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PERIODIC SOLUTIONS OF LINEAR NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS

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In this paper, we consider the neutral integrodifferential equations

$$(1) \qquad \frac{d}{dt}\left(Z(t) - \int_0^t D(t-s)Z(s)ds\right) = AZ(t) + \int_0^t C(t-s)Z(s)ds,\ Z(0) = I\,,$$

$$(2) \qquad \frac{d}{dt}\left(y(t) - \int_0^t D(t-s)y(s)ds\right) = Ay(t) + \int_0^t C(t-s)y(s)ds + f(t)\,,$$

$$(3) \qquad \frac{d}{dt}\left(x(t) - \int_{-\infty}^t D(t-s)x(s)ds\right) = Ax(t) + \int_{-\infty}^t C(t-s)x(s)ds + f(t)\,,$$

where $x, y \in R^n$, Z, A, C, D and I are $n \times n$ matrices with A constant, C and D continuous on $(-\infty, \infty)$, I the identity matrix, and $f: (-\infty, \infty) \to R^n$ is continuous.

Our aim is to get nice formula for periodic solutions of these equations, and so this paper can be considered as an extension of [2], [3] and [4].

Let us first consider the Volterra integral equations

$$(4) \qquad H(t) = I + \int_0^t E(t-s)H(s)ds\,, \qquad H \text{ is } n \times n\,,$$

$$(5) \qquad g(t) = F(t) + \int_0^t E(t-s)g(s)ds\,, \qquad g \in R^n\,,$$

$$(6) \qquad g(t) = F(t) + \int_{-\infty}^t E(t-s)g(s)ds\,, \qquad g \in R^n\,,$$

where E is an $n \times n$ matrix of functions continuous on $(-\infty, \infty)$, and $F: (-\infty, \infty) \to R^n$ is continuous.

REMARK. It is easy to see that $g(t)$ is a solution of (5) on $(-\infty, 0]$ if and only if $g^*(t^*) := g(-t^*)$, $t^* \geqq 0$, is a solution of

$$g^*(t^*) = F^*(t^*) + \int_0^{t^*} E^*(t^* - s)g^*(s)ds$$

on $[0, \infty)$, where $F^*(t^*) := F(-t^*)$, $E^*(s) := E(-s)$. This fact shows that if we have some properties of solutions of (5) on $[0, \infty)$, then we have similar properties on $(-\infty, 0]$.

The following theorem generalizes an analogous theorem of Burton (see [1], [2]).

THEOREM 1. *If $F(t)$ and $E(t)$ are continuous on $(-\infty, \infty)$, then*
(i) *there is one and only one solution $H(t)$ of* (4) *on $(-\infty, \infty)$,*
(ii) *there is one and only one solution $g(t)$ of* (5) *on $(-\infty, \infty)$,*
(iii) *the unique solution $H(t)$ of* (4) *is given by*

$$H(t) = I + \int_0^t G(s)ds \,, \tag{7}$$

where $G(t)$ is the $n\times n$ matrix solution of

$$G(t) = E(t) + \int_0^t E(t-s)G(s)ds \,. \tag{8}$$

Therefore, $H'(t)$ is continuous and satisfies

$$H'(t) = E(t) + \int_0^t E(t-s)H'(s)ds \,, \tag{8*}$$

(iv) *the unique solution $g(t)$ of* (5) *is*

$$g(t) = F(t) + \int_0^t H'(t-s)F(s)ds \,. \tag{9}$$

Moreover, if $F'(t)$ is continuous, then $g(t)$ can be rewritten as

$$g(t) = H(t)F(0) + \int_0^t H(t-s)F'(s)ds \,. \tag{10}$$

PROOF. Combining the analogous theorem of Burton [2, Theorem 1.5] with the remark above, we can show that the solution $g(t)$ of (5) exists and is unique on $(-\infty, \infty)$ and so does $H(t)$.

To prove (iii), we can show as in the cases (i) and (ii) that the solution $G(t)$ of (8) exists and is unique on $(-\infty, \infty)$. Then we have by substitution,

$$\begin{aligned}
H(t) &= I + \int_0^t G(s)ds \\
&= I + \int_0^t\Big(E(v) + \int_0^v E(v-s)G(s)ds\Big)dv \\
&= I + \int_0^t\Big(E(v) + \int_0^v E(s)G(v-s)ds\Big)dv \\
&= I + \int_0^t E(v)dv + \int_0^t\Big(\int_s^t E(s)G(v-s)dv\Big)ds \\
&= I + \int_0^t E(s)\Big(I + \int_0^{t-s} G(v)dv\Big)ds
\end{aligned}$$

$$= I + \int_0^t E(t-s)\Big(I + \int_0^s G(v)dv\Big)ds$$
$$= I + \int_0^t E(t-s)H(s)ds \ .$$

This shows that $H(t) = I + \int_0^t G(s)ds$ is a solution of (4).

To prove part (iv), we only need to show that

$$g(t) = F(t) + \int_0^t H'(t-s)F(s)ds$$

is a solution of (5). Indeed, we have by substitution,

$$\begin{aligned}
&F(t) + \int_0^t E(t-v)g(v)dv \\
&= F(t) + \int_0^t E(t-v)\Big(F(v) + \int_0^v H'(v-s)F(s)ds\Big)dv \\
&= F(t) + \int_0^t E(t-s)F(s)ds + \int_0^t\Big(\int_s^t E(t-v)H'(v-s)dv\Big)F(s)ds \\
&= F(t) + \int_0^t E(t-s)F(s)ds + \int_0^t\Big(\int_0^{t-s} E(t-s-u)H'(u)du\Big)F(s)ds \\
&= F(t) + \int_0^t\Big(E(t-s) + \int_0^{t-s} E(t-s-u)H'(u)du\Big)F(s)ds \\
&= F(t) + \int_0^t H'(t-s)F(s)ds \quad \text{(by (8*))} \\
&= g(t) \ .
\end{aligned}$$

This proves Theorem 1.

Following Burton [2] and Miller [5], we can also get the following theorem.

THEOREM 2. *If $F(t+T) = F(t)$ for some $T > 0$, and if $g(t)$ is a bounded solution of* (5) *on* $[0, \infty)$ *with $E \in L^1[0, \infty)$, then there is a sequence of positive integers $\{n_j\}$, $n_j \to \infty$ as $j \to \infty$, such that $\{g(t + n_jT)\}$ converges uniformly on compact subsets of $(-\infty, \infty)$ to a function $g^*(t)$ which is a solution of* (6).

Burton [2, p. 1.15] asserted that if H and $E \in L^1[0, \infty)$ and $H(t) \to 0$ as $t \to \infty$, then $g(t + n_jT)$ converges to $\int_{-\infty}^t H(t-s)F'(s)ds = g^*(t)$ which is a periodic solution of (6). But this is not consistent with his assumptions. For, $E \in L^1[0, \infty)$ and $H(t) \to 0$ imply $\int_0^t E(t-s)H(s)ds \to 0$, which implies

$$H(t) = I + \int_0^t E(t-s)H(s)ds \to I \neq 0 \quad \text{as} \quad t \to \infty ,$$

a contradiction.

Our next results improve this situation and put the ideas in the correct context.

THEOREM 3. *Let $F(t)$ be T-periodic with F' continuous, and let $E \in L^1[0, \infty)$. If there is a T-periodic matrix $H^*(t)$ such that $H(t)-H^*(t) \in L^1[0, \infty)$ with $H(t) - H^*(t) \to 0$ as $t \to \infty$, and that $\int_0^t H^*(t-s)F'(s)ds$ is T-periodic, then* (6) *has a T-periodic solution*

$$g^*(t) = H^*(t)F(0) + \int_0^t H^*(t-s)F'(s)ds + \int_{-\infty}^t (H(t-s) - H^*(t-s))F'(s)ds .$$

PROOF. For

$$\begin{aligned} g(t) &= H(t)F(0) + \int_0^t H(t-s)F'(s)ds \\ &= (H(t) - H^*(t))F(0) + H^*(t)F(0) + \int_0^t H^*(t-s)F'(s)ds \\ &\quad + \int_0^t (H(t-s) - H^*(t-s))F'(s)ds , \end{aligned}$$

we have that $g(t)$ is bounded on $[0, \infty)$ under the assumptions of the theorem. Then by Theorem 2, we have

$$\begin{aligned} g(t+n_jT) &= (H(t+n_jT) - H^*(t+n_jT))F(0) + H^*(t+n_jT)F(0) \\ &\quad + \int_0^t H^*(t-s)F'(s)ds + \int_{-n_jT}^t (H(t-s) - H^*(t-s))F'(s)ds \\ &\to H^*(t)F(0) + \int_0^t H^*(t-s)F'(s)ds \\ &\quad + \int_{-\infty}^t (H(t-s) - H^*(t-s))F'(s)ds = g^*(t) , \end{aligned}$$

which is a T-periodic solution of (6).

REMARK. In this case, $H^*(t)$ is a T-periodic solution of

$$H^*(t) = I + \int_{-\infty}^t E(t-s)H^*(s)ds .$$

THEOREM 4. *Suppose that $F(t)$ is T-periodic with $E \in L^1[0, \infty)$ and that there is a T-periodic matrix $H'^*(t)$ such that $H'(t) - H'^*(t) \in L^1[0, \infty)$ and that $\int_0^t H'^*(t-s)F(s)ds$ is T-periodic. Then* (6) *has a T-periodic* *solution*

$$g^*(t) = F(t) + \int_0^t H'^*(t-s)F(s)ds + \int_{-\infty}^t (H'(t-s) - H'^*(t-s))F(s)ds \; .$$

REMARK. In this case, if $E(t) \to 0$ as $t \to \infty$, then $H'^*(t)$ is a T-periodic solution of

$$H'^*(t) = \int_{-\infty}^t E(t-s)H'^*(s)ds \; .$$

EXAMPLE 1. Consider the scalar integral equations

$$H(t) = 1 + \int_0^t e^{-(t-s)}(3\cos(t-s) + \sin(t-s) - 2)H(s)ds \; , \tag{11}$$

$$g(t) = \sin 2t + \int_{-\infty}^t e^{-(t-s)}(3\cos(t-s) + \sin(t-s) - 2)g(s)ds \; . \tag{12}$$

It is easy to see that the unique solution $H(t)$ of (11) is

$$H(t) = (e^{-2t} + 7\sin t - \cos t)/5 + 1 \; ,$$

and that all the conditions of Theorem 3 with $H^*(t) = (7\sin t - \cos t)/5 + 1$ hold. Then (12) has a periodic solution

$$\begin{aligned} g^*(t) &= \int_0^t ((7\sin(t-s) - \cos(t-s))/5 + 1)(2\cos 2s)ds \\ &\quad + \int_{-\infty}^t (1/5)e^{-2(t-s)}(2\cos 2s)ds \\ &= (28\cos t + 4\sin t - 25\cos 2t + 25\sin 2t)/30 \; . \end{aligned}$$

Moreover, it is easy to verify that for each $A, B \in R$, $g(t) = \mathrm{A}\cos t + B\sin t$ is a periodic solution of

$$g(t) = \int_{-\infty}^t e^{-(t-s)}(3\cos(t-s) + \sin(t-s) - 2)g(s)ds \; .$$

Then we have that the periodic solutions of (12) are

$$g^*(t) = \mathrm{A}\cos t + B\sin t + (5/6)(\sin 2t - \cos 2t) \; ,$$

where A and B are arbitrary constants.

For this example, Theorem 4 is also applicable to (12), where

$$H'^*(t) = (7\cos t + \sin t)/5 \; .$$

EXAMPLE 2. Consider the scalar integral equations

$$H(t) = 1 + \int_0^t (3 - 2(t-s))e^{-2(t-s)} H(s)ds\ , \tag{13}$$

$$g(t) = \cos t + \int_{-\infty}^t (3 - 2(t-s))e^{-2(t-s)} g(s)ds\ . \tag{14}$$

It is easy to see that the unique solution $H(t)$ of (13) is

$$H(t) = 4t + e^{-t}\ ,$$

and then we have

$$H'(t) - 4 = -e^{-t} \in L^1[0, \infty)\ .$$

Thus, all the conditions of Theorem 4 with $H'^*(t) = 4$ hold. Then (14) has a periodic solution

$$g^*(t) = \cos t + 4 \sin t + \int_{-\infty}^t (-e^{-(t-s)}) \cos s ds = (\cos t + 7 \sin t)/2\ .$$

Note that $H(t) - H^*(t) \notin L^1[0, \infty)$ for all 2π-periodic $H^*(t)$ and that Theorem 3 is not applicable. This fact makes a difference between Theorem 3 and Theorem 4.

Our next result concerns the fundamental properties of solutions of (1) and (2).

THEOREM 5. *There exists a unique matrix solution $Z(t)$ of* (1) *on* $(-\infty, \infty)$ *and for each $y_0 \in R^n$ there is a unique solution $y(t) = y(t, 0, y_0)$ of* (2) *on* $(-\infty, \infty)$ *with*

$$y(t) = Z(t)y_0 + \int_0^t Z(t-s)f(s)ds\ .$$

PROOF. Note that (1) and (2) are equivalent to the integral equations

$$Z(t) = I + \int_0^t E(t-s)Z(s)ds\ ,$$

and

$$y(t) = F(t) + \int_0^t E(t-s)y(s)ds$$

respectively, where $F(t) = y_0 + \int_0^t f(s)ds$ and $E(t) = A + D(t) + \int_0^t C(s)ds$.
Now, our assertions follow from Theorem 1 directly.

THEOREM 6. *Let $C, D \in L^1[0, \infty)$ and $f(t + T) = f(t)$ for some $T > 0$. If $y(t) = y(t, 0, y_0)$ is a bounded solution of* (2) *on* $[0, \infty)$, *then there is a sequence of positive integers $\{n_j\}$, $n_j \to \infty$ as $j \to \infty$, such that $\{y(t + n_jT)\}$ converges uniformly on compact subsets of $(-\infty, \infty)$ to a function $x^*(t)$ which is a solution of* (3).

Note that $C, D \in L^1[0, \infty)$ does not imply $E(t) = A + D(t) + \int_0^t C(s)ds \in L^1[0, \infty)$, and so this theorem can be considered as a counterpart to Theorem 2 above.

PROOF OF THEOREM 6. Let $C, D \in L^1[0, \infty)$, and let $y(t)$ be a bounded solution of (2) on $[0, \infty)$. We want to show that $\{y(t + nT): n = 1, 2, \cdots\}$ is equicontinuous and uniformly bounded on any fixed interval $[-k, k]$.

For $t_2 \geqq t_1 \geqq -nT$, we integrate (2) from $t_1 + nT$ to $t_2 + nT$ and get

$$\begin{aligned} &y(t_2 + nT) - y(t_1 + nT) \\ &\quad = \int_0^{t_2+nT} D(t_2 + nT - s)y(s)ds - \int_0^{t_1+nT} D(t_1 + nT - s)y(s)ds \\ &\qquad + \int_{t_1+nT}^{t_2+nT}\left(Ay(t) + \int_0^t C(t - s)y(s)ds + f(t)\right)dt \,. \end{aligned}$$

$y(t)$ and $f(t)$ are bounded, hence there exists an M with $|f(t)| \leqq M$, $|y(t)| \leqq M$ for $t \geqq 0$. Moreover, since $C \in L^1[0, \infty)$, we have $\int_0^\infty |C(s)|ds = N < \infty$. Thus

$$\int_{t_1+nT}^{t_2+nT}\left|Ay(t) + \int_0^t C(t - s)y(s)ds + f(t)\right|dt \leqq M_1|t_2 - t_1| \,,$$

where $M_1 = M(|A| + 1 + N)$. Moreover, since $D \in L^1[0, \infty)$, for any $\varepsilon > 0$, there is a $k > 0$ such that

$$\int_t^\infty |D(s)|ds < \varepsilon/8M \quad \text{for} \quad t \geqq k \,,$$

and so

$$\int_k^\infty |D(t_2 - t_1 + v) - D(v)|dv < \varepsilon/4M \,.$$

By the continuity to D, there exists a $\delta_1 > 0$ such that $v \in [0, k]$ and $0 \leqq t_2 - t_1 \leqq \delta_1$ imply

$$|D(t_2 - t_1 + v) - D(v)| < \varepsilon/4kM$$

and

$$\int_0^{t_2-t_1} |D(v)|dv < \varepsilon/4M \,.$$

Thus

$$\begin{aligned} &\left|\int_0^{t_2+nT} D(t_2 + nT - s)y(s)ds - \int_0^{t_1+nT} D(t_1 + nT - s)y(s)ds\right| \\ &\quad \leqq \int_0^{t_1+nT} |D(t_2 + nT - s) - D(t_1 + nT - s)|\,|y(s)|ds \\ &\qquad + \int_{t_1+nT}^{t_2+nT} |D(t_2 + nT - s)|\,|y(s)|ds \end{aligned}$$

$$\leqq M\int_0^k |D(t_2 - t_1 + v) - D(v)|dv + M\int_k^\infty |D(t_2 - t_1 + v) - D(v)|dv$$
$$+ M\int_0^{t_2-t_1} |D(v)|dv$$
$$\leqq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4$$

if $0 \leqq t_2 - t_1 \leqq \delta_1$. Let $\delta = \min(\delta_1, \varepsilon/4M_1)$. Then we have

$$|y(t_2 + nT) - y(t_1 + nT)| \leqq 3\varepsilon/4 + \varepsilon/4 = \varepsilon$$

if $0 \leqq t_2 - t_1 \leqq \delta$. Obviously

$$|y(t + nT)| \leqq M \qquad \text{for} \quad n = 1, 2, \cdots .$$

This implies that $\{y(t + nT)\}$ is equicontinuous and uniformly bounded on any fixed interval $[-k, k]$, $k = 1, 2, \cdots$. Thus it contains a subsequence $\{y(t + n_jT)\}$ converging uniformly on $[-1, 1]$, which contains a subsequence converging uniformly on $[-2, 2]$. In this way we obtain a subsequence, say $\{y(t + n_jT)\}$ again, converging uniformly on any fixed interval $[-k, k]$ to a continuous function $x^*(t)$.

Now, we want to show that $x^*(t)$ is a solution of (3). Integrating (2) from n_jT to $t + n_jT$, we have

$$\begin{aligned}
&y(t + n_jT) - y(n_jT) \\
&\quad = \int_0^{t+n_jT} D(t + n_jT - s)y(s)ds - \int_0^{n_jT} D(n_jT - s)y(s)ds \\
&\qquad + \int_{n_jT}^{t+n_jT}\Big(Ay(v) + \int_0^v C(v - s)y(s)ds + f(v)\Big)dv \\
&\quad = \int_{-n_jT}^{t} D(t - v)y(v + n_jT)dv - \int_{-n_jT}^{0} D(-v)y(v + n_jT)dv \\
&\qquad + \int_0^t\Big(Ay(u + n_jT) + \int_{-n_jT}^{u} C(u - v)y(v + n_jT)dv + f(u)\Big)du .
\end{aligned}$$

Since $C, D \in L^1[0, \infty)$, by Lebesgue's dominated convergence theorem, letting $j \to \infty$, we have

$$\begin{aligned}
x^*(t) - x^*(0) &= \int_{-\infty}^{t} D(t - v)x^*(v)dv - \int_{-\infty}^{0} D(-v)x^*(v)dv \\
&\quad + \int_0^t\Big(Ax^*(u) + \int_{-\infty}^{u} C(u - v)x^*(v)dv + f(u)\Big)du .
\end{aligned}$$

Therefore by differentiation, we have

$$\frac{d}{dt}\Big(x^*(t) - \int_{-\infty}^{t} D(t - v)x^*(v)dv\Big) = Ax^*(t) + \int_{-\infty}^{t} C(t - v)x^*(v)dv + f(t) ,$$

and so the limit function $x^*(t)$ is a solution of (3).

Our next theorem can be considered as a counterpart of Theorem 3 above.

THEOREM 7. *Suppose that* $C, D \in L^1[0, \infty)$ *and* $f(t+T) = f(t)$. *If there is a T-periodic matrix* $Z^*(t)$ *such that* $Z(t) - Z^*(t) \in L^1[0, \infty)$, $Z(t) - Z^*(t) \to 0$ *as* $t \to \infty$, *and that* $\int_0^t Z^*(t-s)f(s)ds$ *is T-periodic, then* (3) *has a T-periodic solution*

$$x^*(t) = Z^*(t)y_0 + \int_0^t Z^*(t-s)f(s)ds + \int_{-\infty}^t (Z(t-s) - Z^*(t-s))f(s)ds\ ,$$

where $y_0 \in R^n$ *is an arbitrary constant.*

The proof of this theorem is very similar to that of Theorem 3 and therefore is omitted.

EXAMPLE 3. Consider the scalar equations

$$\frac{d}{dt}\left(Z(t) - \int_0^t e^{-4(t-s)}Z(s)ds\right) = -Z(t) + \int_0^t e^{-4(t-s)}Z(s)ds,\ Z(0) = 1\ , \tag{15}$$

$$\frac{d}{dt}\left(x(t) - \int_{-\infty}^t e^{-4(t-s)}x(s)ds\right) = -x(t) + \int_{-\infty}^t e^{-4(t-s)}x(s)ds + 2\cos t + \sin t\ . \tag{16}$$

Here $C(t) = D(t) = e^{-4t} \in L^1[0, \infty)$ with $f(t) = 2\cos t + \sin t$ periodic.

It is not difficult to show that

$$Z(t) = (3/2)e^{-t} - (1/2)e^{-3t}$$

is the unique solution of (15) and that all the conditions of Theorem 7 with $Z^*(t) = 0$ hold. Then (16) has a periodic solution

$$\begin{aligned} x^*(t) &= \int_{-\infty}^t Z(t-s)f(s)ds \\ &= \int_{-\infty}^t ((3/2)e^{-(t-s)} - (1/2)e^{-3(t-s)})(2\cos s + \sin s)ds \\ &= 2\sin t + (1/2)\cos t. \end{aligned}$$

EXAMPLE 4. Consider the scalar equations

$$Z'(t) = Z(t) - \int_0^t e^{-(t-s)}(\cos(t-s) + 2\sin(t-s))Z(s)ds,\ Z(0) = 1\ , \tag{17}$$

$$x'(t) = x(t) - \int_{-\infty}^t e^{-(t-s)}(\cos(t-s) + 2\sin(t-s))x(s)ds + \sin 2t\ . \tag{18}$$

Here $C(t) = -e^{-t}(\cos t + 2\sin t) \in L^1[0, \infty)$ and $D(t) \equiv 0$.

It is easy to see that the unique solution $Z(t)$ of (17) is

$$Z(t) = (e^{-t} + \cos t + 3\sin t)/2$$

and that all the conditions of Theorem 7 with $Z^*(t) = (\cos t + 3\sin t)/2$ hold. Then (18) has a periodic solution

$$\begin{aligned} x^*(t) &= k(\cos t + 3\sin t) + \int_0^t (1/2)(\cos(t-s) + 3\sin(t-s))\sin 2s ds \\ &\quad + \int_{-\infty}^t (1/2)e^{-(t-s)}(\sin 2s)ds \\ &= (3k+1)(\cos t + 3\sin t)/3 - 2(3\sin 2t + 4\cos 2t)/15 , \end{aligned}$$

where k is an arbitrary constant.

Moreover, it is easy to see that for each $a, b \in R$, $x(t) = a\cos t + b\sin t$ is a periodic solution of

$$x'(t) = x(t) - \int_{-\infty}^t e^{-(t-s)}(\cos(t-s) + 2\sin(t-s))x(s)ds .$$

So, the periodic solutions of (18) are

$$x^*(t) = a\cos t + b\sin t - 2(3\sin 2t + 4\cos 2t)/15 ,$$

where a, b are arbitrary constants.

The following theorem can be considered as a counterpart of Theorem 4 above.

THEOREM 8. *Let $C, D \in L^1[0, \infty)$, and let $F(t) = y_0 + \int_0^t f(s)ds$ be T-periodic. If there is a T-periodic $n\times n$ matrix $Z'^*(t)$ such that $Z'(t) - Z'^*(t) \in L^1[0, \infty)$ and that $\int_0^t Z'^*(t-s)F(s)ds$ is T-periodic, then*

$$x^*(t) = F(t) + \int_0^t Z'^*(t-s)F(s)ds + \int_{-\infty}^t (Z'(t-s) - Z'^*(t-s))F(s)ds$$

is a T-periodic solution of (3).

The proof of this theorem is quite similar to that before and is omitted.

For Example 4, Theorem 8 is also applicable to (18) with $Z'^*(t) = (3\cos t - \sin t)/2$.

EXAMPLE 5. Consider the scalar equations

$$\frac{d}{dt}\left(Z(t) - \int_0^t 4e^{-2(t-s)}Z(s)ds\right) = -Z(t) + \int_0^t 4(t-s)e^{-2(t-s)}Z(s)ds , \tag{19}$$

$$Z(0) = 1 ,$$

$$(20)\qquad \frac{d}{dt}\Big(x(t) - \int_{-\infty}^{t} 4e^{-2(t-s)}x(s)ds\Big) = -x(t) + \int_{-\infty}^{t} 4(t-s)e^{-2(t-s)}x(s)ds + \sin t\ .$$

Here $C(t) = 4te^{-2t} \in L^1[0, \infty)$, $D(t) = 4e^{-2t} \in L^1[0, \infty)$, and $F(t) = y_0 + \int_0^t \sin s ds = (y_0 + 1) - \cos t$ is 2π-periodic.

It is easy to see that the unique solution $Z(t)$ of (19) is

$$Z(t) = 4t + e^{-t}\ .$$

Then we have

$$Z'(t) - 4 = -e^{-t} \in L^1[0, \infty)\ .$$

Let $Z'^*(t) = 4$, and let $y_0 = -1$. Then

$$\int_0^t Z'^*(t-s)F(s)ds = \int_0^t 4(-\cos s)ds = -4\sin t\ ,$$

which is 2π-periodic. Thus, all the conditions of Theorem 8 hold, and (20) has a periodic solution

$$\begin{aligned} x^*(t) &= -\cos t - 4\sin t + \int_{-\infty}^{t} (-e^{-(t-s)})(-\cos s)ds \\ &= -(7\sin t + \cos t)/2\ . \end{aligned}$$

We now consider the question of the existence of T-periodic solutions of (2).

THEOREM 9. *Suppose that $C, D, Z \in L^1[0, \infty)$ and $Z(t) \to 0$ as $t \to \infty$, and that $f(t)$ is T-periodic. Then*

(i) *all solutions of (2) approach a periodic solution of (3) as $t \to \infty$,*

(ii) *if (2) has a T-periodic solution $y^*(t)$, then $y^*(t)$ is unique and is also a T-periodic solution of (3).*

PROOF. (i) By Theorem 7 with $Z^* = 0$, (3) has a T-periodic solution

$$x^*(t) = \int_{-\infty}^{t} Z(t-s)f(s)ds\ .$$

For any solution $y(t)$ of (2), we have by Theorem 5

$$y(t) = Z(t)y(0) + \int_0^t Z(t-s)f(s)ds\ .$$

Then

$$\begin{aligned} y(t) - x^*(t) &= Z(t)y(0) - \int_{-\infty}^{0} Z(t-s)f(s)ds \\ &= Z(t)y(0) - \int_t^{\infty} Z(u)f(t-u)du \to 0 \quad \text{as} \quad t \to \infty\ , \end{aligned}$$

since $Z(t) \to 0$ as $t \to \infty$, $\int_t^\infty |Z(u)|du \to 0$ as $t \to \infty$, and f is bounded.

(ii) From (i) above, we have

$$y^*(t) - x^*(t) \to 0 \quad \text{as} \quad t \to \infty ,$$

which implies $y^*(t) = x^*(t)$, since $y^*(t)$ and $x^*(t)$ are both T-periodic.

THEOREM 10. *Suppose that all the conditions of Theorem 9 hold. Then*

(i) (2) *has a T-periodic solution if and only if*

$$\int_{-\infty}^0 (Z(t-s) - Z(t)Z(-s))f(s)ds \equiv 0 , \tag{21}$$

(ii) (2) *has a T-periodic solution for any continuous and T-periodic function $f(t)$ if and only if*

$$Z(t-s) \equiv Z(t)Z(-s) .$$

PROOF. For the proof we refer to [3].

In addition to Example 3, we consider the following scalar equation

$$\frac{d}{dt}\left(y(t) - \int_0^t e^{-4(t-s)}y(s)ds\right) = -y(t) + \int_0^t e^{-4(t-s)}y(s)ds + 2\cos t + \sin t. \tag{22}$$

It is easy to verify that (21) holds, that is,

$$\int_{-\infty}^0 \Big(Z(t-s) - Z(t)Z(-s)\Big)f(s)ds$$
$$= (e^{-t} + e^{-3t})\int_{-\infty}^0 (e^{3s} - e^{s})(2\cos s + \sin s)ds \equiv 0 .$$

Hence there is a periodic solution of (22) by Theorem 10 which must be equal to the periodic solution $x^*(t) = 2\sin t + (1/2)\cos t$ of (16) by Theorem 9.

Finally, we want to point out that (22) is reduced to

$$y(t) = (3/4)\int_0^t (e^{-4(t-s)} - 1)y(s)ds + y(0) + 2\sin t - \cos t + 1 ,$$

but Theorem 3 is not applicable, since $E(t) = e^{-4t} - 1 \notin L^1[0, \infty)$.

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