dxdy < \infty .$$

In this note, further, we require that $A(G, \delta) \neq \{0\}$. As is known, this is equivalent to $A(\Gamma, \sigma) \neq 0$. Thus this requirement eliminates the following cases, where U_Γ is U with all the fixed points in U of elements of Γ removed: U_Γ/Γ is $\hat{C}$ with three points removed; $\Gamma = 1$ and $\sigma = \{0, 1, \infty\}$; the limit set $\Lambda(\Gamma)$ of Γ is empty or consists of a single point and $(\sigma \setminus \Lambda(\Gamma))/\Gamma$ consists of a single point. We denote by $A(G, \delta)_1$ the set of those $\phi \in A(G, \delta)$ with $\|\phi\|_G = 1$. For every ϕ in $A(G, \delta) \setminus \{0\}$, following [16], we put

$$D[\phi, \kappa] = \iint_{F(D)/G} |\phi(z)| \, |1 - \kappa(z)\phi(z)/|\phi(z)| \, |^2 (1 - |\kappa(z)|^2)^{-1} dxdy ,$$

$$E[\phi, \kappa] = \iint_{F(E)/G} |\phi(z)| \, |1 - \kappa(z)\phi(z)/|\phi(z)| \, |^2 (1 - |\kappa(z)|^2)^{-1} B(f(z)) dxdy ,$$

where $B(w) = (1 + b(w))(1 - b(w))^{-1}$.

We denote by $N(G, \delta)$ the space consisting of all the elements $\alpha \in L_\infty(G)$ which satisfy $L_G(\alpha)(\phi) = 0$ for every $\phi \in A(G, \delta)$, where

$$L_G(\alpha)(\phi) \equiv \mathrm{Re} \iint_{U/G} \alpha(z)\phi(z) dxdy . \tag{2.4}$$

Let τ be an element in $M(G)$ such that $F_\tau|_\delta$ is identical with the identity automorphism of δ. All such τ form a subset $M_0(G, \delta)$ of $M(G)$.

Now we give the summarizing statements of main theorems as follows.

THEOREM 1. *The condition* (I) *below is sufficient for the condition* (II) *below to hold. The condition* (II) *is sufficient for μ to be extremal within M_μ.*

Condition (I): *Either there exists $\phi_0 \in A(G, \delta)_1$ such that*

$$\begin{aligned} \kappa(z) &= b(f(z)) |\phi_0(z)|/\phi_0(z) \quad \text{a.e. in} \quad F(E) , \quad \text{and} \\ \kappa(z) &= \|\mu|_D\|_\infty |\phi_0(z)|/\phi_0(z) \quad \text{a.e. in} \quad F(D) \end{aligned} \tag{2.5}$$

or there exists a sequence $\{\phi_n\}$ in $A(G, \delta)_1$ such that

$$\lim_{n\to\infty} \phi_n(z) = 0 \quad \textit{locally uniformly in } U \cup (\hat{R} \setminus \delta)\,, \quad \textit{and} \tag{2.6}$$
$$\lim_{n\to\infty} L_G(\kappa)(\phi_n) = \|\mu|_D\|_\infty\,.$$

Condition (II):

$$\inf\{K(\mu|_D)D[\phi, \kappa] + E[\phi, \kappa]\} = 1\,, \tag{2.7}$$

where the infimum is taken over all $\phi \in A(G, \delta)_1$.

REMARK 1. If $\|\mu|_D\|_\infty > 0$, then by [17, Corollary 1], we can easily check that the condition (I) is satisfied if and only if the condition (III) below is satisfied.

Condition (III):

$$\sup L_G(\hat{\kappa})(\phi) = \|\mu|_D\|_\infty\,, \tag{2.8}$$

where the supremum is taken over all $\phi \in A(G, \delta)$ with

$$\iint_{(U \setminus F(E_0))/G} |\phi|\, dxdy = 1\,,$$

and where $\hat{\kappa}$ is the extension of $\kappa|_{F(D \cup E_0)}$ to U satisfying (2.3).

THEOREM 2. *Suppose that*

$$\textit{either} \quad c_1 = 0 \quad \textit{in } (1.2) \textit{ or } \quad c_0' > 0 \quad \textit{in } (2.2)\,. \tag{2.9}$$

If $\|\mu|_D\|_\infty > 0$, *then each one of the conditions* (I), (II) *and* (III) *is necessary and sufficient for* μ *to be extremal within* M_μ.

COROLLARY 1. *Suppose that* $c_0 > 0$ *in* (1.5) *and that* $\|\mu|_D\|_\infty > 0$. *Then each one of the conditions* (I), (II) *and* (III) *is necessary and sufficient for* μ *to be extremal within* M_μ.

It is well-known that $\dim A(\Gamma, \sigma) < \infty$ if and only if Γ is finitely generated and the set $(\sigma \setminus \Lambda(\Gamma))/\Gamma$ is finite. Thus, by (2.1), $\dim A(\Gamma, \sigma) < \infty$ if and only if $\dim A(G, \delta) < \infty$. We note that (2.6) does not occur provided that $\dim A(G, \delta) < \infty$. Thus, by Theorem 2, we have the following.

COROLLARY 2. *Suppose* (2.9) *and that* $\dim A(\Gamma, \sigma) < \infty$. *If* $\|\mu|_D\|_\infty > 0$, *then each one of the conditions* (2.5), (II) *and* (III) *is necessary and sufficient for* μ *to be extremal within* M_μ.

REMARK 2. The above Corollary 1 (resp. Corollary 2) can be viewed as a generalized form of [16, Theorems 3 and 4] (resp. [11, Theorem 1]).

Here we note that the following proposition plays an important role in the later proof of Theorem 2. The proof of Gardiner [9, Theorem 1] shows that, if $\dim A(G, \delta) < \infty$, then the proposition holds even if $F(E_0)$ in the proposition is replaced by any G-invariant measurable subset S of

U such that the measure of the set $U \setminus S$ is positive. Careful examination and a slight modification of the proof of [9, Theorem 1], however, show that the proposition is still valid even if $\dim A(G, \delta) = \infty$ under our hypothesis that $F(E_0)/G$ is relatively compact in $\{U \cup (\hat{R} \setminus \delta)\}/G$. For the sake of completeness, we give the proof of the proposition in Section 4. The author thanks the referee and Dr. H. Ohtake very much for their valuable suggestions and comments on the proof.

PROPOSITION. *Suppose that* $\alpha \in N(G, \delta)$ *vanishes on* $F(E_0)$. *Then there exists a curve* $\tau(t) \in M_0(G, \delta)$, *defined in an interval* $(0, t_0)$, *which satisfies*

$$\tau(t) \quad \text{vanishes on} \quad F(E_0)\,, \tag{2.10}$$

and

$$\tau(t) = t\alpha + o(t) \quad \text{as} \quad t \to 0\,, \tag{2.11}$$

where $o(t)$ *term is uniform with respect to* $z \in U$.

THEOREM 3. *Let* $M_\mu = M_\mu(\Gamma, \sigma, E, b)$ *be a class such that* $\|\mu|_D\|_\infty > 0$. *Suppose that* $\dim A(\Gamma, \sigma) < \infty$ *and that* E *is an open set such that the boundary* ∂E *of* E *is a set of measure zero. Suppose, further, that* b *is continuous on* E. *Then each one of the conditions* (2.5), (II) *and* (III) *is necessary and sufficient for* μ *to be extremal within* M_μ.

PROOF. By Theorem 1 and Remark 1, it suffices to prove that the condition (2.5) is satisfied under the hypotheses of our theorem, provided that μ is extremal within M_μ.

We define b_n by $b_n(w) = b(w)$ for every $w \in E$ with $b(w) > 1/n$, $b_n(w) = 1/n$ for every $w \in E$ with $b(w) \leqq 1/n$. Let μ_n be an extremal element within $M_\mu(\Gamma, \sigma, E, b_n)$ and put $F_n = F_{\mu_n}$, $f_n = F_n^{-1}$ and $\kappa_n = (f_n)_{\bar{z}}/(f_n)_z$. It follows from definition that $b(w) \leqq b_{n+1}(w) \leqq b_n(w)$ on E. Thus we have

$$\|\mu_n|_D\|_\infty \leqq \|\mu_{n+1}|_D\|_\infty \leqq k(M_\mu)\,, \tag{2.12}$$

where M_μ means $M_\mu = M_\mu(\Gamma, \sigma, E, b)$. By a normal family argument of quasiconformal mappings, there exists a subsequence of $\{F_n\}$, which we call it again $\{F_n\}$, such that it converges to F_ν for some $\nu \in M(\Gamma)$ uniformly in $\bar{U}$ with respect to the spherical metric. It is obvious that $F_\nu|_\sigma = F_\mu|_\sigma$. It follows from [18, Satz on page 469] that

$$|\nu(w)| \leqq \limsup_{n\to\infty} |\mu_n(w)| \quad \text{a.e. in } U\,. \tag{2.13}$$

By (2.13), we have $|\nu(w)| \leqq \limsup_{n\to\infty} b_n(w) = b(w)$ a.e. in E. This means that ν belongs to M_μ. Moreover, it follows from (2.12) and (2.13) that

$|\nu(w)| \leqq \limsup_{n\to\infty} \|\mu_n|_D\|_\infty \leqq k(M_\mu)$ a.e. in D. Thus we see that ν is extremal within $M_\mu = M_\mu(\Gamma, \sigma, E, b)$ and that

$$k(M_\mu) = \|\nu|_D\|_\infty = \lim_{n\to\infty} \|\mu_n|_D\|_\infty . \tag{2.14}$$

Now we assume that μ is extremal within M_μ. Then we have $k(M_\mu) = \|\mu|_D\|_\infty > 0$. Thus, by (2.14), we may assume that $\|\mu_n|_D\|_\infty > 0$ for every n. Then, by Corollary 2, there exists a sequence $\{\phi_n\}$ in $A(G, \delta)_1$ such that

$$\begin{aligned} \kappa_n(z) &= b_n(f_n(z))|\phi_n(z)|/\phi_n(z) \quad \text{a.e. in} \quad F_n(E)\ , \quad \text{and} \\ \kappa_n(z) &= \|\mu_n|_D\|_\infty|\phi_n(z)|/\phi_n(z) \quad \text{a.e in} \quad F_n(D)\ . \end{aligned} \tag{2.15}$$

Since $\dim A(G, \delta) < \infty$, there exists a subsequence of $\{\phi_n\}$, which we call it again $\{\phi_n\}$, such that it converges locally uniformly in $U\cup(\hat{R}\setminus\delta)$ to some $\phi_0 \in A(G, \delta)_1$.

Let E_1 be a connected component of the open set E. By [12, p. 76], we know that $F_\nu(E_1)$ coincides with a component of $\bigcup_{m=1}^{\infty} \mathrm{Int}(\bigcap_{n=m}^{\infty} F_n(E_1))$. Thus, for each $z \in F_\nu(E_1)$, there exists an open neighborhood V of z and a natural number n_0 such that $V\subset F_n(E_1)$ for every $n \geqq n_0$. Therefore, by (2.15) and continuity of b, we see that

$$\lim_{n\to\infty} \kappa_n(z) = b(\tilde{f}(z))|\phi_0(z)|/\phi_0(z) \quad \text{a.e. in} \quad F_\nu(E)\ , \tag{2.16}$$

where $\tilde{f}$ denotes the inverse of F_ν. Since the sequence $\{F_n\}$ converges to F_ν uniformly in $\bar{E}$ with respect to the spherical metric, for each $z \in U\setminus F_\nu(\bar{E})$, there exists a natural number n_1 such that $z \in U\setminus F_n(\bar{E})$ for every $n \geqq n_1$. Then, by (2.14) and (2.15), we see that

$$\lim_{n\to\infty} \kappa_n(z) = \|\nu|_D\|_\infty|\phi_0(z)|/\phi_0(z) \quad \text{a.e. in} \quad U\setminus F_\nu(\bar{E})\ . \tag{2.17}$$

It follows from our hypotheses that $\partial E = \bar{E}\setminus E$ and that the measure of the set $F_\nu(\partial E) = F_\nu(\bar{E}\setminus E)$ is zero. Thus, by (2.16), (2.17) and [12, Chap. IV, Theorem 5.2], we have the following equalities up to a set of measure zero, where $\tilde{\kappa}$ is the complex dilatation of $\tilde{f}$;

$$\begin{aligned} \tilde{\kappa}(z) &= b(\tilde{f}(z))|\phi_0(z)|/\phi_0(z) \quad \text{for} \quad z \in F_\nu(E)\ , \quad \text{and} \\ \tilde{\kappa}(z) &= \|\nu|_D\|_\infty|\phi_0(z)|/\phi_0(z) \quad \text{for} \quad z \in F_\nu(D)\ . \end{aligned}$$

In this case, by Remark 3 below, we see that ν is uniquely extremal within M_μ. Therefore, ν is identical with μ. This means that μ satisfies the condition (2.5).

REMARK 3. If κ is of the form (2.5) and if $\nu \in M_\mu$ is different from μ, then $|\nu(w)| > |\mu(w)|$ on a set of positive measure. In particular, μ is uniquely extremal within M_μ. In the case $\Gamma = 1$ and $\sigma = \hat{R}$, this result follows from Reich and Strebel [13, Theorem 3]. We can, however, easily check that the proof in [13] goes through verbatim for arbitrarily prescribed Γ and σ.

3. Extremal elements with a substantial boundary point. Recalling (2.1), we let h (resp. j): $\hat{R} \to \hat{R}$ be the boundary mapping induced by F (resp. f). Let $x \in \sigma$ and let q be a quasiconformal mapping of an open neighborhood $V(x)$ of x in $\bar{U}$ onto an open neighborhood of $F(x)$ in $\bar{U}$, which satisfies $q(V(x) \cap \hat{R}) \subset \hat{R}$ and $q|_{V(x) \cap \sigma} = h|_{V(x) \cap \sigma}$. Following Strebel [19] and Fehlmann [7], we define the local dilatation $H_h^\sigma(x)$ of h at x with respect to σ as the infimum of the maximal dilatations $K(q)$ of all q with the above properties. Since $K(q) = K(q^{-1})$, we have

$$H_j^\sigma(F(x)) = H_h^\sigma(x) \quad \text{for every} \quad x \in \sigma . \tag{3.1}$$

In this section, we suppose that Γ is a Fuchsian group of the second kind, that is, $\Lambda(\Gamma) \subsetneqq \hat{R}$. Let $M_\mu = M_\mu(\Gamma, \sigma, E, b)$ be a class such that $\sigma \supsetneqq \Lambda(\Gamma)$. If there exists some $x_0 \in \sigma \setminus \Lambda(\Gamma)$ such that

$$H_h^\sigma(x_0) = K(\mu|_D) , \tag{3.2}$$

then we say that $\mu|_D$ has a substantial boundary point x_0 with respect to (Γ, σ). In this case, we have the following.

THEOREM 4. *Let Γ be a Fuchsian group of the second kind. Let $M_\mu = M_\mu(\Gamma, \sigma, E, b)$ be a class such that $\sigma \supsetneqq \Lambda(\Gamma)$. Suppose that $\|\mu|_D\|_\infty > 0$ and that $\mu|_D$ has a substantial boundary point $x_0 \in \sigma \setminus \Lambda(\Gamma)$ with respect to (Γ, σ). Then there exists a sequence $\{\phi_n\}$ in $A(G, \delta)_1$ which satisfies* (2.6) *and which converges to* 0 *locally uniformly in the complement in $\bar{U}$ of the closure of the set $\{g(F(x_0)); g \in G\}$.*

PROOF. Let $\kappa^* \in M(G)$ be the extension of $\kappa|_{F(D)}$ to U such that $\kappa^*(z) = 0$ on $F(E)$. Let j^*: $\hat{R} \to \hat{R}$ be the boundary mapping induced by F_{κ^*}. By (3.1) and our assumption (3.2), we have

$$H_{j^*}^\sigma(F(x_0)) = K(\kappa|_{F(D)}) = K(\kappa^*) . \tag{3.3}$$

Since $F(x_0) \notin \Lambda(G)$, there exists an open neighborhood V of $F(x_0)$ in $\bar{U}$ such that $g(V) \cap V = \emptyset$ if $g \in G$ is not the identity transformation of $PSL(2, \boldsymbol{R})$. Thus we see that $V \cap U \subset F(D)$ for sufficiently small V, since otherwise the closure of $F(E)/G$ in $\{U \cup (\hat{R} \setminus \delta)\}/G$ is not a compact proper subset of $\{U \cup (\hat{R} \setminus \delta)\}/G$. Thus we have $\kappa^* = \kappa$ on $V \cap U$. Therefore, on $V \cap U$, F_{κ^*} is represented as F_κ followed by a conformal mapping. This implies

$$H_j^\delta(F(x_0)) = H_{j*}^\delta(F(x_0)) \ . \tag{3.4}$$

Let δ_n be the closure in $\hat{R}$ of the set $\{g(x); g \in G, x \in I_n \cap \delta\}$, where I_n is the open interval in $\hat{R}$ with the midpoint $F(x_0)$ such that the spherical length of I_n is equal to $1/n$. Then, for each n, by (3.3), (3.4) and definition of $H_{j*}^\delta(F(x_0))$, we see that $K(\kappa^*) \leqq K(\lambda)$ for every $\lambda \in M(1)$ such that $F_\lambda|_{\overline{I_n} \cap \delta} = F_{\kappa^*}|_{\overline{I_n} \cap \delta}$. Since $\|\kappa^*\|_\infty > 0$, as is known, this implies the following; for each n, there exists a sequence $\{\Psi_{n,m}\}$ in $A(1, \overline{I_n} \cap \delta)_1$ such that $\|\kappa^*\|_\infty = \lim_{m\to\infty} L_1(\kappa^*)(\Psi_{n,m})$ (see [17, Lemma 1] and the references quoted there). For each n, we choose sufficiently large m and put $\Phi_n = \Psi_{n,m}$. We may assume that $\{\Phi_n\}$ converges locally uniformly in $\bar{U} \setminus \{F(x_0)\}$ to the limit function Φ and that

$$\|\kappa^*\|_\infty = \lim_{n\to\infty} L_1(\kappa^*)(\Phi_n) \ . \tag{3.5}$$

The holomorphic function Φ is real on $\bar{U} \setminus \{F(x_0)\}$ and satisfies $\|\Phi\|_1 \leqq \liminf_{n\to\infty} \|\Phi_n\|_1 = 1$. In this case, as noted in Reich [15, p. 400], we see that $\Phi = 0$. The Poincaré series $\hat{\phi}_n = \theta_G \Phi_n$ of Φ_n is defined by $\hat{\phi}_n = \sum_{g\in G} \Phi_n(g(z))g'(z)^2$. It is known that $\hat{\phi}_n$ belongs to $A(G, \delta_n)$ and that $\|\hat{\phi}_n\|_G \leqq 1$ (see [4]). By (3.5) and [17, Theorem 2], we see that

$$\|\kappa^*\|_\infty = \lim_{n\to\infty} L_G(\kappa^*)(\hat{\phi}_n) \ , \tag{3.6}$$

$$\lim_{n\to\infty} \|\hat{\phi}_n\|_G = 1 \ . \tag{3.7}$$

Considering a conjugate group of G, if necessary, we easily check that it suffices to prove our theorem under the hypothesis that $\infty \in \hat{R} \setminus \Lambda(G)$. Then it is well-known that $\sum_{g\in G} g'(z)^2$ converges absolutely and locally uniformly in $\bar{U} \setminus \Lambda(G)$. Thus the sequence $\{\hat{\phi}_n\}$ converges to 0 locally uniformly in the complement in $\bar{U}$ of the closure of the set $\{g(F(x_0)); g \in G\}$. If we put $\phi_n = \hat{\phi}_n/\|\hat{\phi}_n\|_G$, then, by (3.6) and (3.7), the sequence $\{\phi_n\}$ has the desired properties.

REMARK 4. Let the hypotheses of Theorem 4 be satisfied. Then it follows from Theorems 1 and 4 that μ is extremal within M_μ. But this is directly and more easily checked. In fact, since $x_0 \notin \Lambda(\Gamma)$, as noted in the proof of Theorem 4, we see that there exists an open neighborhood V of x_0 in $\bar{U}$ such that $V \cap U \subset D$. From this and (3.2), it easily follows that μ is extremal within M_μ.

We have the reverse implication of the above Theorem 4, provided that Γ is finitely generated and of the second kind. To be specific, we have the following theorem. We note that the essential part of the proof of the theorem is due to Fehlmann [8, Theorem 2.1]. For the sake of completeness, we give the proof.

THEOREM 5. *Let Γ be a finitely generated Fuchsian group of the second kind. Let $M_\mu = M_\mu(\Gamma, \sigma, E, b)$ be a class such that $\sigma \supsetneqq \Lambda(\Gamma)$. Suppose that the condition* (2.6) *is satisfied. Then there exists some $x_0 \in \sigma \setminus \Lambda(\Gamma)$ such that $K(\mu|_D) = H_h^\sigma(x_0)$.*

PROOF. Let κ^* and j^* be defined as in the proof of Theorem 4. By (3.1) and (3.4), it suffices to prove that there exists some $x_1 \in \delta \setminus \Lambda(G)$ such that

$$H_{j^*}^\delta(x_1) = K(\kappa^*) . \tag{3.8}$$

First assume that Γ is torsion free and that E is empty. In this case, [8, Theorem 2.1] says that if the condition (2.6) is satisfied, then there exists some $x_1 \in \delta \setminus \Lambda(G)$ such that $H_j^\delta(x_1) = K(\kappa)$. Further, since E is empty, we have $\kappa = \kappa^*$, $j = j^*$. Thus we see that (2.6) implies (3.8).

Next we consider the general case. Since G is finitely generated, we know as the Selberg theorem that there exists a torsion free subgroup G_1 of G of finite index m (see [4, p. 20] and the references quoted there). It clearly follows that $A(G, \delta) \subset A(G_1, \delta)$, $\Lambda(G) = \Lambda(G_1)$ and that $\|\phi\|_G = \|\phi\|_{G_1}/m$ for every ϕ in $A(G, \delta)$. Furthermore, G_1 is also finitely generated and of the second kind.

Assume that the condition (2.6) is satisfied. Then the sequence $\{\phi_n\}$ in $A(G, \delta)_1$ in (2.6) satisfies

$$\lim_{n\to\infty} L_G(\kappa^*)(\phi_n) = \|\kappa^*\|_\infty . \tag{3.9}$$

But we can rewrite (3.9) as follows:

$$\lim_{n\to\infty} L_{G_1}(\kappa^*)(\phi_n/m) = \|\kappa^*\|_\infty . \tag{3.10}$$

Here the sequence $\{\phi_n/m\}$ in $A(G_1, \delta)_1$ converges to 0 locally uniformly in $U \cup (\hat{R} \setminus \delta)$. Thus, by (3.10) and the former part of the proof, we see that (3.8) holds for some $x_1 \in \delta \setminus \Lambda(G_1) = \delta \setminus \Lambda(G)$.

4. Proofs of Theorems 1 and 2.

In this section, we give the proofs of Theorems 1, 2 and Proposition. For this purpose, first we recall some results from Teichmüller space theory and prove some lemmas.

Let $\tau \in M_0(G, \delta)$ and let R_τ be the right translation of $M(G)$ which is defined by the relation $R_\tau \rho = \lambda$ for $\rho, \lambda \in M(G)$ if and only if $F_\rho \circ F_\tau = F_\lambda$. By these translations, the group $M_0(G, \delta)$ acts on $M(G)$ on the right. The generalized Teichmüller space $T(G, \delta)$ is defined as the factor space $M(G)/M_0(G, \delta)$. Let Ω denote $\hat{C} \setminus \delta$ (resp. the lower half-plane) if $\delta \neq \hat{R}$ (resp. if $\delta = \hat{R}$). Let λ_Ω be the Poincaré metric on Ω. We define $B(G, \delta)$

as the space consisting of all the quadratic differentials ϕ for G on Ω which satisfy $\phi(\bar{z}) = \bar{\phi}(z)$ if $\delta \neq \hat{R}$, and

$$\sup_{z \in \Omega} \lambda_\Omega(z)^{-2}|\phi(z)| < \infty .$$

The following theorem is known as an implicit result of Bers [1], [2] in the case $\delta = \hat{R}$, and is due to Earle [5], [6] in the case where δ is identical with the limit set $\Lambda(G)$ of G. We can check that the arguments developed in [5] and [6] still work for arbitrarily prescribed G and δ such that $A(G, \delta) \neq \{0\}$.

THEOREM A. *There exists an open real analytic (resp. complex analytic) mapping* J_δ: $M(G) \to B(G, \delta)$ *if* $\delta \neq \hat{R}$ *(resp. if* $\delta = \hat{R}$*) satisfying the following properties*:

(a) $J_\delta(M(G))$ *is a bounded domain in* $B(G, \delta)$,

(b) *the kernel of* J_δ *is identical with* $M_0(G, \delta)$,

(c) *for every* λ *in* $M(G)$, *the differential mapping* $DJ_\delta(\lambda)$: $L_\infty(G) \to B(G, \delta)$ *is surjective and has a continuous linear section*,

and

(d) *the kernel of* $DJ_\delta(0)$ *is identical with* $N(G, \delta)$.

Theorem A implies that $M_0(G, \delta)$ is an analytic closed submanifold of $M(G)$ and that the tangent space of $M_0(G, \delta)$ at 0 is identical with $N(G, \delta)$. Furthermore, $T(G, \delta)$ carries an analytic structure induced by J_δ. The tangent space of $T(G, \delta)$ at $[0]$ is $L_\infty(G)/N(G, \delta)$ which is isomorphic to the dual space of $A(G, \delta)$ by the pairing (2.4).

PROOF OF PROPOSITION. Let $V = U \setminus F(E_0)$ and let $L_\infty(V, G)$ be the subspace of $L_\infty(G)$ consisting of those $\lambda \in L_\infty(G)$ which vanish on $F(E_0)$. We denote by $M(V, G)$ the open unit ball of $L_\infty(V, G)$. Put $\chi = J_\delta|_{M(V,G)}$. As noted in the proof of [9, Theorem 1], it suffices to prove that the mapping $D\chi(0)$: $L_\infty(V, G) \to B(G, \delta)$ is surjective and that the kernel of $D\chi(0)$ splits in $L_\infty(V, G)$. Let λ be an arbitrary element in $L_\infty(G)$. We define λ_1 by $\lambda_1(z) = \lambda(z)$ on $F(E_0)$, $\lambda_1(z) = 0$ on V. Put $\lambda_2 = \lambda - \lambda_1$. We abbreviate the space $\{\phi|_V; \phi \in A(G, \delta)\}$ to $A(G, \delta)|_V$. Since, the closure of $F(E_0)/G$ in $\{U \cup (\hat{R} \setminus \delta)\}/G$ is a compact proper subset of $\{U \cup (\hat{R} \setminus \delta)\}/G$, we easily see that the linear functional $T_1(\lambda)$ (resp. $T_2(\lambda)$) on $A(G, \delta)|_V$ sending $\phi \in A(G, \delta)|_V$ into $-L_G(\lambda_1)(\phi)$ (resp. $L_G(\lambda_2)(\phi)$) is bounded. Then, by the Hahn-Banach and Riesz representation theorems, there exist $\lambda_i^* \in L_\infty(V, G)$, $i = 1, 2$, which satisfy $\|\lambda_i^*\|_\infty = \|T_i(\lambda)\|$ and $L_G(\lambda_i^*)(\phi) = T_i(\lambda)(\phi)$ for every $\phi \in A(G, \delta)$, where $\|T_i(\lambda)\|$ means the operator norm of $T_i(\lambda)$ on $A(G, \delta)|_V$. Let $\alpha \in L_\infty(G)$ be the extension of $\lambda|_{F(E_0)}$ which satisfies $\alpha(z) = \lambda_1^*(z)$ on V. By our construction, we see that $\alpha \in N(G, \delta)$ and that

$\lambda - \alpha \in L_\infty(V, G)$. Thus, by Theorem A, we have $D\chi(0)(\lambda - \alpha) = DJ_\delta(0)(\lambda)$. This implies that the mapping $D\chi(0)\colon L_\infty(V, G) \to B(G, \delta)$ is surjective, because the choice of $\lambda \in L_\infty(G)$ is arbitrary and the mapping $DJ_\delta(0)\colon L_\infty(G) \to B(G, \delta)$ is surjective.

Let $\lambda \in L_\infty(V, G)$. Then clearly we see that $\lambda - \lambda_2^*$ belongs to the kernel of $D\chi(0)$ and that $\|\lambda_2^*\|_\infty = \|T_2(\lambda_2^*)\|$. Noting these facts and Theorem A, we can easily check that the kernel of $D\chi(0)$ is closed in $L_\infty(V, G)$ and that it has a closed complementary subspace $\{\lambda \in L_\infty(V, G); \|\lambda\|_\infty = \|T_2(\lambda)\|\}$ in $L_\infty(V, G)$. This completes the proof of Proposition.

As was shown in [16, Lemma 2.1], we obtain (4.1) below as an immediate consequence of the "main inequality" of Reich and Strebel for the configuration (G, δ) (see Strebel [20], Bers [4, Theorem 2] and Gardiner [10, Theorem 4.2] for the "main inequality"). The inequality (4.2) follows from (4.1), (1.3) and (1.4).

LEMMA 1. *Suppose that $\nu \in M_\mu$. Then the following two inequalities hold for every $\phi \in A(G, \delta) \setminus \{0\}$:*

$$\|\phi\|_G \leqq K(\nu|_D)D[\phi, \kappa] + E[\phi, \kappa]\ , \tag{4.1}$$

$$\|\phi\|_G \leqq K(M_\mu)D[\phi, \kappa] + E[\phi, \kappa]\ . \tag{4.2}$$

LEMMA 2. *There exists a positive number r such that*

$$D[\phi, \kappa] \geqq r \quad \textit{for every} \quad \phi \in A(G, \delta)_1\ .$$

PROOF. Suppose the contrary. Then there exists a sequence $\{\phi_n\}$ in $A(G, \delta)_1$ such that $\lim_{n\to\infty} D[\phi_n, \kappa] = 0$. By the mean value property of holomorphic functions, $A(G, \delta)_1$ forms a normal family with respect to locally uniform convergence. Thus we may assume that the sequence $\{\phi_n\}$ converges to some $\phi_0 \in A(G, \delta)$ locally uniformly in $U \cup (\hat{R} \setminus \delta)$.

It follows from definition that

$$D[\phi_n, \kappa] \geqq K(\mu|_D)^{-1} \iint_{F(D)/G} |\phi_n|\, dxdy\ .$$

Thus we have

$$\begin{aligned} 0 = \lim_{n\to\infty} D[\phi_n, \kappa] &\geqq K(\mu|_D)^{-1} \liminf_{n\to\infty} \iint_{F(D)/G} |\phi_n|\, dxdy \\ &\geqq K(\mu|_D)^{-1} \iint_{F(D)/G} |\phi_0|\, dxdy\ . \end{aligned}$$

Since the set D/Γ has a positive measure, so does $F(D)/G$. Hence the above inequality implies that $\phi_0 = 0$. Therefore we see that $\lim_{n\to\infty} E[\phi_n, \kappa] = 0$ and that $\lim_{n\to\infty} K(\mu|_D)D[\phi_n, \kappa] + E[\phi_n, \kappa] = 0$. This contradicts (4.1). Thus we have the lemma.

LEMMA 3. *Let* $\{\phi_n\}$ *be a sequence in* $A(G, \delta)_1$ *which converges to* 0 *locally uniformly in* $U \cup (\hat{R} \setminus \delta)$. *Then the conditions* (2.6) *and* (4.3) *below are simultaneously true or false*:

$$\lim_{n\to\infty} L_G(\kappa(1-|\kappa|^2)^{-1})(\phi_n) = \|\mu|_D\|_\infty(1-\|\mu|_D\|_\infty^2)^{-1} . \tag{4.3}$$

PROOF. Let κ^* be the extension of $\kappa|_{F(D)}$ to U such that $\kappa^*(z) = 0$ on $F(E)$. Since $\|\mu|_D\|_\infty = \|\kappa^*\|_\infty$ and the integral of $|\phi_n|$ over $F(E)/G$ converges to 0 as n tends to ∞, we may replace κ (resp. $\|\mu|_D\|_\infty$) in (2.6) and (4.3) by κ^* (resp. $\|\kappa^*\|_\infty$). Thus we have only to consider the case $k_0 = \|\kappa^*\|_\infty > 0$. We put $S_c = \{z \in U; |\kappa^*(z)| \leqq c\}$ for $0 < c < k_0$. Then, in the same way as in Reich and Strebel [14, pp. 382–383], we easily see that, given any $\varepsilon > 0$ there exists some c, $0 < c < k_0$, such that

$$\begin{aligned}|L_G(\kappa^*(1-|\kappa^*|^2)^{-1})(\phi_n) - (1-k_0^2)^{-1}L_G(\kappa^*)(\phi_n)|\\ \leqq 2k_0(1-k_0^2)^{-1}\iint_{S_c/G}|\phi_n|\,dxdy + \varepsilon k_0 .\end{aligned}$$

Thus it suffices to prove that if any one of (2.6) and (4.3) is satisfied, then, for every c, $0 < c < k_0$, the integral of $|\phi_n|$ over S_c/G converges to 0 as n tends to ∞. But, as noted in [14, p. 381], we can easily check this.

Now we give the proof of Theorem 1.

PROOF OF THEOREM 1. If (2.5) holds, then it follows from definition that $K(\mu|_D)D[\phi_0, \kappa] + E[\phi_0, \kappa] = 1$. Thus, by (4.1), we have (2.7).

Now we assume (2.6). Then, by Lemma 3, we have (4.3). Thus, by (4.1), we can easily check that $\lim_{n\to\infty} E[\phi_n, \kappa] = 0$ and that $\lim_{n\to\infty} D[\phi_n, \kappa] = 1/K(\mu|_D)$. Therefore we have (2.7).

Next we assume (2.7) and show that μ is extremal within M_μ. Take an arbitrary positive number ε. By (4.2) and (2.7), there exists $\phi_1 \in A(G, \delta)_1$ such that

$$K(\mu|_D)D[\phi_1, \kappa] + 1 - K(M_\mu)D[\phi_1, \kappa] \leqq K(\mu|_D)D[\phi_1, \kappa] + E[\phi_1, \kappa] < 1 + \varepsilon .$$

Thus, by Lemma 2, we see that there exists a positive number r such that

$$K(\mu|_D) - K(M_\mu) < \varepsilon/D[\phi_1, \kappa] \leqq \varepsilon/r .$$

Since ε is arbitrary, we see that $K(\mu|_D) \leqq K(M_\mu)$. This implies that μ is extremal within M_μ.

After proving one more lemma, we give the proof of Theorem 2.

LEMMA 4. *Suppose that* $k_0 = \|\mu|_D\|_\infty > 0$. *Let* $\tau(t) \in M(G)$ *be a curve defined in an interval* $(0, t_0)$ *which satisfies* (2.10) *and* (2.11) *for some* $\alpha \in L_\infty(G)$. *Let* $H = F_{\tau(t)}$ *and let* $\nu(t)$ *be the complex dilatation of* $H \circ F$. *Suppose that the conditions* (2.9) *and* (4.4) *below hold*:

$$\|(\hat{\kappa} - \alpha)|_{U \setminus F(E_0)}\|_\infty \equiv a < k_0 \, . \tag{4.4}$$

Then there exists a positive number t_1 *such that the following inequalities* (4.5) *and* (4.6) *hold for every* $t \leqq t_1$:

$$\|\nu(t)|_D\|_\infty < k_0 \, , \tag{4.5}$$

$$|\nu(t)(w)| \leqq b(w) \quad a.e. \ in \quad E \, . \tag{4.6}$$

PROOF. Noting (2.1), we put $w = f(z)$, $p = f_z$. Then it is well-known that

$$\nu(t)(w) = (\tau(t)(z) - \kappa(z))(1 - \tau(t)(z)\bar{\kappa}(z))^{-1}p/\bar{p} \tag{4.7}$$

holds a.e. in U. By (2.11) and (4.7), we have

$$\begin{aligned} |\nu(t)(w)| &= |(\kappa(z) - t\alpha(z))(1 + t\alpha(z)\bar{\kappa}(z)) + o(t)| \\ &= |\{1 - t(1 - |\kappa(z)|^2)\}\kappa(z) + t(1 - |\kappa(z)|^2)(\kappa(z) - \alpha(z))| + o(t) \\ &\leqq k_0 - t(1 - |\kappa(z)|^2)(k_0 - a) + o(t) \end{aligned}$$

for almost every point w in D. Thus, by (4.4), we have (4.5). We note that the above procedure of the estimate of $|\nu(t)(w)|$ is due to Bers [3, p. 42].

It remains to prove (4.6). By (2.10) and (4.7), it is obvious that

$$|\nu(t)(w)| = |\kappa(z)| = |\mu(w)| \leqq b(w) = 0 \quad \text{a.e. in} \quad E_0 \, . \tag{4.8}$$

In the case $c_1 = 0$, (4.8) is none other than (4.6). Thus we have only to consider the case $c_0' > 0$.

We modify the argument developed in [16, p. 109]. We put

$$V_1 = \{z \in F(E \setminus E_0); |\kappa(z)| \leqq (k_0 + a)b(f(z))/2k_0\} \, ,$$

$$V_2 = \{z \in F(E \setminus E_0); (k_0 + a)b(f(z))/2k_0 < |\kappa(z)| \leqq b(f(z))\} \, .$$

Let $V_i^* = f(V_i)$ for $i = 1, 2$. Then we have

$$b(f(z)) - |\kappa(z)| \geqq (k_0 - a)b(f(z))/2k_0 \geqq (k_0 - a)c_0'/2k_0 > 0 \tag{4.9}$$

a.e. in V_1. By (2.11), (4.7) and (4.9), we can easily check that there exist positive numbers r_1 and s_1 such that

$$|\nu(t)(w)| \leqq b(w) - r_1 t \quad \text{for every} \quad t \leqq s_1 \, , \tag{4.10}$$

and almost every point w in V_1^*.

Since $c_0' > 0$, $|\kappa(z)|$ is bounded away from 0 on V_2 up to a set of measure zero. Expanding (4.7), we have

(4.11) $$|\nu(t)(w)| = |\kappa(z)| - t(1 - |\kappa(z)|^2)\,\mathrm{Re}\,(\alpha(z)\bar{\kappa}(z))/|\kappa(z)| + o(t)\ ,$$

where $o(t)$ term is uniform with respect to almost every point z in V_2. We put $\beta = \hat{\kappa} - \alpha$ on V_2. Then, by (2.3) and (4.4), we have

$$\alpha\bar{\kappa} = (\hat{\kappa} - \beta)\bar{\kappa} = k_0|\kappa|^2/(b\circ f) - \beta\bar{\kappa}\ ,$$
$$\mathrm{Re}\,\alpha\bar{\kappa} \geqq k_0|\kappa|^2/(b\circ f) - |\beta||\kappa| \geqq |\kappa|(k_0 - a)/2 > 0$$

a.e. in V_2. Thus we have

(4.12) $$(1 - |\kappa(z)|^2)\,\mathrm{Re}\,(\alpha(z)\bar{\kappa}(z))/|\kappa(z)| \geqq (1 - c_1^2)(k_0 - a)/2 > 0$$

a.e. in V_2. Therefore, by (4.11) and (4.12), we see that there exist positive numbers r_2 and s_2 such that

(4.13) $$|\nu(t)(w)| \leqq b(w) - r_2 t \quad \text{for every} \quad t \leqq s_2$$

and almost every point w in V_2^*.

By (4.8), (4.10) and (4.13), we have (4.6). This completes the proof of Lemma 4.

PROOF OF THEOREM 2. Since $||\mu|_D||_\infty > 0$, by Theorem 1 and Remark 1, we have only to prove (2.8) provided that μ is extremal within M_μ. Assume the contrary. Then the value a of the left hand side of (2.8) is less than $k_0 = ||\mu|_D||_\infty$. We note that $\hat{\kappa}|_{F(E_0)} = 0$. By the Hahn-Banach and Riesz representation theorems, there exists $\beta \in L_\infty(G)$ which vanishes on $F(E_0)$ and which satisfies the conditions $||\beta||_\infty = a$ and $L_G(\hat{\kappa})(\phi) = L_G(\beta)(\phi)$ for every $\phi \in A(G, \delta)$. Put $\alpha = \hat{\kappa} - \beta$. Then α is an element in $N(G, \delta)$ which vanishes on $F(E_0)$ and which satisfies

(4.14) $$||\hat{\kappa} - \alpha||_\infty = a < k_0\ .$$

By Proposition, there exists a curve $\tau(t) \in M_0(G, \delta)$ which satisfies (2.10) and (2.11). Let $\nu(t)$ be the complex dilatation of $F_{\tau(t)} \circ F$. Since $\tau(t)$ belongs to $M_0(G, \delta)$, we have $F_{\nu(t)}|_\sigma = F|_\sigma$. Thus, by (4.14) and Lemma 4, we see that $\nu(t) \in M_\mu$ and that $||\nu(t)|_D||_\infty < k_0$ for a sufficiently small t. Therefore μ is not extremal within M_μ. This contradiction proves Theorem 2.

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