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NECESSARY CONDITIONS FOR QUASIRADIAL FOURIER MULTIPLIERS

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0. In this paper we give necessary conditions for quasiradial functions $m \circ \rho$ to be Fourier multipliers in $M_p^q(\boldsymbol{R}^n)$. Here m is defined in $(0, \infty)$; ρ is an A_t-homogeneous distance function; that is, $\rho(x) > 0$, $x \neq 0$, and ρ is homogeneous with respect to the dilations $A_t = t^P$, $t > 0$: $\rho(A_t x) = t\rho(x)$. P is a real $n \times n$-matrix whose eigenvalues have positive real parts. The trace of P is denoted by ν.

Our results extend and inprove those of Gasper and Trebels [9] for radial multipliers. They can be used to produce counterexamples in many concrete cases without explicitly computing asymptotic expansions.

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1. Let us first introduce some notation. On S, the space of rapidly decreasing C^∞-functions the Fourier transform is defined by

$$F[f](\xi) = f\hat{\ }(\xi) = \int f(x) e^{-i\xi\cdot x} dx$$

(where the integration is extended over all of $\boldsymbol{R}^n$); by F^{-1} we denote its inverse. Let L^p be the standard Lebesgue spaces over $\boldsymbol{R}^n$ with norm $\|\cdot\|_p$. A tempered distribution $\mu \in S'$ is called a Fourier multiplier of type (p, q) if

$$\|\mu\|_{M_p^q} = \sup\{\|F^{-1}[\mu f\hat{\ }]\|_q / \|f\|_p ;\ 0 \neq f \in S\}$$

is finite. We set $M_p = M_p^p$. For standard properties of the M_p^q-spaces see Hörmander [10]. In particular, if $q \leqq 2$, M_p^q contains only locally integrable functions.

In order to formulate our results we need the notion of Besov spaces $B_{\alpha q}^p(\boldsymbol{R})$. Let χ be a nonnegative C^∞-function with support in $(1/2, 2)$ and $\sum_{k \in \boldsymbol{Z}} \chi(2^{-k}t) = 1$, $t > 0$. Let $\eta_k = F^{-1}[\chi(2^{-k}|\cdot|)]$, $k \geqq 1$; $\eta_0 = F^{-1}[1 - \sum_{k\geqq 1} \eta_k]$. Then $B_{\alpha q}^p$ is the space of all L^p-functions with finite norm

$$\|f\|_{B_{\alpha q}^p} = \Big(\sum_{k \geqq 0} 2^{k\alpha q} \|\eta_k * f\|_p^q\Big)^{1/q} .$$

For embedding properties and identification as smoothness spaces we refer to Bergh and Löfström ([2, ch. 6]).

We throughout work with functions g compactly supported in $\boldsymbol{R}_+ = (0, \infty)$. In this case we extend g to an even function $\tilde{g}$ in $\boldsymbol{R}$ and define

$$\|g\|_{B^p_{\alpha q}(\boldsymbol{R}_+)} = \|\tilde{g}\|_{B^p_{\alpha q}(\boldsymbol{R})} .$$

We set $\boldsymbol{R}^n_0 = \boldsymbol{R}^n \setminus \{0\}$. S^{n-1} denotes the sphere $\{x \in \boldsymbol{R}^n;\ |x| = 1\}$ with surface measure $d\theta$. φ will always be a C^∞-bump-function in $\boldsymbol{R}_+$. By c we denote positive constants which may vary in different occurences. Our main result is the following:

THEOREM 1. *Let $\rho \in C^\infty(\boldsymbol{R}^n_0)$ be an A_t-homogeneous distance function. If $m \circ \rho$ is a Fourier multiplier in $M_p(\boldsymbol{R}^n)$, $1 \leqq p \leqq 2$; then for $\alpha = (n-1)(1/p - 1/2)$ it holds that*

$$\|m\|_\infty + \sup_{t>0} \|\varphi m(t\cdot)\|_{B^{p'}_{\alpha p}} \leqq c\, \|m \circ \rho\|_{M_p} .$$

2. Before proving this theorem we briefly discuss the function spaces occuring in its statement. Define the localized Besov space $T(p, \alpha, q)$ as the space of all $L^1_{\mathrm{loc}}(\boldsymbol{R}_+)$-functions m, whose norms

$$\|m\|_{T(p,\gamma,q)} = \sup_{t>0} \|\varphi m(t\cdot)\|_{B^p_{\alpha q}(\boldsymbol{R}_+)}$$

are finite. These spaces are related to localized spaces and WBV-spaces considered in Connett and Schwartz [7], Gasper and Trebels [8], Carbery, Gasper and Trebels [5]. The following lemmas provide some properties; the proofs are easy modifications of those for similar results in the cited papers; hence we omit them.

LEMMA 1. (a) *The definition of $T(p, \alpha, q)$ does not depend on any specific choice of φ.*

(b) *Let $\gamma = k + \delta$, $k \in \boldsymbol{N} \cup \{0\}$, $0 < \delta < 2$. Then*

$$\|m\|_{T(p,\gamma,q)} \approx \sup_{t>0} \left\{ \left(\int_t^{2t} |m(v)|^p \frac{dv}{v} \right)^{1/p} + \left(\int_0^t \left[\int_t^{2t} v^{\gamma p} |\Delta^2_s m^{(k)}(v)|^p \frac{dv}{v} \right]^{q/p} \frac{ds}{s^{1+\delta q}} \right)^{1/q} \right\} .$$

LEMMA 2. (a) *Let $\psi \in C^N(\boldsymbol{R})$. Then we have for $0 < \gamma \leqq N$*

$$\|\psi g\|_{B^p_{\gamma q}} \leqq c \sum_{j=0}^N \|\psi^{(j)}\|_\infty \|g\|_{B^p_{\gamma q}} .$$

(b) *Suppose that χ is a strictly monotone C^N-function in a compact subinterval I of $(0, \infty)$, the image J containing* $\operatorname{supp} \varphi$ *in its interior. Then for $0 < \gamma \leqq N$*

$$\|(\varphi g) \circ \chi\|_{B^p_{\gamma q}} \leqq c(\chi)\, \|\varphi g\|_{B^p_{\gamma q}} ;$$

$c(\chi)$ remains bounded if χ and χ^{-1} are chosen from a bounded subset of $C^N(I)$ and $C^N(J)$ respectively.

3. Proof of Theorem 1. $\sum_\rho = \{\xi; \rho(\xi) = 1\}$ is a closed manifold, hence there is a point $x_0 \in \sum_\rho$ with nonzero Gaussian curvature $K(x_0)$. Define Φ as a C^∞-function with support in a small $\sum_\rho$-neighbourhood of x_0 (how small will be specified later); then extend Φ to $\boldsymbol{R}_0^n$ via $\Phi(A_t x) = \Phi(x)$, $x \in \sum_\rho$, $t > 0$. Choose $\varphi \in C_0^\infty(\boldsymbol{R})$ with support in $(1-\delta, 1+\delta)$, δ sufficiently small. Then $\varphi \circ \rho\Phi$ is a Schwartz-function and, by uniqueness of Fourier transforms, $\|F^{-1}[\varphi \circ \rho\Phi]\|_p > 0$.

Now assume $m \circ \rho \in M_p$. Let $g_t(s) = \varphi(s)m(ts)$. Then

$$(1) \qquad \begin{aligned} \sup_{t>0} \|F^{-1}[g_t \circ \rho\Phi]\|_p &\leqq \sup_{t>0} \|m \circ t\rho\|_{M_p} \|F^{-1}[\varphi \circ \rho\Phi]\|_p \\ &\leqq c \sup_{t>0} \|m \circ \rho(A_t \cdot)\|_{M_p} = c\|m \circ \rho\|_{M_p} . \end{aligned}$$

We introduce polar coordinates via the map

$$\boldsymbol{R}^n \ni x \mapsto (t, x') \in \boldsymbol{R}_+ \times \sum_\rho \quad \text{with} \quad \rho(x) = t \ , \quad x' = A_{1/\rho(x)} x \ .$$

The transformation of the Euclidean measure is given by $dx = t^{\nu-1} dt d\sigma(x')/|\nabla\rho(x')|$, $d\sigma$ being surface measure on $\sum_\rho$. If $\sum_\rho$ is parametrized near x_0 by

$$\boldsymbol{R}^{n-1} \ni y \mapsto x(y) \in \sum_\rho \quad \text{with} \quad x(0) = x_0$$

and if $G(y) = [\det(\partial x/\partial y_i, \partial x/\partial y_j)]^{1/2}$ then we can write

$$(2) \qquad \begin{aligned} (2\pi)^n F^{-1}[g_t \circ \rho\Phi](\xi) &= \int_0^\infty g_t(s) s^{\nu-1} \int_{\Sigma_\rho} \Phi(x') e^{i\langle A_s x', \xi\rangle} |\nabla\rho(x')|^{-1} d\sigma(x') ds \\ &= \int_0^\infty g_t(s) s^{\nu-1} \int \Phi(x(y)) |\nabla\rho(x(y))|^{-1} G(y) e^{i\langle x(y), A_s^*\xi\rangle} dy ds \ . \end{aligned}$$

It is well known that the method of stationary phase can be used to obtain an asymptotic expansion for the inner integral in (2) (see [11, ch. 7]). To apply this we examine the occuring phase functions. Let

$$f(y, s, \xi') = \langle x(y), A_s^*\xi'\rangle \ , \quad \text{where} \quad |\xi'| = 1 \ .$$

For fixed (s, ξ'), f has a nondegenerate critical point if $\xi'_s = A_s^*\xi'/|A_s^*\xi'|$ is a unit normal vector to $\sum_\rho$ in $x(y)$, provided the Gaussian curvature does not vanish there. This is the case in a $\sum_\rho$-neighborhood V_0 of x_0, where the normal map n is a diffeomorphism onto a neighbourhood W_0 of $n(x_0) \in S^{n-1}$. By continuity there is a neighbourhood $W \subset W_0$ of $n(x_0)$ and $\delta > 0$ so that $\xi'_s \in W_0$ for all $\xi' \in W$ and $s \in I_\delta = (1-\delta, 1+\delta)$. In view of Euler's homogeneity relation $\langle \nabla\rho(x), Px\rangle = \rho(x)$ we may assume that

$\langle Px, \xi' \rangle \geqq d_0 > 0$ for all $x \in V = n^{-1}(W)$, $\xi' \in W_0$. Now let $\tilde{y}(s, \xi')$ implicitly be defined by $\nabla_y f(y, s, \xi') = 0$ and let

$$u(s, \xi') = \langle n^{-1}(\xi'_s), A_s^* \xi' \rangle = f(\tilde{y}(s, \xi'), s, \xi') ,$$

near $y = 0$. Then $u_s(s, \xi') = \langle n^{-1}(\xi'_s), P^* A_s^* \xi'/s \rangle$, and we have $u_s(s, \xi') \geqq d_1 > 0$, $(s, \xi') \in I_\delta \times W$. Hence $u(\cdot, \xi')$ can be inverted in I_δ; that is $s = \sigma(u(s, \xi'), \xi')$, where the inverse σ depends smoothly on (u, ξ'). Further $u_s(s, \xi')$ is bounded away from zero for $(s, \xi') \in I_\delta \times W$.

Now assume that $\operatorname{supp} \Phi|_{\Sigma_\rho} \subset V$ and $\operatorname{supp} \varphi \subset I_\delta$. We consider the asymptotic expansion of the inner integral of (2) in the truncated cone $C_b = \{\omega \xi' ; \xi' \in W, \omega \geqq b\}$. It is given by (cf. [11, ch. 7])

$$\left| \int \Phi(x(y)) |\nabla \rho(x(y))|^{-1} G(y) e^{i\langle x(y), A_s^* \xi' \rangle \omega} dy - e^{i\langle n^{-1}(\xi'_s), A_s^* \xi' \rangle \omega} \sum_{j=0}^{N} \Psi_j(s, \xi') \omega^{-(n-1)/2 - j} \right|$$
$$\leqq c\omega^{-(n-1)/2-\beta} , \quad \beta \leqq N+1 .$$

The precise form of $\Psi_j \in C^\infty$, $j \geqq 1$, is not important here, $\Psi_j(\cdot, \xi')$ lies in a bounded subset of $C^N(I_\delta)$, whenever $\xi' \in W$. Ψ_0 is given by

$$[\Phi |\nabla \rho|^{-1} |K|^{-1/2}](n^{-1}(\xi'_s)) \, |A_s^* \xi'|^{-(n-1)/2} . \tag{3}$$

We introduce spherical polar coordinates and obtain

$$(2\pi)^n \|F^{-1}[g_t \circ \rho \Phi]\|_p \geqq (2\pi)^n \Big(\int |F^{-1}[g_t \circ \rho \Phi](\xi)|^p d\xi \Big)^{1/p} \geqq I_0 - \sum_{j=1}^{N} I_j - II . \tag{4}$$

where

$$I_j = \Big(\int_W \int_b^\infty \Big| \int_0^\infty g_t(s) s^{\nu-1} \Psi_j(s, \xi') e^{i\langle n^{-1}(\xi'_s), A_s^* \xi' \rangle \omega} \omega^{-(n-1)/2 - j} ds \Big|^p \omega^{n-1} d\omega d\theta(\xi') \Big)^{1/p}$$

and

$$II = \int_0^\infty |g_t(s) s^{\nu-1}| ds \Big(\int_W d\theta(\xi') \Big)^{1/p} \Big(\int_b^\infty \omega^{-((n-1)/2+\beta)p} \omega^{n-1} d\omega \Big)^{1/p} .$$

Let us first estimate the main term I_0. It is useful to substitute $u = \langle n^{-1}(\xi'_s), A_s^* \xi' \rangle$; this was seen to be correct for $(s, \xi') \in I_\delta \times W$. Abbreviate

$$\tilde{\Psi}_j(u, \xi') = \sigma^{\nu-1}(u, \xi') \frac{d\sigma}{du}(u, \xi') \Psi_j(\sigma(u, \xi'), \xi') . \tag{5}$$

Let $\alpha = (n-1)(1/p - 1/2)$, $2^{m+2} \leqq b < 2^{m+3}$, and χ, η_k as in the definition of the Besov spaces.

We apply the one-dimensional Hausdorff-Young inequality to obtain

$$I_0 \geqq c \Big(\int_W \int_{|\omega| \geqq b} |[g_t \circ \sigma(\cdot, \xi') \tilde{\Psi}_0(\cdot, \xi')]^\wedge(\omega) \omega^\alpha|^p d\omega d\theta(\xi') \Big)^{1/p}$$

$$\geqq c\Big(\int_W \sum_{k\geqq m}\int_0^\infty \Big|\chi\Big(\frac{\omega}{2^k}\Big)2^{k\alpha}[g_t\circ\sigma(\cdot,\xi')\tilde{\Psi}_0(\cdot,\xi')]\hat{}\,(\omega)\Big|^p d\omega d\theta(\xi')\Big)^{1/p}$$
$$\geqq c\Big(\int_W \sum_{k\geqq m} 2^{k\alpha p}\|\eta_k * \{g_t\circ\sigma(\cdot,\xi')\tilde{\Psi}_0(\cdot,\xi')\}\|_{p'}^p d\theta(\xi')\Big)^{1/p},$$

hence

$$I_0 \geqq c\Big(\int_W \|g_t\circ\sigma(\cdot,\xi')\tilde{\Psi}_0(\cdot,\xi')\|_{B_{\alpha p}^{p'}}^p d\theta(\xi')\Big)^{1/p}$$
$$- c_1 b^\alpha\Big(\int_W \|g_t\circ\sigma(\cdot,\xi')\tilde{\Psi}_0(\cdot,\xi')\|_{p'}^p d\theta(\xi')\Big)^{1/p}.$$

The derivatives of $\tilde{\Psi}_0(\cdot,\xi')$ and $[\tilde{\Psi}_0(\cdot,\xi')]^{-1}$ remain bounded if $\xi'\in W$, as an inspection of (3), (5) shows. Our previous discussion allows to apply Lemma 2 to deduce

$$I_0 \geqq c'\|g_t\|_{B_{\alpha p}^{p'}} - c'' b^\alpha \|g_t\|_{p'}. \tag{6}$$

We are left with the remainder terms. If $1\leqq j\leqq N$, it follows by Hölder's inequality and the Plancherel identity

$$I_j \leqq c\Big(\int_W\int_b^\infty |[g_t\circ\sigma(\cdot,\xi')\tilde{\Psi}_j(\cdot,\xi')]\hat{}\,(-\omega)\omega^{\alpha-1}|^p d\omega d\theta(\xi')\Big)^{1/p} \tag{7}$$
$$\leqq c\Big(\int_W\Big(\int_b^\infty \omega^{-2/(2-p)}d\omega\Big)^{1-p/2}\Big(\int_0^\infty |\omega^{\alpha-1/p'}[g_t\circ\sigma(\cdot,\xi')\tilde{\Psi}_j(\cdot,\xi')]\hat{}\,(\omega)|^2 d\omega\Big)^{p/2} d\theta(\xi')\Big)^{1/p}$$
$$\leqq cb^{-1/2}\Big(\int_W \|g_t\circ\sigma(\cdot,\xi')\tilde{\Psi}_j(\cdot,\xi')\|_{B_{\alpha-1/p',2}^2}^p d\theta(\xi')\Big)^{1/p}$$
$$\leqq cb^{-1/2}\|g_t\|_{B_{\alpha-1/p',2}^2}$$

Here Lemma 2 was applied; further we have used the fact that the Besov space $B_{\gamma,2}^2$ coincide with the potential space L_γ^2.

Choosing in (4) N sufficiently large we may achieve $\beta > n(1/p-1/2)+1/2$; hence

$$II \leqq cb^{n(1/p-1/2)+1/2-\beta}\|g_t\|_{p'} \leqq c'\|g_t\|_{p'}. \tag{8}$$

Collecting the estimates (6), (7), (8) we get

$$\|F^{-1}[g_t\circ\rho\Phi]\|_p \geqq c\|g_t\|_{B_{\alpha p}^{p'}} - c_0 b^\alpha\|g_t\|_{p'} - c_1 b^{-1/2}\|g_t\|_{B_{\alpha-1/p',2}^2}, \tag{9}$$
$$g_t = \varphi m(t\cdot).$$

The constants are independent of $t>0$. Observe that

$$T(p',\alpha,p)\subset T(2,\alpha,p)\subset T(2,\alpha-1/p',2).$$

The first inclusion follows via Hölder's inequality and Lemma 1(b), the second inclusion by an embedding property of Besov spaces. Now choose b in (9) sufficiently large. Since $\operatorname{supp} g_t$ is compact, $\|g_t\|_{p'}$ is dominated by $c\|m\|_\infty$. Since $M_p \subset M_2 = L^\infty$, by (1) and (9) the assertion of the Theorem follows. □

4. The proof of Theorem 1 suggests to give criteria also for M_p^q-multipliers; especially for $p = 1$, $q > 1$ (recall that in this case $M_1^q = [L^q]^\wedge$).

THEOREM 2. *Let ρ be as in Theorem 1 and $m \circ \rho \in M_p^q$, $1 \leqq p \leqq q \leqq 2$,*

$$\alpha = (n-1)(1/q - 1/2) .$$

Then

(a) $\sup_{t>0} t^{\nu(1/p-1/q)} \|\varphi m(t\cdot)\|_{B_{\alpha q}^{q'}} \leqq c\|m \circ \rho\|_{M_p^q}$.

(b) *If $p = 1$, $1 < q \leqq 2$, the following sharper inequality is valid:*

$$\left(\int_0^\infty [t^{\nu/q'}\|\varphi m(t\cdot)\|_{B_{\alpha q}^{q'}}]^2 \frac{dt}{t}\right)^{1/2} \leqq c\|F^{-1}[m \circ \rho]\|_q .$$

PROOF. (a) We use the same notations as in the proof of Theorem 1. For every $t > 0$ we have

$$\|m \circ \rho\|_{M_p^q} = t^{\nu(1/q-1/p)}\|m \circ t\rho\|_{M_p^q} \geqq ct^{\nu(1/q-1/p)}\|F^{-1}[g_t \circ \rho\Phi]\|_q .$$

The proof of Theorem 1 leads us to the inequality (9), with p replaced by q; from this it follows

$$\|g_t\|_{B_{\alpha q}^{q'}} \leqq c\|F^{-1}[g_t \circ \rho\Phi]\|_{L^q(\boldsymbol{R}^n)} + \|g_t\|_{L^{q'}(\boldsymbol{R})} .$$

We introduce polar coordinates (with respect to ρ). An application of the Hausdorff-Young-inequality (in $\boldsymbol{R}^n$) gives

$$\|g_t\|_{L^{q'}(\boldsymbol{R})} \leqq c\|g_t \circ \rho\Phi\|_{L^{q'}(\boldsymbol{R}^n)} \leqq c\|F^{-1}[g_t \circ \rho\Phi]\|_{L^q(\boldsymbol{R}^n)}$$

This is enough to deduce the assertion in (a).

(b) Choose a C^∞-function ψ with compact support in $(0, \infty)$, $\psi(t) = 1$, if $t \in \operatorname{supp} \varphi$. By (a),

$$t^{\nu/q'}\|\varphi m(t\cdot)\|_{B_{\alpha q}^{q'}} \leqq ct^{\nu/q'}\|F^{-1}[\psi \circ \rho m \circ t\rho]\|_q = c\left\|F^{-1}\left[\psi \circ \frac{\rho}{t} m \circ \rho\right]\right\|_q .$$

We integrate and use Minkowski's inequality and Littlewood-Paley-theory (see Madych [12]) to obtain

$$\left(\int_0^\infty [t^{\nu/q'}\|\varphi m(t\cdot)\|_{B_{\alpha q}^{q'}}]^2 \frac{dt}{t}\right)^{1/2} \leqq c\left(\int_0^\infty \left\|F^{-1}\left[\psi \circ \frac{\rho}{t} m \circ \rho\right]\right\|_q^2 \frac{dt}{t}\right)^{1/2}$$

$$\leqq c\left\|\left(\int_0^\infty \left|F^{-1}\left[\psi \circ \frac{\rho}{t} m \circ \rho\right]\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_q \leqq c\|F^{-1}[m \circ \rho]\|_q . \quad \square$$

There are also versions of our theorems for convolution operators acting on anisotropic H^p-spaces. Here H^p is defined with respect to the A_t^*-dilations (see Calderón and Torchinsky [3]). From the proofs of the above theorems we obtain the following:

COROLLARY. *Let $\rho \in C^\infty(\boldsymbol{R}_0^n)$ be an A_t-homogeneous distance function and $m \in L^1_{\mathrm{loc}}(0, \infty)$.*

(a) *If for every $f \in H^p$*

$$\|F^{-1}[m \circ \rho f\hat{\ }]\|_{H^q} \leqq A\|f\|_{H^q}\,, \quad 0 < p \leqq q \leqq 2,\ q \geqq 1\,,$$

then

$$\sup_{t>0} t^{\nu(1/p-1/q)}\|\varphi m(t\cdot)\|_{B^{q'}_{\alpha q}} \leqq cA\,; \quad \alpha = (n-1)(1/q - 1/2)\,.$$

If $0 < p \leqq q < 1$, one has to replace $B^{q'}_{\alpha q}$ by $B^{\infty}_{\alpha q}$.

(b) If *$m \circ \rho$ is the Fourier transform of an H^q-distribution* $(0 < q \leqq 1)$, *then*

$$\left(\int_0^\infty [t^{\nu(1-1/q)}\|\varphi m(t\cdot)\|_{B^\infty_{\alpha q}}]^2 \frac{dt}{t}\right)^{1/2} \leqq c\|F^{-1}[m \circ \rho]\|_{H^q}\,, \quad \alpha = (n-1)\left(\frac{1}{q} - \frac{1}{2}\right).$$

5. Remarks. (a) The proof of Theorem 1 shows that the global C^∞-assumption can be weakened; it suffices to assume that ρ is smooth near a point $x_0 \in \sum_\rho$, where the Gaussian curvature does not vanish. This is the case in most applications. The proof works if we require that $\rho \in C^L$ near x_0, $L > (2n/p) - (n-5)/2$. Also the assumption $\rho(x) > 0$, $x \neq 0$ is not really necessary; e.g. all results remain valid if $\rho(x) = \prod_{i=1}^n |\xi_i|^{\alpha_i}$, $\alpha_i > 0$.

(b) Gasper and Trebels [9], [8] proved

$$\sup_{t>0}\|\varphi m(t\cdot)\|_{L^{p'}_\alpha} \leqq c\|m(|\cdot|)\|_{M_p}\,, \quad 1 \leqq p \leqq 2\,, \quad \alpha = (n-1)\left(\frac{1}{p} - \frac{1}{2}\right).$$

Theorem 1 is slightly sharper even for radial multipliers, because of the embedding $B^{p'}_{\alpha p} \subset L^{p'}_\alpha$, $1 < p \leqq 2$, ([2, p. 152]). An analogous remark applies to more general Hankel multipliers, considered in [9].

(c) Theorem 1 can be used to accomplish some known results on quasiradial multipliers ([4], [6], [14]): If $\rho \in C^\infty(\boldsymbol{R}_0^n)$, $\sum_\rho$ strictly convex, then the following inequality holds, provided $1 < p \leqq 2(n+1)/(n+3)$.

$$\|m \circ \rho\|_{M_p} \leqq c \sup_{t>0} \|\varphi m(t\cdot)\|_{L^2_\gamma}\,, \quad \gamma > n(1/p - 1/2)\,. \tag{10}$$

In dimension two, (10) is valid for $1 < p \leqq 4/3$ $(\rho(\xi) = |\xi|)$. Well known counterexamples show that (10) is false, if $\gamma < n(1/p - 1/2) =: \gamma_c$ (see e.g. [4]). What about $\gamma = \gamma_c$? Consider

$$m_{\beta,\gamma,q}(t) = \psi(t)(1-t)_+^{\beta-1/q}|\log(1-t)|^{-\gamma}\,,$$

where ψ is a C^∞-bump function in $(0, \infty)$, $\psi(1) > 0$.
Then $m_{\beta,\gamma,q} \in L^q_\beta \setminus B^{p'}_{\beta+(1/p'-1/q),p}$, if $1/q < \gamma < 1/p$, $p < q \leqq 2$. Hence Theorem 1 shows that (10) is false for the critical index γ_c.

(d) Theorem (2b) should be compared with the following inequality which furnishes an $[L^q]^\wedge$-criterion for quasiradial multipliers: Let $\gamma = n(1/q - 1/2)$, $\rho \in C^N(\boldsymbol{R}^n_0)$, $N > \gamma$, $1 \leqq q \leqq 2$. Then

$$\|F^{-1}[m \circ \rho]\|_q \leqq c\left(\int_0^\infty [t^{\nu/q'}\|\varphi m(t\cdot)\|_{B^2_{\gamma,q}}]^q \frac{dt}{t}\right)^{1/q}. \tag{11}$$

If $q = 2$, this immediately follows by the Plancherel identity; if $q = 1$ the inequality is a dilation invariant version of Bernstein's theorem (see Peetre [13]), specialized to quasiradial multipliers. The case $1 < q < 2$ follows by a complex interpolation argument (cf. [7], [5]). The inequality (11) and counterexamples (see (c)) show that the smoothness condition in Theorem (2b) cannot be improved in the context of Besov spaces.

(e) The following criterion is a special case of an anisotropic version of Baernstein's and Sawyer's sharp multiplier theorem ([1, p. 20]).

$$\|F^{-1}[m \circ \rho f^\wedge]\|_{H^p} \leqq c \sup_{t>0} \|\varphi m(t\cdot)\|_{B^2_{\gamma p}} \|f\|_{H^p}, \tag{12}$$

$0 < p < 1$, $\gamma = n(1/p - 1/2)$ (H^p, ρ as in Section 4).

The necessary conditions in the corollary provide new counterexamples to the results of Baernstein and Sawyer. In particular it follows that $B^2_{\gamma p}$ in (12) cannot be replaced by any larger $B^2_{\beta q}$-space.

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