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FEJÉR-RIESZ INEQUALITIES FOR LOWER DIMENSIONAL SUBSPACES

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Introduction. The purpose of this note is to obtain inequalities of the Fejér-Riesz type for subspaces of the unit ball and the generalized half-plane in $\boldsymbol{C}^n$, $n \geqq 2$. Let $B = \{z \in \boldsymbol{C}^n \mid |z| < 1\}$ denote the unit ball in $\boldsymbol{C}^n$, where $|z|^2 = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$. We write L_{2k+1} for the space $\boldsymbol{R} \times \boldsymbol{C}^k \times \{0\} \times \cdots \times \{0\} \subset \boldsymbol{C}^n$ and L_{2k} for $\boldsymbol{C}^k \times \{0\} \times \cdots \times \{0\}$, where $\boldsymbol{R}$ means the real line in $\boldsymbol{C}$. We denote by B_m the unit ball in $\boldsymbol{R}^m$. Surface measures on ∂B and ∂B_m, respectively, will be denoted by $d\sigma$ and $d\sigma_m$. For $f \in H^p(B)$, $0 < p < +\infty$, we define

$$\|f\|_p^p = \sup_{0<r<1} \int_{\partial B} |f(r\zeta)|^p d\sigma(\zeta) .$$

In the following, (1) is known for $k = n-1$ ([1]), and (2) is also known for $k = n-1$ ([6, 7.2.4, (b)]). A similar inequality is given in [5], where the subspace is $\boldsymbol{R} \times \boldsymbol{R}$ in $\boldsymbol{C}^2$. Moreover, analogous inequalities are known for harmonic functions on the unit ball in $\boldsymbol{R}^n$ ([4]).

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THEOREM 1. *There is a constant C, depending on n, k, such that the following holds for every $f \in H^p(B)$, $0 < p < +\infty$:*

$$(1) \qquad \int_{B \cap L_{2k+1}} |f(z)|^p (1-|z|^2)^{n-k-1} dz \leqq C\|f\|_p^p , \quad 0 \leqq k \leqq n-1 .$$

There is a constant C such that

$$(2) \qquad \int_{B \cap L_{2k}} |f(z)|^p (1-|z|^2)^{n-k-1} dz \leqq C\|f\|_p^p , \quad 1 \leqq k \leqq n-1 .$$

$n-k-1$ is the smallest exponent in each case. The best possible constant $C_0(n; 2k+1)$ for (1) satisfies

$$\frac{\Gamma(n/2)\Gamma(n-k)}{2\Gamma(n/2+1/2)\pi^{n-k-1/2}} \leqq C_0(n; 2k+1) \leqq \frac{\Gamma(n-k)}{2\pi^{n-k-1}} .$$

For (2), $C_0(n; 2k) = (2\pi^{n-k})^{-1}\Gamma(n-k)$ is the best possible constant.

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Let $D = \{(z_1, z') \in \boldsymbol{C} \times \boldsymbol{C}^{n-1} \mid \operatorname{Im} z_1 - |z'|^2 > 0\}$ and let

$$\|f\|_p^p = \sup_{y_1>0} \int_{\boldsymbol{R}\times \boldsymbol{C}^{n-1}} |f(x_1 + iy_1 + i|z'|^2, z')|^p dx_1 dz'$$

for $f \in H^p(D)$, $0 < p < +\infty$. In the following, $C_0(n; j)$ denote the constants in Theorem 1. (3) is known for $k = n - 1$ ([2]). Again, analogous inequalities for harmonic functions on $\boldsymbol{R}^n \times (0, +\infty)$ are given in [3].

THEOREM 2. *For every $x_1 \in \boldsymbol{R}$, every $f \in H^p(D)$, $0 < p < +\infty$, and $0 \leqq k \leqq n - 1$, we have*

$$\int_0^{+\infty} dy_1 \int_{L_{2k}} |f(x_1 + iy_1 + i|z'|^2, z')|^p y_1^{n-k-1} dz' \leqq C_0(n; 2k + 1)\|f\|_p^p . \tag{3}$$

The following holds for $f \in H^p(D)$, $0 < p < +\infty$, and $1 \leqq k \leqq n - 1$:

$$\int_0^{+\infty} dy_1 \int_{L_{2k-1}} |f(x_1 + iy_1 + i|z'|^2, z')|^p y_1^{n-k-1} dx_1 dz' \leqq 2C_0(n; 2k)\|f\|_p^p . \tag{4}$$

The exponent $n - k - 1$ is unique in each case.

1. **Proof of Theorem 1.** We write $f_r(z) = f(rz)$, $0 < r < 1$. To prove (1), we may suppose that $0 \leqq k \leqq n - 2$, since the case $k = n - 1$ is included in [1]. Let $f \in H^p(B)$. Then for an arbitrarily fixed point $(x_1, z') \in B_{2k+1}$, where $x_1 \in \boldsymbol{R}$ and $z' = (z_2, \cdots, z_{k+1})$, the function $|f_r(x_1, z', (1 - x_1^2 - |z'|^2)^{1/2} z'')|^p$ of the variable $z'' = (z_{k+2}, \cdots, z_n)$ is plurisubharmonic in a neighborhood of $\bar{B}_{2n-2k-2}$, hence

$$|f_r(x_1, z', 0'')|^p \leqq |B_{2n-2k-2}|^{-1} \int_{B_{2n-2k-2}} |f_r(x_1, z', (1 - x_1^2 - |z'|^2)^{1/2} z'')|^p dz'' .$$

By Fubini's theorem we can see that

$$\begin{aligned} I_r := \int_{B_{2k+1}} & |f_r(x_1, z', 0'')|^p (1 - x_1^2 - |z'|^2)^{n-k-1} dx_1 dz' \\ & \leqq |B_{2n-2k-2}|^{-1} \int_{B_{2n-1}} |f_r(x_1, z', z'')|^p dx_1 dz' dz'' . \end{aligned}$$

[1, Theorem 1, (3)] shows that the right side does not exceed $2^{-1}\Gamma(n - k)\pi^{-(n-k-1)}\|f\|_p^p$. From

$$I_r = r^{-2n+1} \int_{|(x_1, z')|<r} |f(x_1, z', 0'')|^p (r^2 - x_1^2 - |z'|^2)^{n-k-1} dx_1 dz'$$

we obtain (1) by letting $r \to 1$. Next, for $N \geqq 1$, $\beta > -1$, $\rho > 0$, note that

$$\int_{|x|<\rho, x \in \boldsymbol{R}^N} (\rho^2 - |x|^2)^\beta dx = A(N, \beta)\rho^{N+2\beta} , \tag{5}$$

where $A(N, \beta) = \Gamma^{-1}(N/2 + \beta + 1)\Gamma(\beta + 1)\pi^{N/2}$. Let $\alpha > -1$, $c < n$. Then, for $1 \leqq k \leqq n - 1$, by Fubini's theorem and (5),

$$\begin{aligned} I_{2k+1} := & \int_{B \cap L_{2k+1}} |1 - z_1|^{-c}(1 - |z|^2)^\alpha dz \\ = & A(2k, \alpha) \int_{-1}^{1} (1 + x_1)^{\alpha+k}(1 - x_1)^{\alpha+k-c} dx_1 . \end{aligned}$$

This is valid for $k = 0$ with $A(0, \alpha) = 1$. Now suppose that $-1 < \alpha < n - k - 1$. Then, for $c = \alpha + k + 1$, the function $(1 - z_1)^{-c}$ belongs to $H^1(B)$ and we have $I_{2k+1} = +\infty$. Thus $n - k - 1$ is the smallest exponent that guarantees (1). Finally, take $c < n$ and put $\alpha = n - k - 1$, $0 \leqq k \leqq n - 1$. Then, by using Legendre's duplication formula, we have

$$I_{2k+1} = A(2k, n - k - 1)\frac{2^{2n-c-1}\Gamma(n - c)\Gamma(n)}{\Gamma(2n - c)} = \frac{\Gamma(n - c)\Gamma(n - k)\pi^{k+1/2}}{\Gamma(n - c/2)\Gamma(n - c/2 + 1/2)} .$$

On the other hand, we know from [6, p. 54] that

$$J_n := \int_{\partial B} |1 - \zeta_1|^{-c} d\sigma(\zeta) = \frac{2\Gamma(n - c)\pi^n}{\Gamma^2(n - c/2)} ,$$

hence

$$\frac{\Gamma(n - c/2)\Gamma(n - k)}{2\Gamma(n - c/2 + 1/2)\pi^{n-k-1/2}} \leqq C_0(n;\ 2k + 1) \leqq \frac{\Gamma(n - k)}{2\pi^{n-k-1}} . \tag{6}$$

Letting $c \to n$, we obtain the desired estimate. Now we shall prove (2) using a formula which will be treated in the next section. Write z in the form (z', z'') with $z' = (z_1, \cdots, z_k)$, $z'' = (z_{k+1}, \cdots, z_n)$. The plurisubharmonicity of $|f_r(z', (1 - |z'|^2)^{1/2}z'')|^p$ as a function of z'' implies that

$$|f_r(z', 0'')|^p \leqq |\partial B_{2n-2k}|^{-1} \int_{\partial B_{2n-2k}} |f_r(z', (1 - |z'|^2)^{1/2}\zeta'')|^p d\sigma_{2n-2k}(\zeta'') ,$$

hence by the formula (7) we have

$$\int_{B_{2k}} |f_r(z', 0'')|^p(1 - |z'|^2)^{n-k-1} dz' \leqq |\partial B_{2n-2k}|^{-1} \int_{\partial B} |f_r(\zeta)|^p d\sigma(\zeta) .$$

Equality holds for $f = 1$, so the constant is best possible. Next suppose that $-1 < \alpha < n - k - 1$ and let $c = \alpha + k + 1$. Then, by (5),

$$\int_{B \cap L_{2k}} |1 - w_1|^{-c}(1 - |w|^2)^\alpha dw = C(k, \alpha) \int_{B_2} |1 - w_1|^{-c}(1 - |w_1|^2)^{\alpha+k-1} dw_1 ,$$
$$1 \leqq k \leqq n - 1 .$$

Letting $w_1 = (z_1 + i)^{-1}(z_1 - i)$, where $z_1 = x_1 + iy_1$ with $x_1 \in \boldsymbol{R}$, $y_1 > 0$, we can see that

$$\int_{B_2} |1 - w_1|^{-c}(1 - |w_1|^2)^{\alpha+k-1} dw_1$$
$$= C'(k, \alpha)\int_0^{+\infty} dy_1 \int_{-\infty}^{+\infty} (x_1^2 + (y_1 + 1)^2)^{-(\alpha+k+1)/2} y_1^{\alpha+k-1} dx_1 = +\infty .$$

REMARK. If, formally, we let $n = 1$, $k = 0$ in (6), then $C_0(1, 1)$ becomes 1/2. It may be worth mentioning that this is actually so. Indeed, letting $e^{i\theta} = (t + i)^{-1}(t - i)$, $-\infty < t < +\infty$, in

$$J_1 = \int_0^{2\pi} |1 - e^{i\theta}|^{-c} d\theta , \quad c < 1 ,$$

we see from $d\theta = 2(t^2 + 1)^{-1}dt$ that $J_1 = 2^{1-c}\Gamma^{-1}(1 - c/2)\Gamma((1 - c)/2)\pi^{1/2}$ and, together with $I_1 = 2^{1-c}(1 - c)^{-1}$, we get (6).

2. An integral formula. Let $n \geqq 2$, $1 \leqq k \leqq n - 1$. For $f \in L^1(\partial B, d\sigma)$ we shall show that

$$(7) \qquad \int_{\partial B} f(\zeta) d\sigma(\zeta)$$
$$= \int_{B_{2k}} (1 - |z'|^2)^{n-k-1} dz' \int_{\partial B_{2n-2k}} f(z', (1 - |z'|^2)^{1/2}\zeta'') d\sigma_{2n-2k}(\zeta'') ,$$

where $z' = (z_1, \cdots, z_k)$, $\zeta'' = (\zeta_{k+1}, \cdots, \zeta_n)$. Note that, if we employ normalized measures, this yields 1.4.4 and 1.4.7, (2) of [6]. In the rest of this section, Lebesgue measure on $\boldsymbol{R}^j$ will be denoted by $d\nu_j$.

LEMMA. *Let* $m \geqq 3$, $1 \leqq k \leqq m - 2$. *Let* $f \in L^1(\partial B_m, d\sigma_m)$. *Then*

$$\int_{\partial B_m} f(x) d\sigma_m(x)$$
$$= \int_{B_k} (1 - |x'|^2)^{(m-k-2)/2} d\nu_k(x') \int_{\partial B_{m-k}} f(x', (1 - |x'|^2)^{1/2}x'') d\sigma_{m-k}(x'') ,$$

where $x' = (x_1, \cdots, x_k)$, $x'' = (x_{k+1}, \cdots, x_m)$.

PROOF. Let Ψ be the usual parametrization for ∂B_m defined by $\Psi(\theta) = \Psi(\theta_1, \cdots, \theta_{m-1}) = (x_1, \cdots, x_m)$, where

$$x_1 = \cos\theta_1 ,$$
$$x_j = \sin\theta_1 \cdots \sin\theta_{j-1}\cos\theta_j , \quad 2 \leqq j \leqq m - 1 ,$$
$$x_m = \sin\theta_1 \cdots \sin\theta_{m-2}\sin\theta_{m-1} ,$$

$0 < \theta_1, \cdots, \theta_{m-2} < \pi$, $0 \leqq \theta_{m-1} < 2\pi$. Then $d\sigma_m = \prod_{j=1}^{m-2} (\sin\theta_j)^{m-j-1} d\theta_1 \cdots d\theta_{m-1} =: J(\theta)d\theta$. Put $\Psi_k(\theta') = (x_1, \cdots, x_k)$, $\theta' = (\theta_1, \cdots, \theta_k)$. Then the mapping Ψ_k: $(0, \pi)^k \to B_k$ gives a parametrization for the ball B_k, and $d\nu_k = \prod_{j=1}^{k} (\sin\theta_j)^{k-j+1} d\theta_1 \cdots d\theta_k =: J'(\theta')d\theta'$. Finally define the mapping ψ by

$$\psi(\theta'') = (y_{k+1}, \cdots, y_m) , \quad \theta'' = (\theta_{k+1}, \cdots, \theta_{m-1}) \in (0, \pi)^{m-k-2} \times [0, 2\pi) ,$$

where $y_{k+1} = \cos\theta_{k+1}, \cdots, y_m = \sin\theta_{k+1}\cdots\sin\theta_{m-1}$. Then ψ gives a parametrization for ∂B_{m-k}, and $d\sigma_{m-k} = \prod_{j=1}^{m-k-2}(\sin\theta_{k+j})^{m-k-j-1}d\theta_{k+1}\cdots d\theta_{m-1} =: J''(\theta_{k+1}, \cdots, \theta_{m-2})d\theta''$. Note that $\prod_{j=1}^{k}\sin\theta_j = (1 - |\Psi_k(\theta')|^2)^{1/2}$, so $\Psi(\theta) = (\Psi_k(\theta'), (1 - |\Psi_k(\theta')|^2)^{1/2}\psi(\theta''))$. On the other hand, we can write

$$\begin{aligned} J(\theta) &= \left(\prod_{j=1}^{k}\sin\theta_j\right)^{m-k-2} J'(\theta')J''(\theta_{k+1}, \cdots, \theta_{m-2}) \\ &= (1 - |\Psi_k(\theta')|^2)^{(m-k-2)/2}J'(\theta')J''(\theta_{k+1}, \cdots, \theta_{m-2}) . \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\partial B_m} f(x)d\sigma_m(x) &= \int_{(0,\pi)^{m-2}\times[0,2\pi)} f(\Psi(\theta))J(\theta)d\theta \\ &= \int_{(0,\pi)^k} (1 - |\Psi_k(\theta')|^2)^{(m-k-2)/2}J'(\theta')d\theta' \int_{(0,\pi)^{m-k-2}\times[0,2\pi)} (*)d\theta'' , \end{aligned}$$

where $(*) = f(\Psi_k(\theta'), (1 - |\Psi_k(\theta')|^2)^{1/2}\psi(\theta''))J''(\theta_{k+1}, \cdots, \theta_{m-2})$.

3. Proof of Theorem 2. The Cayley transform $\Psi: D \to B$ is defined by $\Psi(z_1, \cdots, z_n) = (w_1, \cdots, w_n)$, where $w_1 = (z_1 + i)^{-1}(z_1 - i)$ and $w_j = 2z_j(z_1 + i)^{-1}$, $2 \leqq j \leqq n$. For $f \in H^p(D)$, $0 < p < +\infty$, there is a unique $g \in H^p(B)$ such that

$$f(z) = (g \circ \Psi)(z)(z_1 + i)^{-2n/p} , \quad z \in D , \tag{8}$$

and this correspondence determines an isomorphism of $H^p(D)$ onto $H^p(B)$. Moreover, $\|g\|_p^p = A(n)\|f\|_p^p$ for a constant $A(n)$ ([7]). We note here that $A(n) = 2^{2n-1}$, which is seen from a computation by letting $g = 1$. Now, to see (3), it suffices to assume that $x_1 = 0$. Take $f \in H^p(D)$. Then by (1) and (8)

$$I := \int_{B \cap L_{2k+1}} |g(w)|^p(1 - |w|^2)^{n-k-1}dw \leqq 2^{2n-1}C_0(n; 2k + 1)\|f\|_p^p .$$

Put $w = \Psi(z)$, where $z = (iy_1, z_2, \cdots, z_{k+1}, 0, \cdots, 0)$, $y_1 > |z'|^2$. Then Ψ maps $D \cap (i\boldsymbol{R} \times \boldsymbol{C}^k \times \{0\} \times \cdots \times \{0\})$ onto $B \cap L_{2k+1}$ and the Jacobian determinant is seen to be $2^{2k+1}(y_1 + 1)^{-2k-2}$. Note that $1 - |\Psi(z)|^2 = 4(y_1 - |z'|^2)(y_1 + 1)^{-2}$. Thus

$$\begin{aligned} I &= 2^{2n-1}\int_{y_1 > |z'|^2} |f(iy_1, z')|^p(y_1 - |z'|^2)^{n-k-1}dy_1dz' \\ &= 2^{2n-1}\int_0^{+\infty} dy_1 \int_{L_{2k}} |f(iy_1 + i|z'|^2, z')|^p y_1^{n-k-1}dz' , \end{aligned}$$

and this proves (3). To verify (4) it is enough to see that Ψ maps $D \cap L_{2k}$ onto $B \cap L_{2k}$ and that the Jacobian determinant is $2^{2k}|z_1 + i|^{-2k-2}$. Next we shall show that $n - k - 1$ is the unique admissible exponent. First, in (3), consider y_1^α with $\alpha > n - k - 1$, and let $c = \alpha + k + 1$. Then

$(z_1+i)^{-c}\in H^1(D)$ as is seen from (8), and changing variables we can see that, for $0\leqq k\leqq n-1$,

$$\int_0^{+\infty} dy_1\int_{C^k}(y_1+|z'|^2+1)^{-c}y_1^\alpha dz' = C\int_0^{+\infty}(y_1+1)^{-(\alpha+1)}y_1^\alpha dy_1 = +\infty\ .$$

In the case $\alpha<n-k-1$, put $c=\alpha+k+1$. Then $f(z):=z_1^{-c}(z_1+i)^{-2n+c}\in H^1(D)$ by (8). If $1\leqq k\leqq n-1$, we see that

$$\begin{aligned}\int_0^{+\infty} dy_1\int_{L_{2k}}|f(iy_1+i|z'|^2,z')|y_1^\alpha dz' &> C\int_0^1 dy_1\int_{|z'|<1}(y_1+|z'|^2)^{-c}y_1^\alpha dz'\\ &> C\int_0^1 y_1^{-1}dy_1 = +\infty\ .\end{aligned}$$

The case $k=0$ is straightforward. In (4) consider y_1^α with $\alpha>n-k-1$ and put $c=\alpha+k+1$. Then

$$\begin{aligned}\int_0^{+\infty} dy_1\int_{L_{2k-1}}(x_1^2+(y_1+|z'|^2+1)^2)^{-c/2}y_1^\alpha dx_1dz' &= C\int_0^{+\infty}(y_1+1)^{-(\alpha+1)}y_1^\alpha dy_1\\ &= +\infty\ .\end{aligned}$$

Finally, let $\alpha<n-k-1$ and put $c=\alpha+k+1$. We suppose that $2\leqq k\leqq n-1$, the case $k=1$ being similar. For $f(z)=z_1^{-c}(z_1+i)^{-2n+c}$ we have

$$\begin{aligned}I:=\int_0^{+\infty} dy_1\int_{L_{2k-1}}|f(x_1+iy_1+i|z'|^2,z')|y_1^\alpha dx_1dz'\\ > C\int_0^1 dy_1\int_{|z'|<1}dz'\int_{|x_1|<1}(x_1^2+(y_1+|z'|^2)^2)^{-c/2}y_1^\alpha dx_1\ ,\end{aligned}$$

where

$$\int_{-1}^1 (x_1^2+(y_1+|z'|^2)^2)^{-c/2}dx_1 > C(y_1+|z'|^2)^{1-c}\ ,$$

$$\int_{|z'|<1}(y_1+|z'|^2)^{1-c}dz' > Cy_1^{-\alpha-1}\ .$$

Thus $I=+\infty$.

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