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FOLIATIONS AND SUBSHIFTS

Dedicated to Ky Fan on his retirement

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Introduction. Let $(M, \mathscr{F})$ be a closed, transversely orientable, C^2-foliated manifold of codimension one. Let $O(\mathscr{F})$ denote the family of open, $\mathscr{F}$-saturated subsets of M. Let $U \in O(\mathscr{F})$, and let L be a leaf of $\mathscr{F}|U$. Smoothness of class C^2 implies that there exists a compact, transverse one-manifold $R \subset U$ such that every leaf of $\bar{L} \cap U$ meets $\mathrm{int}(R)$ [C-C 1, (3.7)]. Consequently, $\bar{L} \cap U$ contains a minimal set of $\mathscr{F}|U$ [C-C 1, (3.0)].

DEFINITION. An $\mathscr{F}$-saturated subset $X \subseteq M$ is a local minimal set (LMS) of $\mathscr{F}$ if there exists $U \in O(\mathscr{F})$ such that X is a minimal set of $\mathscr{F}|U$.

Every proper leaf is a LMS, with $U = M \setminus (\bar{L} \setminus L)$. If $U \in O(\mathscr{F})$ and each leaf of $\mathscr{F}|U$ is dense in U, then U itself is a LMS. Finally, an exceptional LMS is one of neither of these types. If X is exceptional, then the transverse manifold $R \subset U$ can be chosen so that $C = X \cap R$ is a Cantor set and misses ∂R.

These LMS play a key role in the structure theory of compact, C^2-foliated manifolds of codimension one [C-C 1]. Our very incomplete understanding of the exceptional type constitutes a major gap in the theory.

Let X be an exceptional LMS, with U, R, and C as above. The holonomy of $\mathscr{F}|U$ induces a C^2 pseudogroup Γ on R for which C is a Γ-minimal set. Let $\Gamma|C$ denote the induced pseudogroup on C. It frequently happens that the choice of R can be made so that $\Gamma|C$ is generated by the local restrictions of a single transformation $\tau: C \to C$ which, in a sense to be made precise in §1, is essentially a one-sided subshift of finite type (also known as a topological Markov chain [Wa, p. 119]).

DEFINITION. If there exists $\tau: C \to C$ as above, then C is a Markov

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Γ-minimal set and X is called a Markov LMS.

While there exist examples of exceptional LMS that are not Markov LMS (see §7), it is likely that the Markov ones are the fundamental "building blocks" for the general case (in a sense that we cannot presently make precise). At any rate, examples of Markov minimal sets abound in the literature (cf. [Sa 1], [Ra], [He], [G-S], [In], [Ma]).

We will settle, for Markov LMS, a number of questions that have been open for the general exceptional LMS.

Let $L \subset X$ be a leaf, let $x \in L \cap R$, and let $\Gamma_x \subset \Gamma$ be the subpseudogroup fixing x.

DEFINITION. The holonomy group of L relative to X is the group $H_x(L, X)$ of germs at x of all $\gamma \in \Gamma_x | C$.

THEOREM 1. *Let X be a Markov* LMS *and let $L \subset X$ be a leaf. Then $H_x(L, X)$ is either trivial or infinite cyclic and generated by the germ of a contraction that is unique in a suitable neighborhood of x in C. Exactly a countable infinity of leaves in X have $H_x(L, X) \cong \boldsymbol{Z}$ and among these are all of the semiproper leaves.*

Recall that a leaf is *semiproper* if it is asymptotic to itself from at most one side (hence proper leaves are also semiproper).

Dippolito [Di, §9] has asked whether, in an exceptional LMS, $H_x(L, X)$ is always cyclic. For the Markov case, our result is stronger. Indeed, whenever $H_x(L, X)$ is nontrivial, the generator is a unique contraction.

THEOREM 2. *If X is a Markov* LMS, *then X contains only finitely many semiproper leaves.*

This theorem answers, for the Markov case, an open question of Hector (proposed in [Sch, Problem 28.1]).

A special case of the following result was proven independently by Matsumoto [Ma].

THEOREM 3. *If X is a Markov* LMS, *then it has Lebesgue measure $|X| = 0$.*

By Duminy's localization of the Godbillon-Vey class to an $H^3(M)$-valued measure $\mathrm{gv}(*, \mathscr{F})$ [Du], Theorem 3 implies that $\mathrm{gv}(X, \mathscr{F}) = 0$ whenever X is a Markov LMS. We expect this to generalize to general exceptional LMS, although we are less confident that Theorem 3 itself will generalize.

Remark that the Denjoy example [De] violates the conclusions of Theorem 1 and Theorem 3, but is only of class C^1. Also, there are C^0

Markov examples, probably C^1-smoothable, that violate the conclusions of all three of our theorem.

1. Markov pseudogroups. Let $\mathscr{S} = (\{I_1, \cdots, I_m\}, \{h_1, \cdots, h_m\}, P)$, where each $I_j \subset \boldsymbol{R}$ is a compact, nondegenerate interval, each h_j is a C^2 diffeomorphism with range $R(h_j)$ and domain $D(h_j)$, both being open, bounded intervals, and $P = (p_{ij})$ is an $m \times m$ matrix with entries $p_{ij} \in \{0, 1\}$. Assume that $m \geqq 2$.

DEFINITION. If the following properties hold, then $\mathscr{S}$ is called a Markov system and the pseudogroup $\Gamma = \Gamma_{\mathscr{S}}$, generated by $(h_j)_{j=1}^m$, is called a Markov pseudogroup.

(1) $R(h_i) \cap R(h_j) = \varnothing$, $i \neq j$.

(2) $I_j \subset R(h_j)$, for all j. Set $X_j = h_j^{-1}(I_j)$.

(3) $p_{ij} = 1$ implies that $I_j \subseteqq X_i$.

(4) $p_{ij} = 0$ implies that $I_j \cap D(h_i) = \varnothing$.

EXAMPLE. Let $h_1\colon]-1/2, 3/2[\to]-1/6, 1/2[$ and $h_2\colon]-1/2, 3/2[\to]1/2, 7/6[$ be defined by $h_1(x) = x/3$ and $h_2(x) = (x+2)/3$. Let $I_1 = [0, 1/3]$, $I_2 = [2/3, 1]$, hence $X_1 = X_2 = [0, 1]$ and $p_{ij} = 1$, $1 \leqq i, j \leqq 2$.

Let $\mathscr{S}$ be a Markov system. Then $w = h_{i_1} \circ \cdots \circ h_{i_n} \in \Gamma$ is defined at a point of X_{i_n} if and only if $p_{i_k i_{k+1}} = 1$, $1 \leqq k \leqq n-1$, in which case $X_{i_n} \subset D(w)$ and we denote $w(X_{i_n})$ by I_w or by $I_{i_1 \cdots i_n}$. Set $|w| = n$ and

$$Z = \bigcap_{n=1}^{\infty}\left(\bigcup_{|w|=n}(I_w)\right)$$
$$Z_0 = Z \setminus \operatorname{int}(Z)\ .$$

In the above example, $Z = Z_0$ is a Cantor set and a minimal set of Γ. It is not hard to modify the example so that $\operatorname{int}(Z) \neq \varnothing$ and Z_0 is still a Γ-minimal Cantor set. This property of Z_0 is typical but is not implied by the definitions. For example, if P is the identity matrix, then Z_0 is a finite point set.

DEFINITION. If $\mathscr{S}$ is a Markov system and $x \in R(h_i)$, set $T(x) = h_i^{-1}(x) \in D(h_i)$. This well defines

$$T\colon \bigcup_{i=1}^{m} R(h_i) \to \bigcup_{i=1}^{m} D(h_i)\ ,$$

locally a C^2 diffeomorphism, such that $T(Z_0) \subseteqq Z_0$. Set $\tau = T|Z_0\colon Z_0 \to Z_0$, a continuous map that is locally a homeomorphism.

DEFINITION. A sequence $(i_n)_{n=1}^{\infty}$ such that $p_{i_n i_{n+1}} = 1$, for all $n \geqq 1$, is called P-admissible. The set of all such sequences is denoted by $\mathscr{K}_P$.

Let $(i_n)_{n=1}^{\infty} \in \mathscr{K}_P$. Let $w_0 = \mathrm{id} \in \Gamma$ and let $w_n = w_{n-1} \circ h_{i_n}$. Then $I_{w_{n+1}} \subseteqq I_{w_n}$ and the set $I_{i_1 i_2 \cdots i_n \cdots} = \bigcap_{n=1}^{\infty} I_{w_n}$ is either a singleton or a nondegenerate, compact interval. Remark that Z is the disjoint union of all of these sets $I_{i_1 i_2 \cdots i_n \cdots}$.

The set-theoretic boundary $\dot{I}_{i_1 i_2 \cdots}$ reduces to $I_{i_1 i_2 \cdots}$ whenever this set is a singleton and, otherwise, is the pair of endpoints. It is clear that

$$Z_0 = \bigcup_{\iota \in \mathscr{K}_P} \dot{I}_\iota ,$$

so denoting $x \in Z_0$ by x_ι if $x \in \dot{I}_\iota$ leads to at most a countable infinity of pairs of points with the same P-admissible index. We formalize this as a surjection, $h\colon Z_0 \to \mathscr{K}_P$, $h(x_\iota) = \iota$, that is two to one on at most a countable subset of Z_0 and, elsewhere, is one to one.

It is customary to topologize $\mathscr{K}_P$ as a closed subset of the Cartesian product $\mathscr{K} = \{1, 2, \cdots, m\}^N$, where N denotes the natural numbers. Then $\mathscr{K}$ is a Cantor set and, in the more interesting cases, so is $\mathscr{K}_P$. In any case, the surjection $h\colon Z_0 \to \mathscr{K}_P$ is continuous.

DEFINITION. The (one-sided) shift $\sigma\colon \mathscr{K} \to \mathscr{K}$ is defined by $\sigma(j_1, j_2, j_3, \cdots) = (j_2, j_3, \cdots)$ and $\sigma = \sigma | \mathscr{K}_P\colon \mathscr{K}_P \to \mathscr{K}_P$ is called a subshift of finite type or a topological Markov chain.

Evidently, $\sigma \circ h = h \circ \tau$, so τ is semiconjugate to σ. It is also evident that τ belongs locally to $\Gamma | Z_0$ and that the one-one restrictions of τ to suitable open subsets of Z_0 generate $\Gamma | Z_0$. Let Γ_σ denote the pseudogroup on $\mathscr{K}_P$ that is similarly generated by σ.

(1.1) LEMMA. *The set $\mathscr{K}_P$ is a Cantor set and Γ_σ-minimal if and only if there exists an exceptional Γ-minimal set $C \subseteqq Z_0$ such that $h(C) = \mathscr{K}_P$. In this case, $Z_0 \setminus C$ is a union of at most countably many Γ-orbits, each of which accumulates exactly on C.*

PROOF. Assume that $\mathscr{K}_P$ is a Γ_σ-minimal Cantor set. Let $x = (i_1, i_2, \cdots) \in \mathscr{K}_P$ and let $y \in \mathscr{K}_P$. Then $\Gamma_\sigma(y)$ clusters at x. This implies that, for all $n \geqq 1$, there exists $g_n \in \Gamma$ such that $I_y \subset D(g_n)$, $g_n(I_y) \subset I_{i_1 \cdots i_n} \supset I_x$, and $I_x \cap g_n(I_y) = \emptyset$. Therefore, at least one point of $\dot{I}_x$ is a cluster point of $\Gamma(a)$, for all $a \in I_y$. It follows that, if $b \in Z_0$ is not a cluster point of some $\Gamma | Z_0$-orbit, then there exists $z \in \mathscr{K}_P$ such that I_z is nondegenerate, $b \in \dot{I}_z$, and no $\Gamma | Z_0$-orbit clusters on b. The set B of such $b \in Z_0$ is Γ-invariant and either empty or countable. The set $C = Z_0 \setminus B$ is an exceptional Γ-minimal set and $h(C) = \mathscr{K}_P$. If $b \in B$, the above observations imply that $\Gamma(b)$ clusters exactly on C.

The converse is trivial. □

REMARK. A condition that guarantees the hypotheses in (1.1) is that, for all $(i, j) \in \{1, \cdots, m\}^2$, there exists $k \geqq 1$ such that the (i, j)-th entry of P^k (the k-th power of the matrix) is $\geqq 2$.

DEFINITION. Let Γ be a pseudogroup on an open subset of $\boldsymbol{R}$, $Y \subset \boldsymbol{R}$ a compact, totally disconnected, Γ-invariant set. Let $\tau: Y \to Y$ be a transformation belonging locally to $\Gamma | Y$ and generating that pseudogroup. Let $\sigma: \mathscr{K}_P \to \mathscr{K}_P$ be a subshift of finite type and $h: Y \to \mathscr{K}_P$ a continuous surjection such that $h \circ \tau = \sigma \circ h$. Suppose that, for all $y \in Y$, $h^{-1}(h(y))$ is either a singleton or is the pair of endpoints of the closure of a bounded component of $\boldsymbol{R} \setminus Y$. Then τ is said to be essentially conjugate to the subshift σ (and, less precisely, τ is said to be an essential subshift). If there exists an exceptional Γ-minimal set $C \subseteqq Y$ such that $h(C) = \mathscr{K}_P$, then C is called a Markov Γ-minimal set.

For example, if Γ is a Markov pseudogroup, then $\tau: Z_0 \to Z_0$ is essentially conjugate to $\sigma: \mathscr{K}_P \to \mathscr{K}_P$. For $\tau: C \to C$ as in (1.1), C is a Markov Γ-minimal set. The following lemma is essentially the converse.

(1.2) LEMMA. *Let $\tilde{\Gamma}$ be a C^2 pseudogroup on an open subset of $\boldsymbol{R}$, $\tilde{C}$ a Markov $\tilde{\Gamma}$-minimal set. Then there exists an open neighborhood W of $\tilde{C}$, a smooth imbedding $\iota: W \to \boldsymbol{R}$, and a C^2 Markov pseudogroup Γ on a neighborhood of $C = \iota(\tilde{C})$ in $\boldsymbol{R}$, such that $\iota \circ (\tilde{\Gamma} | \tilde{C}) \circ \iota^{-1} = \Gamma | C$, Γ defines the essential subshift $\tau: Z_0 \to Z_0$, and $C \subseteqq Z_0$ is a Markov Γ-minimal set.*

PROOF. By a *segment* $Q \subseteqq \tilde{C}$, we mean a set of the form $Q = \tilde{C} \cap [a, b] = \tilde{C} \cap]a - \varepsilon, b + \varepsilon[$, $\varepsilon > 0$ and $a < b$. Clearly $\tilde{C}$ decomposes in many ways into disjoint segments $\tilde{C}_1, \cdots, \tilde{C}_r$ and, for $i \neq j$, either every point of $\tilde{C}_i$ precedes every point of $\tilde{C}_j$ (and we write $\tilde{C}_i < \tilde{C}_j$) or vice versa. There is an open neighborhood W of $\tilde{C}$ in $\boldsymbol{R}$ and a smooth imbedding $\iota: W \to \boldsymbol{R}$ such that $\iota(\tilde{C}_i) < \iota(\tilde{C}_j)$ whenever $i < j$.

Let $\sigma: \mathscr{K}_{\tilde{P}} \to \mathscr{K}_{\tilde{P}}$ be a subshift to which $\tilde{\tau}$ is essentially conjugate, $\tilde{P} = (\tilde{p}_{\lambda\mu})$, $1 \leqq \lambda, \mu \leqq \tilde{m}$. For $1 \leqq \lambda \leqq \tilde{m}$, let φ_λ be the branch of σ^{-1} that is defined by

$$\varphi_\lambda(i_1, i_2, \cdots) = (\lambda, i_1, i_2, \cdots)$$

whenever $\tilde{p}_{\lambda i_1} = 1$. Choose the numbering so that there is an integer $q \leqq \tilde{m}$ with the property that $1 \leqq \lambda \leqq q$ is the necessary and sufficient condition that there exists μ with $\tilde{p}_{\lambda\mu} = 1$. Set $\mathscr{K}_\lambda = R(\varphi_\lambda)$, $1 \leqq \lambda \leqq q$. Then $\mathscr{K}_{\tilde{P}} = \bigcup_{1 \leqq \lambda \leqq q} \mathscr{K}_\lambda$ is a disjoint union of open, compact subsets, and $D(\varphi_\lambda) = \bigcup_{\tilde{p}_{\lambda\mu}=1} \mathscr{K}_\mu$. Set $\mathscr{K}_{\lambda\mu} = \varphi_\lambda(\mathscr{K}_\mu)$ whenever $\tilde{p}_{\lambda\mu} = 1$, so $\mathscr{K}_\lambda = \bigcup_{\tilde{p}_{\lambda\mu}=1} \mathscr{K}_{\lambda\mu}$ is a union of disjoint, open, compact subsets.

Via the essential conjugacy map h, transfer the above definitions to give $\tilde{\varphi}_\lambda \in \tilde{\Gamma}|\tilde{C}$ and $\tilde{C}_\lambda$ and $\tilde{C}_{\lambda\mu}$. Let

$$\Lambda = \{(\lambda, \mu) | 1 \leqq \lambda \leqq q, 1 \leqq \mu \leqq \tilde{m}, \tilde{p}_{\lambda\mu} = 1\} .$$

Under lexicographic order, Λ is order-isomorphic to $\{1, \cdots, m\}$, where $m = \mathrm{card}(\Lambda)$.

By decomposing each $\tilde{C}_{\lambda\mu}$ into disjoint segments and proceeding as in the first paragraph, one obtains an open neighborhood W of $\tilde{C}$ and an imbedding $\iota: W \to \boldsymbol{R}$ such that, whenever $(\lambda_1, \mu_1) < (\lambda_2, \mu_2)$ in Λ, then $\iota(\tilde{C}_{\lambda_1\mu_1}) < \iota(\tilde{C}_{\lambda_2\mu_2})$. After conjugating $\tilde{\Gamma}|W$ by ι, we can let $\tilde{C}_{\lambda\mu}$ denote $\iota(\tilde{C}_{\lambda\mu})$, etc., for notational simplicity.

Thus, $\tilde{C}_{\lambda\mu}$ and $\tilde{C}_\lambda$ are segments and we let $I_{\lambda\mu}$ and X_λ denote the minimal compact intervals such that $\tilde{C}_{\lambda\mu} = \tilde{C} \cap I_{\lambda\mu}$ and $\tilde{C}_\lambda = \tilde{C} \cap X_\lambda$. Also write $X_{\lambda\mu} = X_\mu$ for $(\lambda, \mu) \in \Lambda$. Let $h_{\lambda\mu}$ be a local C^2 diffeomorphism in $\boldsymbol{R}$ which extends the map $\tilde{\varphi}_\lambda | \tilde{\varphi}_\lambda^{-1}(\tilde{C}_{\lambda\mu}) = \tilde{\varphi}_\lambda | \tilde{C}_\mu$ so that $R(h_{\lambda\mu})$ is an open interval containing $I_{\lambda\mu}$. Make sure that the intervals $R(h_{\lambda\mu})$ are disjoint. Clearly,

$$D(h_{\lambda\mu}) \supset X_{\lambda\mu} = h_{\lambda\mu}^{-1}(I_{\lambda\mu}) \supset \bigcup_{(\mu,\gamma) \in \Lambda} (I_{\mu\gamma}) .$$

Also, by choosing each $R(h_{\lambda\mu})$ slightly smaller, we assume that $D(h_{\lambda\mu}) \cap I_{\nu\gamma} = \varnothing$ whenever $\nu \neq \mu$. Finally, $P = (P_{\lambda\mu,\nu\gamma})$ is the $m \times m$ matrix with $p_{\lambda\mu,\mu\gamma} = 1$ and $p_{\lambda\mu,\nu\gamma} = 0$ if $\nu \neq \mu$, where all (λ, μ), $(\nu, \gamma) \in \Lambda$.

It is clear that the pseudogroup Γ, generated by $\{h_{\lambda\mu}\}_{(\lambda,\mu) \in \Lambda}$, is as required. □

The precise meaning of the definition of Markov LMS for a foliated manifold, as indicated in the introduction, should now be clear. By (1.2), it will be sufficient to prove our theorems under the assumption that the holonomy pseudogroup Γ on a neighborhood of C in $\mathrm{int}(R)$ is a Markov pseudogroup and that $C \subsetneqq Z_0$ is a Markov Γ-minimal set.

REMARKS. (1) Given a Markov pseudogroup Γ as in (1.1), one can realize Γ as the holonomy in a neighborhood of an exceptional minimal set X in a suitable C^2-foliated manifold. One method, that of "branched staircases", is due to Takamura [Ta] and Inaba [In]. Another produces X in the nonsingular part of a singular foliation and then removes the singularities. Sometime Γ can be "completed" to a subgroup of $\mathrm{Diff}^2(S^1)$ [Sa 1], [G-S] and X can then be realized in a suitable foliated S^1-bundle.

(2) Having realized Γ as the holonomy of a Markov minimal set X, one can turbulize along a closed transversal meeting X to produce a Markov LMS at level one. Indeed, it is possible to realize Γ as the holonomy of a Markov LMS at any desired finite level.

2. Some estimates on derivatives. Let Γ be a C^2 pseudogroup in $\boldsymbol{R}$, with generating set $\{h_1, \cdots, h_m\}$. Let

$$g = h_{i(p)}^{\varepsilon(p)} \circ h_{i(p-1)}^{\varepsilon(p-1)} \circ \cdots \circ h_{i(1)}^{\varepsilon(1)} ,$$

with all $\varepsilon(k) = \pm 1$ and $D(g)$ maximal possible. Set

$$g_k = h_{i(k)}^{\varepsilon(k)} \circ \cdots \circ h_{i(1)}^{\varepsilon(1)}$$

and, for $u_0 \in D(g)$, set $u_k = g_k(u_0)$, $1 \leqq k \leqq p$.

DEFINITION. If g as above is a reduced word in the generators $\{h_i^{\pm 1}\}_{1 \leqq i \leqq m}$, then g is called a chain of length $|g| = p$. If also $u_0, u_1, \cdots, u_p$ are all distinct, then g is a simple chain at u_0. If $u_1, \cdots, u_p$ are distinct and $u_p = u_0$, then g is a simple loop at u_0. If $g = q^{-1} \circ f \circ q$, where q is a simple chain at u_0 and f is a simple loop at $q(u_0)$, then g is a basic loop at u_0.

Let $Y \subset \bigcup_{i=1}^m D(h_i)$ be a compact, Γ-invariant set. By passing to $\Gamma | W$, where W is a suitable bounded, open neighborhood of Y in $\boldsymbol{R}$, we can assume that there are positive constants c and b such that $(h_i^{\pm 1})' > c$ and $|(h_i^{\pm 1})''| \leqq b$ everywhere, $1 \leqq i \leqq m$. Set $\theta = b/c$ and $\lambda = \exp(6\theta |W|)$. Here, $|W|$ denotes the sum of the lengths of the components of W.

The following estimate will be found in [Sa 2].

(2.1) LEMMA. *Let $g \in \Gamma$ be a chain of length p and let $u_0, v_0 \in D(g)$. Then*

$$g'(u_0)/g'(v_0) \leqq \exp\left(\theta \cdot \sum_{i=0}^{p-1} |u_i - v_i|\right) .$$

DEFINITION. A gap J of Y is a compact, nondegenerate interval such that $\partial J = J \cap Y$.

If $g \in \Gamma$ is a chain of length p and $K_0 \subset D(g)$ is a compact interval, we set $K_j = g_j(K_0)$, $1 \leqq j \leqq p$. If J_0 is a gap of Y, $u_0 \in \partial J_0$, and if g is a simple chain at u_0 with $D(g) \supset J_0$, then $J_0, J_1, \cdots, J_p$ have disjoint interiors. If g is a simple or basic loop at u_0 with $D(g) \supset J_0$, then each of the intervals $J_0, J_1, \cdots, J_p$ appears at most three times in this list and $\operatorname{int}(J_k) \cap \operatorname{int}(J_i) \neq \varnothing$ if and only if $J_k = J_i$.

(2.2) LEMMA. *Let J_0 be a gap of Y and let K_0 be a compact interval such that $J_0 \cap K_0$ is a singleton $\{u_0\}$ and $|K_0|/|J_0| \leqq 1/\lambda$. If $g \in \Gamma$ is either a simple chain, a simple loop, or a basic loop at u_0 and if $J_0 \cup K_0 \subset D(g)$, then $|g(K_0)| < |g(J_0)|$ and $g'(u)/g'(v) < \lambda$, for all $u, v \in J_0 \cup K_0$.*

PROOF. Evidently $|K_0| < |J_0|$. Inductively, assume that $|K_i| < |J_i|$, $0 \leqq i < s$. Then, by the definition of λ, together with the above remarks and (2.1), we have $g_s'(u)/g_s'(v) < \lambda$, for all $u, v \in J_0 \cup K_0$. Furthermore, there exists $u \in K_0$ and $v \in J_0$ such that

$$|K_s|/|J_s| = g_s'(u)|K_0|/g_s'(v)|J_0| < 1 . \qquad \square$$

REMARK. In this paper, we will only need (2.2) for the case in which g is a simple chain. In this case, the factor 6 in the definition of λ can be reduced to 2. But the general version of (2.2) will be useful elsewhere.

3. The relative holonomy groups. Let $\mathscr{S} = (\{I_1, \cdots, I_m\}, \{h_1, \cdots, h_m\}, P)$ and $\Gamma = \Gamma_{\mathscr{S}}$ be Markovian and assume that $C \subseteqq Z_0$ is a Markov Γ-minimal set.

Remark that chains must reduce to the form

$$g = h_{i_p} \circ \cdots \circ h_{i_k} \circ h_{i_{k-1}}^{-1} \circ \cdots \circ h_{i_1}^{-1} = h_{i_p} \circ \cdots \circ T^{k-1}|D(g) .$$

This is because the generators have disjoint images.

DEFINITION. If $x \in C$, then Γ_x is the pseudogroup of all $g|U$, where g is a chain as above, $g(x) = x$, and U is an open, connected neighborhood of x in $D(g)$.

DEFINITION. Let $x \in C$. Then the group of germs at x of all $\gamma = g|C$, where $g \in \Gamma_x$, is denoted $H_x(\Gamma(x), C)$ and is called the holonomy group at x relative to C or, more simply, the relative holonomy group at x.

Evidently, $H_x(\Gamma(x), C)$ depends, as an abstract group, only on the orbit $\Gamma(x)$ and not on the basepoint x.

DEFINITION. A point $y \in C$ is said to be τ-cyclic if $\tau^k(y) = y$, some $k \geqq 1$. If the integer k is minimal, then $\{\tau^{k-1}(y), \tau^{k-2}(y), \cdots, \tau(y), y\}$ is called a τ-cycle.

(3.1) LEMMA. *Exactly a countable infinity of Γ-orbits in C contain τ-cycles and there is exactly one τ-cycle in each such orbit.*

PROOF. It is enough to prove the corresponding assertions for the subshift $\sigma: \mathscr{K}_P \to \mathscr{K}_P$ to which τ is essentially conjugate. Since $\mathscr{K}_P$ is uncountable, this is an easy exercise in symbolic dynamics. $\square$

(3.2) LEMMA. *Let $x \in C$. If $\Gamma(x)$ does not contain a τ-cycle, then $H_x(\Gamma(x), C) = 0$. If $\Gamma(x)$ contains a τ-cycle, then there is a neighborhood V_x of x in C such that $\Gamma_x|V_x$ contains a contraction $f: V_x \to V_x$ and each element of $\Gamma_x|V_x$ is the restriction of f^k to a suitable neighborhood of x in V_x, some $k \in \mathbf{Z}$.*

PROOF. Since τ generates $\Gamma|C$, the first assertion is immediate. If

$\Gamma(x)$ contains a τ-cyclic point x_0, we lose no generality in assuming that $x = x_0$. Since each h_i is a single-valued branch of τ^{-1}, there is a simple loop at x_0 of the form $g = h_{i_1} \circ \cdots \circ h_{i_p}$. Thus, $h(x_0)$ is the σ-cyclic point $(i_1, \cdots, i_p, i_1, \cdots, i_p, \cdots) \in \mathscr{K}_P$, the neighborhood V_{x_0} can be defined by $h(V_{x_0}) = \{(j_k)_{k=1}^\infty \in \mathscr{K}_P \mid j_1 = i_1\}$, and $h \circ g = \bar{g} \circ h$ where $\bar{g}(i_1, j_2, j_3, \cdots) = (i_1, \cdots, i_p, i_1, j_2, j_3, \cdots)$. From this it is evident that $f = g \mid V_{x_0}$ is a contraction to x_0. It is also evident that the only chains that fix x_0 restrict, in V_{x_0}, to the powers of f. $\square$

(3.3) Corollary. *For each $x \in C$, $H_x(\Gamma(x), C)$ is either trivial or infinite cyclic, generated by a contraction that is unique in a suitable neighborhood of x in C. Those $x \in C$ such that $H_x(\Gamma(x), C) \cong \boldsymbol{Z}$ lie on a countable infinity of distinct orbits.*

So far, nothing in this section has required smoothness of class C^2. The following does.

(3.4) Lemma. *If C clusters at x from only one side, then $\Gamma(x)$ contains a τ-cycle, hence $H_x(\Gamma(x), C) \cong \boldsymbol{Z}$.*

Proof. First suppose that $\Gamma(x)$ has an element $x_0 \in I_j = [a, b]$ such that either $C \cap [a, b] = C \cap [x_0, b]$ or $C \cap [a, b] = C \cap [a, x_0]$. For definiteness, assume that the first is the case. Then $X_j = h_j^{-1}[a, b] = [T(a), T(b)]$ and $C \cap X_j = C \cap [\tau(x_0), T(b)]$. Since $\tau(x_0) \in C \cap I_k$, some $I_k = [c, d]$, it follows that $\{\tau^r(x_0) \mid r \geqq 0\}$ is a finite set, and we are done.

Alternatively, each $y \in \Gamma(x)$ is an endpoint of a gap J_y of C and $J_y \subset \operatorname{int}(I_j)$, some j. In this case, every chain defined at y is defined on all of J_y. Fix $x_0 \in \Gamma(x)$ and $J_0 = J_{x_0}$ such that $|J_0| \geqq |J_y|$, for every $y \in \Gamma(x)$.

Assume that $\Gamma(x)$ contains no τ-cycle. It follows that every reduced chain g, defined at x_0, is a simple chain at x_0 with $J_0 \subset D(g)$. We will show that this leads to a contradiction.

Let K_0 be a compact, nondegenerate interval such that $K_0 \cap J_0 = \{x_0\}$, $\partial K_0 = \{x_0, y_0\} \subset \Gamma(x)$, and $|K_0|/|J_0| \leqq 1/\lambda$, where λ is as in §2. By (2.2), whenever g is a chain such that $K_0 \subset D(g)$ (hence g reduces to a simple chain at x_0 and $J_0 \subset D(g)$), then $|g(K_0)| < |g(J_0)|$.

Let $\delta > 0$ be such that every point δ-close to I_k (respectively, to X_k) lies in $R(h_k)$ (respectively, in $D(h_k)$), $1 \leqq k \leqq m$. Let $r \geqq 1$ be an integer such that every chain g at x_0 with $|g| > r$ satisfies $|g(J_0)| < \delta$. Let $\gamma_1, \cdots, \gamma_p$ be the chains at x_0 with $|\gamma_i| \leqq r$ and, without prejudice to the properties of K_0 listed above, choose that interval so small that $K_0 \subset D(\gamma_i)$, $1 \leqq i \leqq p$. This guarantees, via (2.2), that $K_0 \cup J_0 \subset D(g)$, for each chain g defined at x_0. By induction on $|g|$, the same holds for every chain defined at y_0.

For definiteness, let $K_0 = [x_0, y_0]$. Since every chain defined at x_0 is simple at x_0, we obtain a well defined map $\varphi: \Gamma(x) \to \Gamma(x)$ by setting $\varphi(g(x_0)) = g(y_0)$. Similarly, every chain at $y_0 \in \Gamma(x)$ is simple and defined at x_0, so φ is bijective. Let $z_0 \in \Gamma(x)$ be the point such that $\varphi(z_0) = x_0$. Then there is a chain g such that $[z_0, x_0] = g(K_0) \supset J_0$. By (2.2) and the maximality of $|J_0|$, we obtain $|J_0| \leqq |g(K_0)| < |g(J_0)| \leqq |J_0|$. $\square$

PROOF OF THEOREM 1. Let $X \subset U \in O(\mathscr{F})$ be a Markov LMS as defined in the introduction. Let Γ, R, and $C = X \cap \operatorname{int}(R)$ also be as in the introduction. If Γ is a Markov pseudogroup on a neighborhood of C in R and if C is a Markov Γ-minimal set, then Theorem 1 is a consequence of (3.2), (3.3), and (3.4). By (1.2), no generality is lost in making these assumptions. $\square$

REMARKS. (1) The leaves $L \subset X$ such that $H_x(L, X) \cong \boldsymbol{Z}$ must be *resilient*. That is, such a leaf L has an element of contracting holonomy on at least one side and L itself meets the interior of the support of this contraction. Resiliency figures in many important properties of foliated manifolds. For example, the nonvanishing of the exotic characteristic class $\mathrm{gv}(\mathscr{F})$ implies the presence of a resilient leaf [Du]. Another example is the *entropy* of $(M, \mathscr{F})$, defined in [G-L-W] and proven there to be nontrivial if and only if there is a resilient leaf.

(2) The exceptional minimal set constructed in [Sa 1] is a Markov minimal set. Consequently, the assertion in [H-H, 3.9, p. 114], that this set contains only two leaves with nontrivial holonomy, is erroneous.

4. Markov sub-pseudogroups. In the proofs of Theorem 2 and Theorem 3, the most delicate step will involve passing to a Markov sub-pseudogroup. It will be helpful to have discussed the salient features of this process before getting into the proofs of the theorems.

Let $\mathscr{S} = (\{I_1, \cdots, I_m\}, \{h_1, \cdots, h_m\}, P)$ be a Markov system and let $\Gamma = \Gamma_{\mathscr{S}}$. Fix $n \geqq 2$ and let $W_n = \{w_i\}_{i=1}^q$ be the set of all words of length n in positive powers of the generators $h_1, \cdots, h_m$. For $w \in W_n$, write $w = h_{\lambda(w)} \circ v$, $1 \leqq \lambda(w) \leqq m$, and let h_w be the restriction of $h_{\lambda(w)}$ to an open neighborhood $D(h_w)$ of I_v. Choosing all $D(h_w)$ small enough guarantees that $R(h_w) \cap R(h_{w'}) = \varnothing$, whenever $w \neq w'$. We denote the interval $I_v = h_w^{-1}(I_w)$ by X_w. Let $Q = (q_{ij})$ be the $q \times q$ matrix with entries $\{0, 1\}$ such that $q_{ij} = 0$ if and only if $I_{w_j} \cap D(h_{w_i}) = \varnothing$. Thus, $q_{ij} = 1$ if and only if $I_{w_j} \subseteqq X_{w_i}$. Then $\mathscr{S}_n = (\{I_{w_1}, \cdots, I_{w_q}\}, \{h_{w_1}, \cdots, h_{w_q}\}, Q)$ is a Markov system. Let $\Gamma_n = \Gamma_{\mathscr{S}_n}$ and, for uniformity, let $\Gamma_1 = \Gamma$.

In discussions where more than one Markov pseudogroup, say $\{\Gamma',$

$\Gamma'', \cdots, \Gamma^{(l)}\}$, are involved, we use the notation $Z_0(\Gamma^{(k)})$ to denote the set Z_0 determined by the Markov pseudogroup $\Gamma^{(k)}$ as in §1, $1 \leqq k \leqq l$.

(4.1) LEMMA. *If Γ is a Markov pseudogroup and $n \geqq 1$, then $Z_0(\Gamma_n) = Z_0(\Gamma) = Z_0$ and $\Gamma_n | Z_0 = \Gamma | Z_0$.*

Let $1 \leqq i_1 < \cdots < i_r \leqq m$. Then we can define a Markov system $\mathscr{S}(i_1, \cdots, i_r) = (\{I_{i_j}\}_{j=1}^r, \{h_{i_j}\}_{j=1}^r, (p_{i_j i_k}))$ and this gives rise to a sub-pseudogroup $\Gamma(i_1, \cdots, i_r) \subseteqq \Gamma$.

(4.2) LEMMA. $Z_0(\Gamma(i_1, \cdots, i_r)) \subseteqq Z_0(\Gamma)$.

We combine these two constructions. The elements of $W_n = \{w_1, \cdots, w_q\}$ serve as indices in $\mathscr{S}_n$, so we define $\Gamma^* = \Gamma_n(w_{i_1}, \cdots, w_{i_p})$, $1 \leqq i_1 < \cdots < i_p \leqq q$, obtaining a Markov sub-pseudogroup of Γ. By renumbering, we set $\Gamma^* = \Gamma_n(w_1, \cdots, w_p)$.

Let $Z_0 = Z_0(\Gamma)$, $Z_0^* = Z_0(\Gamma^*)$, and let $h_j^* = h_{w_j}$, $1 \leqq j \leqq p$. By (4.1) and (4.2), we see that $Z_0^* \subseteqq Z_0$. The typical chain $g^* \in \Gamma^*$ is of the form

$$g^* = h_{j_1}^* \circ \cdots \circ h_{j_s}^* \circ (h_{j_{s+1}}^*)^{-1} \circ \cdots \circ (h_{j_t}^*)^{-1}$$

and has canonical extension to a chain

$$g = h_{\lambda(j_1)} \circ \cdots \circ h_{\lambda(j_s)} \circ h_{\lambda(j_{s+1})}^{-1} \circ \cdots \circ h_{\lambda(j_t)}^{-1} \in \Gamma \ .$$

Finally, the essential subshift is $\tau^* = \tau | Z_0^*$.

(4.3) LEMMA. *Let $g^* \in \Gamma^*$ be a chain, $g \in \Gamma$ the canonical extension, and let $x \in (Z_0 \setminus Z_0^*) \cap D(g)$. Then $g(x) \notin Z_0^*$.*

PROOF. Write g^* and g in the above forms and assume that $g(x) \in Z_0^*$. By a finite induction, we readily obtain that $y = \tau^s(g(x)) = (\tau^*)^s(g(x))$ is an element of Z_0^*. But $y = h_{\lambda(j_{s+1})}^{-1} \circ \cdots \circ h_{\lambda(j_t)}^{-1}(x)$, so another finite induction shows that

$$x = h_{\lambda(j_t)} \circ \cdots \circ h_{\lambda(j_{s+1})}(y) = h_{j_t}^* \circ \cdots \circ h_{j_{s+1}}^*(y)$$

which is also an element of Z_0^*, a contradiction. □

(4.4) LEMMA. *Let g and g^* be as in* (4.3). *Let* $\mathrm{int}(J_0^*) \subset \boldsymbol{R} \setminus Z_0^*$ *and let K_0^* be a compact, nondegenerate interval such that $J_0^* \cap K_0^* = \{u_0\}$, $u_0 \in Z_0^*$, and $|K_0^*|/|J_0^*| \leqq 1/\lambda$. Let g^* be a simple chain, a simple loop, or a basic loop at u_0 such that $J_0^* \cup K_0^* \subset D(g)$. Then $|g(K_0^*)| < |g(J_0^*)|$ and $g'(u)/g'(v) < \lambda$, for all points $u, v \in J_0^* \cup K_0^*$.*

PROOF. By (4.3), each of the intervals $J_0^*, \cdots, J_t^*$ (we are using notation from §2) has interior missing Z_0^*, while $u_k \in Z_0^*$, $0 \leqq k \leqq t$. Therefore, if $u_i \neq u_k$, then $\mathrm{int}(J_i^*)$ and $\mathrm{int}(J_k^*)$ lie in distinct components of $\boldsymbol{R}\setminus Z_0^*$. The fact that these open intervals are disjoint is all that is needed in order to mimic the proof of (2.2). $\square$

5. The measure of a Markov set. As usual, let $\mathscr{S}=(\{I_j\}_{j=1}^m, \{h_j\}_{j=1}^m, P)$ be a Markov system and set $\Gamma=\Gamma_{\mathscr{S}}$. We do *not* assume that Z_0 contains a Cantor set since that property can be lost by passing to a subpseudogroup.

For $Y \subseteqq \boldsymbol{R}$ a measurable set, we denote by $|Y|$ its Lebesgue measure.

(5.1) THEOREM. $|Z_0| = 0$.

By (1.2), this result generalizes Theorem 3. The proof of (5.1) will consist of a series of definitions and lemmas.

DEFINITION. If $A \subseteqq Z_0$ and N is a nonnegative integer, then $A_N = \bigcup_{n \geqq N} \tau^{-n}(A)$ and $A_\infty = \bigcap_{N=0}^\infty A_N$.

(5.2) LEMMA. $(A \cup A')_\infty = A_\infty \cup A'_\infty$.

(5.3) LEMMA. $(\tau^{-1}A)_\infty = A_\infty$.

If $g \in \Gamma$, we adopt the notation gA for $g(A \cap D(g))$. We also set $\Gamma_+ = \bigcup_{n=1}^\infty W_n$, the set of nontrivial words in nonnegative powers of $h_1, \cdots, h_m$.

(5.4) LEMMA. $(\gamma A)_\infty \subseteqq A_\infty$, *for each* $\gamma \in \Gamma_+$.

(5.5) LEMMA. $A_\infty = (h_1A)_\infty \cup \cdots \cup (h_mA)_\infty$.

PROOF. Use the fact that $\tau^{-1}(A) = h_1A \cup \cdots \cup h_mA$, together with (5.2) and (5.3). $\square$

(5.6) LEMMA. $(Z_0)_\infty = Z_0$.

(5.7) LEMMA. *Let* $x \in Z_0$. *If* x *is not* τ*-cyclic, then* $\{x\}_\infty = \varnothing$. *Otherwise,* $\{x\}_\infty = \Gamma(x)$.

(5.8) LEMMA. *Let* $X = \{x_0, \cdots, x_p\}$ *be a* τ*-cycle. Let* $y_1, \cdots, y_t$ *be the elements of* $\Gamma(x_0)\setminus X$ *such that* $\tau(y_\alpha) \in X$, $1 \leqq \alpha \leqq t$. *Let* $\gamma_\alpha \in \Gamma_+$ *be of minimal length such that* $\gamma_\alpha(x_0) = y_\alpha$, $1 \leqq \alpha \leqq t$. *Let* J *be an interval, let* $j \in \{1, \cdots, m\}$, *and let* $x_0 \in V = J \cap I_j \cap Z_0$. *Then* $V_\infty = F \cup (\bigcup_{\alpha=1}^t (\gamma_\alpha V)_\infty)$, *where the set* F *is at most countable.*

PROOF. (1) Let $h_{j_i}(x_i) = x_{i+1}$, $0 \leqq i \leqq p$, let $x_{p+1} = x_0$, and let $\gamma_0 = h_{j_p} \circ \cdots \circ h_{j_0}$. By the dynamics of the subshift $\sigma\colon \mathscr{K}_P \to \mathscr{K}_P$, it is clear that

$$\gamma_0\colon I_j \cap Z_0 \to I_j \cap Z_0$$

has either one fixed point x_0 or two such, x_0 and $\bar{x}_0$. In the second case,

$\bar{x}_0$ lies in a τ-cycle $\{\bar{x}_0, \cdots, \bar{x}_p\}$. Consider both cases at once by allowing $x_i = \bar{x}_i$. Then the set $\{x_0, \bar{x}_0\}$ is exactly $\lim_{n\to\infty} \gamma_0^n(Z_0 \cap I_j)$, and we let F denote the set $\Gamma(x_0) \cup \Gamma(\bar{x}_0)$.

(2) By (5.4) and (5.7), $V_\infty \supseteqq F \cup (\gamma_1 V)_\infty \cup \cdots \cup (\gamma_t V)_\infty$.

(3) Let $y \in V_\infty$ and let $\{n(k)\}_{k=1}^\infty$ be the strictly increasing sequence of positive integers such that $\tau^{n(k)}(y) \in V$, for all $k \geqq 1$. Set $V_i = h_{j_{i-1}} \circ \cdots \circ h_{j_0}(V)$, for each $i \geqq 1$, and set $V_0 = V$. Since $V = J \cap I_j \cap Z_0$ and J is an interval, we see that $V_{p+1} \subseteqq V_0$. For a given $k \geqq 1$, $\tau^{n(k)-1}(y) \in \tau^{-1}(\tau^{n(k)}(y)) \subseteqq \tau^{-1}(V_0)$ and this latter set is exactly $V_1 \cup \gamma_{\alpha_1}(V) \cup \cdots \cup \gamma_{\alpha_r}(V)$, for suitable indices $1 \leqq \alpha_1 < \cdots < \alpha_r \leqq t$. If $\tau^{n(k)-1}(y) \in V_1$, repeat this procedure to get $\tau^{n(k)-2}(y) \in V_2 \cup \gamma_{\beta_1}(V) \cup \cdots \cup \gamma_{\beta_q}(V)$, for suitable $1 \leqq \beta_1 < \cdots < \beta_q \leqq t$, etc. There are two cases.

Case 1. For all large values of n, $\tau^n(y) \in \cup_{i=0}^p V_i$. Thus, find $N \geqq 0$ such that $\tau^N(y) \in V$ and $\gamma_0^{-n}(\tau^N(y))$ is defined and belongs to V, for each $n \geqq 0$. By the compactness of I_j and step (1), we conclude that $\tau^N(y) \in \{x_0, \bar{x}_0\}$, hence $y \in F$.

Case 2. There is a strictly increasing sequence of integers $\{m(k)\}_{k=1}^\infty$ such that $\tau^{m(k)}(y) \in \gamma_1(V) \cup \cdots \cup \gamma_t(V)$, for each $k \geqq 1$. Thus $y \in (\gamma_1 V)_\infty \cup \cdots \cup (\gamma_t V)_\infty$. □

DEFINITION. A point $x \in Z_0$ is good if there is a neighborhood V of x in Z_0 such that $|V_\infty| = 0$. The set of good points is denoted G. The bad set is $B = Z_0 \setminus G$.

Remark that B is compact.

(5.9) LEMMA. *$B = \varnothing$ if and only if $|Z_0| = 0$.*

PROOF. If $|Z_0| = 0$, it is evident that $B = \varnothing$. If $B = \varnothing$, the compactness of Z_0 allows us to find a finite cover $V_1, \cdots, V_r$ of Z_0 by relatively open subsets, each with $|(V_i)_\infty| = 0$. Then (5.2) and (5.6) imply that $|Z_0| = 0$. □

(5.10) LEMMA. *Let $V \subseteqq Z_0$ be measurable. If $\sum_{g \in \Gamma_+} |gV|$ converges, then $|V_\infty| = 0$.*

PROOF. Since $\tau^{-n}(V) = \cup_{g \in W_n} gV$, we obtain, for each $N \geqq 1$, the inequalities

$$|V_\infty| = \left|\bigcap_{k=1}^\infty \bigcup_{n \geqq k} \tau^{-n}(V)\right| \leqq \left|\bigcup_{n \geqq N} \tau^{-n}(V)\right| \leqq \sum_{|g| \geqq N} |gV| . \qquad \square$$

(5.11) LEMMA. *Let $N \geqq 1$ be an integer. Then $x \in Z_0$ is a good point if and only if $g(x) \in G$, for each $g \in W_N$.*

PROOF. Use (5.5) and induction on N. □

(5.12) LEMMA. *Let $x \in I_j \cap Z_0$. If $\tau^{-n}(x) = \varnothing$, some $n \geqq 1$, then $(I_j \cap Z_0)_\infty = \varnothing$, hence $x \in G$.*

PROOF. Equivalently, $x \in D(g)$ for only finitely many $g \in \Gamma_+$. That is, for all but these finitely many $g \in \Gamma_+$, $I_j \cap D(g) = \varnothing$. $\square$

(5.13) LEMMA. *Let J_0 be a gap of Z_0 and assume that $J_0 \subset I_j$, some j. Let $K_0 \subset I_j$ be a compact, nondegenerate interval such the $J_0 \cap K_0 = \{x_0\}$ and $|K_0|/|J_0| \leqq 1/\lambda$. Then $|(Z_0 \cap (J_0 \cup K_0))_\infty| = 0$. In particular, $x_0 \in G$.*

PROOF. There are two cases.

Case 1. Assume that x_0 does not lie in a τ-cycle. By (5.12), we can also assume that $x_0 \in D(g)$ for infinitely many $g \in \Gamma_+$, each of which must be a simple chain at x_0 with $J_0 \cup K_0 \subset D(g)$. By (2.2), $|gK_0| < |gJ_0|$ for all such g. The intervals $\{gJ_0\}_{g \in \Gamma_+}$ have disjoint interiors, so

$$\sum_{g \in \Gamma_+} |g(J_0 \cup K_0)| \leqq 2 \cdot \sum_{g \in \Gamma_+} |gJ_0| < \infty .$$

Take $V = Z_0 \cap (J_0 \cup K_0)$ in (5.10).

Case 2. Assume that x_0 is τ-cyclic. Then the neighborhood $V = Z_0 \cap (J_0 \cup K_0)$ and the point x_0 satisfy the hypotheses of (5.8). Since $\gamma_\alpha(x_0)$ does not lie in a τ-cycle, $1 \leqq \alpha \leqq t$, we can apply the argument in Case 1 to all chains of the form $g \circ \gamma_\alpha$, $g \in \Gamma_+$, to conclude that

$$\sum_{g \in \Gamma_+} |g(\gamma_\alpha(J_0 \cup K_0))| < \infty .$$

Then (5.10) implies that $|(Z_0 \cap \gamma_\alpha(J_0 \cup K_0))_\infty| = 0$, $1 \leqq \alpha \leqq t$. By (5.8), $|(Z_0 \cap (J_0 \cup K_0))_\infty| = 0$. $\square$

(5.14) LEMMA. *Let $x \in Z_0$ and assume that Z_0 accumulates on x from at most one side. Then $x \in G$.*

PROOF. By (5.12), we can assume that J_0 and K_0 are as in (5.13) and that $J_0 \cap K_0 = \{x\}$, hence $x \in G$. $\square$

In the following lemma, let $W_n = \{w_1, \cdots, w_{p+r}\}$ and $\Gamma^* = \Gamma_n(w_1, \cdots, w_p)$, as in §4. As usual, $Z_0^* = Z_0(\Gamma^*)$ and $\tau^* = \tau | Z_0^*$. We will also use the notation

$$V_{\infty *} = \bigcap_{N=0}^{\infty} \bigcup_{n \geqq N} (\tau^*)^{-n}(V) ,$$

for all $V \subseteqq Z_0^*$.

(5.15) LEMMA. *Let $A_i = I_{w_{p+i}}$, $1 \leqq i \leqq r$, and let $V \subseteqq Z_0$. Then $V_\infty \subseteqq (\bigcup_{i=1}^r (A_i)_\infty) \cup (\bigcup_{k \geqq 0} \tau^{-k}((V \cap Z_0^*)_{\infty *}))$.*

PROOF. Let $x \in V_\infty$ and consider two cases.

Case 1. Assume that $\tau^k(x) \in Z_0^*$, some $k \geqq 0$. Then one can find infinitely many $n \geqq k$ such that $(\tau^*)^{n-k}(\tau^k(x)) = \tau^{n-k}(\tau^k(x)) = \tau^n(x) \in V$. Therefore, $(\tau^*)^{n-k}(\tau^k(x)) \in Z_0^* \cap V$, so $\tau^k(x) \in (Z_0^* \cap V)_{\infty *}$. That is, $x \in \tau^{-k}((Z_0^* \cap V)_{\infty *})$.

Case 2. Assume that there is no $k \geqq 0$ as in Case 1. Then there are infinitely many integers $n \geqq 0$ such that $\tau^n(x) \notin \bigcup_{i=1}^p I_{w_i}$. For these integers, $\tau^n(x) \in A_1 \cup \cdots \cup A_r$, hence $x \in (A_1)_\infty \cup \cdots \cup (A_r)_\infty$. □

(5.16) PROPOSITION. $B = \varnothing$.

PROOF. Let $y \in B$ and deduce a contradiction as follows.

(1) By (5.14), $y \in \operatorname{int}(I_j)$, for some j, and there is a gap $J_0 = [a, b]$ of Z_0, also in I_j, such that $b < y$. Since B is compact and, by (5.14), $b \notin B$, there is a point $y_0 \in]b, y] \cap B$ such that $]b, y_0[\cap B = \varnothing$.

(2) By (5.14), Z_0 clusters on y_0 from both sides. Thus, for n sufficiently large, there is $w \in W_n$ such that $I_w = [c, d]$, $b < c < y_0 < d$, and $|I_w|/|J_0| \leqq 1/\lambda$.

(3) Enumerate $W_n = \{w_1, \cdots, w_{p+r}\}$ in such a way that $I_{w_j} \cap]a, c[= \varnothing$ exactly for $j = 1, \cdots, p$. Set $A_i = I_{w_{p+i}}$, $1 \leqq i \leqq r$, and note that each $A_i \subset [b, c[$. By the choice of y_0, $|(A_i)_\infty| = 0$, $1 \leqq i \leqq r$.

(4) Let $\Gamma^* = \Gamma_n(w_1, \cdots, w_p)$, as in §4, and let b^* be the minimal element of $Z_0^* \cap [c, d]$. Then b^* is maximal such that $Z_0^* \cap]a, b^*[= \varnothing$. Consider three cases.

Case 1. Let $b^* = d$, hence $]a, d[\cap Z_0^* = \varnothing$. Take $V =]a, d[\cap Z_0$ and use (5.15) to conclude that $V_\infty \subseteqq \bigcup_{i=1}^r (A_i)_\infty$. By step (3), $|V_\infty| = 0$ and this contradicts the fact that $y_0 \in B$.

Case 2. Let $b^* < d$, and $(\tau^*)^{-n}(b^*) = \varnothing$, some $n \geqq 1$. For V, we take any neighborhood of y_0 in the set $[c, d] \cap Z_0 = I_w \cap Z_0$. By (5.12), $(V \cap Z_0^*)_{\infty *} = \varnothing$, so the argument in Case 1 again yields a contradiction.

Case 3. Let $b^* < d$ and $b^* \in D(g^*)$ for infinitely many $g^* \in \Gamma_+^*$. Set $J_0^* = [a, b^*]$, $K_0^* = [b^*, d]$, and $V = Z_0 \cap (J_0^* \cup K_0^*)$. Then $\operatorname{int}(J_0^*) \subset \boldsymbol{R} \setminus Z_0^*$, but $b^* \in Z_0^*$, and $|K_0^*|/|J_0^*| \leqq |I_w|/|J_0| \leqq 1/\lambda$. We would like to use (5.13) to conclude that $|(V \cap Z_0^*)_{\infty *}| = 0$. The difficulty is that $J_0^* \nsubseteqq I_w$, contrary to what is required by the hypotheses of (5.13). This means that, if $g^* \in \Gamma_+^*$ and $b^* \in D(g^*)$, then $J_0^* \nsubseteqq D(g^*)$. The solution to this difficulty is to use the canonical extension $g \in \Gamma_+$ of each such g^*, to note that $J_0^* \subset I_j \subset D(g)$, and to appeal to (4.4) in place of (2.2) in the proof of (5.13). Then $|(V \cap Z_0^*)_{\infty *}| = 0$ and we use (5.15) to conclude that $|V_\infty| = 0$, again contradicting the fact that $y_0 \in B$. □

The proof of (5.1), hence of Theorem 3, is complete.

6. Counting the semiproper leaves. Our present goal is to prove Theorem 2. Let $\mathscr{S}$ and $\Gamma = \Gamma_{\mathscr{S}}$ be as in the previous section.

DEFINITION. Let $A \subseteqq \boldsymbol{R}$, $x \in A$. Then x is semi-isolated in A if there is an open interval $J \subset \boldsymbol{R} \setminus \bar{A}$ such that $x \in \partial \bar{J}$.

DEFINITION. The Γ-orbit of $x \in Z_0$ is semiproper if there is a non-empty, open interval $J \subset \boldsymbol{R} \setminus \overline{\Gamma(x)}$ such that $x \in \partial \bar{J}$.

(6.1) THEOREM. *Only finitely many τ-cyclic points are semi-isolated in Z_0.*

(6.2) COROLLARY. *Let $C \subseteqq Z_0$ be a Markov Γ-minimal set. Then C contains only finitely many semiproper Γ-orbits.*

PROOF. Every Γ-orbit in C is dense in C, hence is semiproper if and only if each of its points is semi-isolated in C. Apply (6.1) and (3.4). □

In particular, (6.2) and (1.2) establish Theorem 2.

The proof of (6.1) will consist of a series of definitions and lemmas.

DEFINITION. A nondegenerate interval A is Γ_+-uniform if there is a number $\nu = \nu(A) > 0$ such that $g'(u)/g'(v) \leqq \nu$, for all $g \in \Gamma_+$ and u, $v \in A \cap D(g)$.

(6.3) LEMMA. *If A_1 and A_2 are Γ_+-uniform intervals and $A_1 \cap A_2 \neq \varnothing$, then $A_1 \cup A_2$ is Γ_+-uniform.*

(6.4) LEMMA. *If $g \in \Gamma_+$ and A is a Γ_+-uniform interval, then gA is also Γ_+-uniform.*

(6.5) LEMMA. *Let $N \geqq 1$ be an integer and let A be an interval. If gA is Γ_+-uniform, for all $g \in W_N$, then A is Γ_+-uniform.*

DEFINITION. The uniform set $U_+ \subseteqq \boldsymbol{R}$ is the union of all open, Γ_+-uniform intervals. The non-uniform set is $B_+ = \boldsymbol{R} \setminus U_+$.

Since intervals not meeting the bounded set $\bigcup_{j=1}^{m} D(h_j)$ are Γ_+-uniform by default, B_+ is compact. Also, if G and B are as in § 5, one easily shows that $Z_0 \cap U_+ \subseteqq G$ and that $Z_0 \cap B_+ \supseteqq B$.

(6.6) LEMMA. *Let J be a gap of Z_0 with an endpoint that is not τ-cyclic. Then, with at most finitely many exceptions, $J \subset U_+$.*

PROOF. Let $x \in \partial J$ be a point that is not τ-cyclic. Since the endpoints of $X_j = h_j^{-1}(I_j)$ are τ-cyclic, $1 \leqq j \leqq m$, provided they pertain to Z_0, we see that $x \in \operatorname{int}(X_j)$ and, with at most finitely many exceptions, $J \subset X_j$, for some j. It follows that $J \cap D(g) \neq \varnothing$ if and only if $J \subset D(g)$, for each $g \in \Gamma_+$. Every such g is a simple chain at x, hence g is also a simple chain at the other point $\tilde{x} \in \partial J$ and $\tilde{x}$ is not τ-cyclic. Therefore, $J \subset \operatorname{int}(X_j)$. Let

K and $\tilde{K}$ be compact, nondegenerate subintervals of X_j such that $K \cap J = \{x\}$, $\tilde{K} \cap J = \{\tilde{x}\}$, and $|K|/|J| \leqq 1/\lambda \geqq |\tilde{K}|/|J|$. By (2.2), $K \cup J$ and $J \subset \tilde{K}$ are Γ_+-uniform, hence, by (6.3), so is $K \cup J \cup \tilde{K}$. □

(6.7) LEMMA. *Let $x \in Z_0$ be τ-cyclic and let $\gamma \in \Gamma_+$ be the simple loop at x. If $\gamma'(x) = 1$ and γ is not germinally the identity at x on at least one side of x, then $x \in B_+$.*

PROOF. In any neighborhood of x in $\boldsymbol{R}$, there is a γ-fixed point y (perhaps x itself) at which γ is either a (one-side) expansion or contraction. Thus, find a sequence $\{y_n\}_{n \geqq 1}$ near y such that either $(\gamma^n)'(y_n) \to \infty$ or $(\gamma^n)'(y_n) \to 0$ as $n \to \infty$. Since $(\gamma^n)'(x) = 1$, for each integer n, it is clear that $x \in B_+$. □

DEFINITION. A gap J of Z_0 is τ-cyclic if $J \subseteqq X_j$, for some $j \in \{1, \cdots, m\}$, and ∂J consists of τ-cyclic points.

Equivalently, there exists $\gamma \in \Gamma_+$ (nontrivial, by the definition of Γ_+) such that $\gamma J = J$.

(6.8) LEMMA. *Let J be a gap of Z_0. Then, with at most finitely many exceptions, J is τ-cyclic if and only if $J \cap B_+ \neq \varnothing$.*

PROOF. (1) Assume that $J \cap B_+ \neq \varnothing$. With at most finitely many exceptions, this implies that ∂J consists of τ-cyclic points (6.6) and $J \subseteqq X_j$, for some $j \in \{1, \cdots, m\}$.

(2) Suppose that J is τ-cyclic and let $\gamma J = J$, for some $\gamma \in \Gamma_+$. We consider three cases.

Case 1. $\gamma | J \neq \mathrm{id}_J$. Find sequences $\{y_n\}_{n \geqq 1}$ and $\{z_n\}_{n \geqq 1}$ in J such that $(\gamma^n)'(y_n) \to 0$ and $(\gamma^n)'(z_n) \to \infty$ as $n \to \infty$. Thus, J is not Γ_+-uniform. By (6.3) and compactness, $J \subset U_+$ would imply that J is Γ_+-uniform.

Case 2. $\gamma | J = \mathrm{id}_J$, but at least one $x \in \partial J$ is not isolated in Z_0. An elementary use of symbolic dynamics shows that γ is a one-sided contraction to x. By (6.7), $x \in B_+$.

Case 3. $\gamma | J = \mathrm{id}_J$ and both elements of ∂J are isolated in Z_0. In this case, it might be that $J \subset U_+$. We must show that at most finitely many such gaps exist. Suppose, on the contrary, that $\{J_n\}_{n \geqq 1}$ is an infinite sequence of distinct such gaps, chosen so that $J_n \to x \in Z_0$ as $n \to \infty$. Write $x = x_{i_1 i_2 \cdots}$, where $(i_k)_{k=1}^{\infty} = h(x) \in \mathscr{K}_P$, as in § 1. The set $V = \{(j_k) \in \mathscr{K}_P \mid j_1 = i_1\}$ is a neighborhood of (i_k) in $\mathscr{K}_P$ and $h(\partial J_n) \subset V$, for all large values of n. The points of $\partial J_n = \{x, \tilde{x}_n\}$ are τ-cyclic, so one writes $h(\tilde{x}_n) = h(x_n) = \alpha \cdot \alpha \cdots \alpha \cdots$ (juxtaposition), where $\alpha = (i_1, j_2, \cdots, j_s)$. In particular, the entry $p_{j_s i_1} = 1$ in the matrix P. It follows that $\alpha \cdot \alpha \cdots \alpha \cdot (i_k) \in \mathscr{K}_P$, for all $r \geqq 1$. This translates to the statement that either x_n or $\tilde{x}_n$

is not isolated in Z_0, contradicting the assumption. □

(6.9) LEMMA. *Let $x_0 \in Z_0 \cap B_+$ and assume that x_0 lies in a τ-cycle $X = \{x_0, x_1, \cdots, x_p\}$. Let $\gamma_0 = h_{j_p} \circ \cdots \circ h_{j_0}$ be the simple loop at x_0 and write $\gamma_{0i} = h_{j_i} \circ \cdots \circ h_{j_0}$, $0 \leqq i \leqq p-1$. If $\Gamma_+(x_0) \cap B_+ \subseteqq X$, then there is a neighborhood V_0 of x_0 in Z_0 such that $z \in V_0 \cap B_+$ if and only if $\Gamma_+(z) \cap B_+ = \{\gamma_{0i} \circ \gamma_0^n(z) \mid n \geqq 0,\ 0 \leqq i \leqq p-1\}$.*

PROOF. Let y_α and γ_α be as in (5.8), $1 \leqq \alpha \leqq t$. In particular, $y_\alpha \notin B_+$, $1 \leqq \alpha \leqq t$. Let U_α be an open, Γ_+-uniform interval about y_α and set $V_\alpha = Z_0 \cap U_\alpha$. Choose the neighborhood V_0 of x_0 in Z_0 such that $x_0 \in D(\gamma_\alpha)$ implies that $V_0 \subset D(\gamma_\alpha)$ and $\gamma_\alpha V_0 \subseteqq V_\alpha$, $0 \leqq \alpha \leqq t$. If $z \in V_0 \cap B_+$, it follows from this choice and from (6.4) that $\Gamma_+(z) \cap B_+ \subseteqq \{\gamma_{0i} \circ \gamma_0^n(z) \mid n \geqq 0,\ 0 \leqq i \leqq p-1\}$. By (6.5) and induction, one obtains the reverse inclusion. □

DEFINITION. The set $\mathscr{J}_+$ consists of all gaps J of Z_0 such that $J \cap B_+ \neq \varnothing$. The set $\mathscr{B}_+$ is $\mathscr{J}_+ \cup (Z_0 \cap B_+)$.

It makes sense to consider, in $\boldsymbol{R}$, the cluster points of $\mathscr{B}_+$, each element of $\mathscr{J}_+$ being treated as a single "point". These cluster points are honest points, necessarily being elements of $Z_0 \cap B_+$.

DEFINITION. The set $B_0 \subseteqq Z_0 \cap B_+$ consists of those points x with a half-open neighborhood $]a, x]$ or $[x, a[$ meeting no element of $\mathscr{B}_+$ except x.

We are going to prove, in (6.15), that every $x \in B_0$ is τ-cyclic. For the present we suppose that x is a counterexample.

(6.10) CLAIM. The non-τ-cyclic point $x \in B_0$ is not semi-isolated in Z_0.

PROOF. Suppose that x is semi-isolated in Z_0 and consider two cases.

Case 1. $\Gamma_+(x)$ is infinite. By (6.5), we assume, without loss of generality, that x is an endpoint of a gap J of Z_0 and that $J \subset \operatorname{int}(X_i)$, $X_i = h_i^{-1}(I_i)$, for some $i \in \{1, \cdots, m\}$. As in the proof of (6.6), it follows that $x \in U_+$.

Case 2. $\Gamma_+(x)$ is finite. Then x has an open, connected neighborhood that meets $D(g)$ for only finitely many $g \in \Gamma_+$. Such an interval is Γ_+-uniform, again contradicting the assumption that $x \in B_+$. □

For definiteness, assume that $]a, x]$ is as in the definition of the set B_0. By (6.10), we can assume that $[a, x] \subset \operatorname{int}(X_i)$, for some $i \in \{1, \cdots, m\}$, and that $J = [a, b]$ is a gap of Z_0 and $b < x$.

Choose n large and $w \in W_n$ such that $I_w = [c, d]$, $b < c < x < d$, and $|I_w|/|J| \leqq 1/\lambda$. Let $W_n = \{w_1, \cdots, w_{p+r}\}$, so ordered that $I_{w_j} \subset [b, c[$ exactly for $j = p + i$, $1 \leqq i \leqq r$. Then $I_{w_j} \cap\,]a, c[\ = \varnothing$, $1 \leqq j \leqq p$. Set $A_i = I_{w_{p+i}}$,

$1 \leqq i \leqq r$.

(6.11) CLAIM. $\bigcup_{s=1}^{r} A_s \subset U_+$.

As in § 4, let $\Gamma^* = \Gamma_n(w_1, \cdots, w_p)$ with generators $h_1^*, \cdots, h_p^*$, and let $Z_0^* = Z_0(\Gamma^*)$, $\tau^* = \tau | Z_0^*$. By means of Γ^*, we define U_+^* and $B_+^* = \boldsymbol{R} \setminus U_+^*$ in exact analogy with U_+ and B_+. Let $D^* = \bigcup_{j=1}^{p} D(h_j^*)$.

(6.12) CLAIM. $U_+^* \cap D^* \subseteqq U_+$.

PROOF. Let $z \in U_+^* \cap D^*$. Let $I \subset U_+^*$ be a compact interval with $z \in \mathrm{int}(I)$. We choose I so small that $z \in D(h_j^*)$ implies that $I \subset D(h_j^*)$, $1 \leqq j \leqq p$. By the definition of a Markov system, it follows that $g^* \in \Gamma_+^*$ and $z \in D(g^*)$ implies that $I \subset D(g^*)$.

Let $\gamma \in \Gamma_+$ be such that $z \in D(\gamma)$ and write $\gamma | I = (v \circ h_k \circ g^*)| I$, where $g^* \in \Gamma_+^*$, $h_k(g^* I) \subset \bigcup_{s=1}^{r} A_s$, and $v \in \Gamma_+$. By (6.11) and the fact that I is Γ_+^*-uniform, it follows easily that I is Γ_+-uniform. □

(6.13) CLAIM. $x \in Z_0^*$.

PROOF. Otherwise, $\tau^k(x) \in \bigcup_{s=1}^{r} \mathrm{int}(A_s) \subset U_+$ (6.11), for some $k \geqq 1$. Let $g \in \Gamma_+$ be the branch of τ^{-k} such that $x \in R(g)$. An application of (6.4) to g gives the contradiction that $x = g(\tau^k(x)) \in U_+$. □

By (6.13), there is a maximal element $b^* \in [c, x]$ such that $Z_0^* \cap [c, b^*[= \varnothing$.

(6.14) CLAIM. b^* is τ^*-cyclic and $b^* < x$.

PROOF. In (4.4), let $J_0^* = [a, b^*]$ and $K_0^* = [b^*, d]$ be such that $|K_0^*|/|J_0^*| \leqq |I_w|/|J| \leqq 1/\lambda$. If b^* is not τ-cyclic, (4.4) implies that $[c, d]$ is Γ_+^*-uniform, hence $x \in \,]c, d[\, \subset U_+^* \cap D^* \subseteqq U_+$ (6.12), contradicting the fact that $x \in B_+$. □

(6.15) LEMMA. *Every element of B_0 is τ-cyclic.*

PROOF. Assume that $x \in B_0$ and that x is not τ-cyclic. Choose all data as in the above discussion and deduce a contradiction as follows.

(1) By (6.14), let $\gamma_0 \in \Gamma_+^*$ be the simple loop at b^*. Since $[c, d] = I_w \subset D(\gamma_0)$, and since x is not τ^*-cyclic, elementary symbolic dynamics shows that $\gamma_0^n(x) \to b^*$ as $n \to \infty$. In particular, $\gamma_0^n(x) \in U_+$, for each integer $n \geqq 1$.

(2) By (6.5), choose $g_k^* \in \Gamma_+^*$, $|g_k^*| = k$, such that $g_k^*(x) \in B_+$, for all $k \geqq 1$. By step (1) and (6.4), each g_k^* is a simple chain at b^*.

(3) By step (2), we can take N so large that $g_N^*(b^*)$ does not lie in the τ^*-cycle containing b^*, hence is not τ^*-cyclic. That is, if $g^* \in \Gamma_+^*$

and $g_N^*(b^*) \in D(g^*)$, then g^* is a simple chain at $g_N^*(b^*)$. By elementary symbolic dynamics, this implies that $g^* \circ g_N^*$ is a simple chain at b^*.

(4) By step (3) and (4.4), we proceed exactly as in the proof of (6.14) to find $(g^* \circ g_N^*)'(u)/(g^* \circ g_N^*)'(v) \leqq \lambda$, for each $u, v \in [c, d]$. Therefore $g_N^*[c, d]$ is Γ_+^*-uniform. By (6.12), it follows that $g_N^*]c, d[\subset U_+$, contradicting the fact that $g_N^*(x) \in B_+$. □

DEFINITION. The set $B_1 \subseteqq Z_0 \cap B_+$ consists of those points x having a half-open neighborhood $]a, x]$ or $[x, a[$ that contains exactly a countable infinity of elements of $\mathscr{B}_+$, this subset of $\mathscr{B}_+$ clustering exactly at x.

(6.16) LEMMA. *Every element of B_1 is τ-cyclic.*

PROOF. The idea is to mimic the proof of (6.15) exactly. However, one needs (6.15) in order to set this up.

Let $x \in B_1$. Exactly as in the proof of (6.10), x is not semi-isolated in Z_0.

Assume that $]a, x]$ is as in the above definition, that $[a, x] \subset \operatorname{int}(X_i)$, for some $i \in \{1, \cdots, m\}$, and that $J = [a, b]$ is a gap of Z_0, $b < x$.

Choose n large, $w \in W_n$ as before, $I_w = [c, d]$, $b < c < x < d$, $|I_w|/|J| \leqq 1/\lambda$, etc. The problem is that (6.11) does not necessarily hold this time. It may be that $\bigcup_{s=1}^r A_s$ contains finitely many elements of $\mathscr{B}_+$, say $y_1, \cdots, y_t \in B_0$ and $J_1, \cdots, J_u \in \mathscr{J}_+$.

Choose $g_k \in \Gamma_+$, $[a, d] \subset D(g_k)$, $|g_k| = k$, such that $g_k(x) \in B_0 \cup B_1$, for all $k \geqq 1$. These exist by (6.4) and (6.5). Also, by (6.4), $\{g_k(y_i)\}_{i=1}^t \cup \{g_k(J_i)\}_{i=1}^u$ are the only possible elements of $\mathscr{B}_+$ in $g_k(\bigcup_{s=1}^r A_s)$.

By (6.6), y_i is τ-cyclic, $1 \leqq i \leqq t$. As in the proof of (6.15), k large enough implies that $g_k(y_i)$ is not τ-cyclic, hence $g_k(y_i) \notin B_0$, hence $g_k(y_i) \notin \mathscr{B}_+$, $1 \leqq i \leqq t$. A similar use of (6.8) shows that $g_k(J_i) \notin \mathscr{B}_+$, $1 \leqq i \leqq u$ and all large values of k.

Take $\tilde{g} = g_N$, N large enough. Then $\tilde{g}(\bigcup_{s=1}^r A_s) \subset U_+$. Replace w by $\tilde{w} = \tilde{g} \circ w \in W_{n+N}$, I_w by $I_{\tilde{w}} = [\tilde{g}(c), \tilde{g}(d)]$, J by $\tilde{J} = [\tilde{g}(a), \tilde{g}(b)]$, W_n by $W_{n+N} = \{\tilde{w}_1, \cdots, \tilde{w}_{q+r}\}$ (note that, generally, $q > p$), A_s by $\tilde{g}(A_s) = I_{\tilde{g} \circ w_{p+s}} = I_{\tilde{w}_{q+s}}$, $1 \leqq s \leqq r$, etc.

At this point, the proof of (6.15) can be carried out with no change. □

(6.17) LEMMA. $B_1 = \varnothing$.

PROOF. Let $x_0 \in B_1$ and deduce a contradiction. By (6.4), $g(x_0) \in B_1 \cup B_0 \cup U_+$, for each $g \in \Gamma_+$. By (6.15) and (6.16), $g(x_0) \in B_1 \cup B_0$ implies that $g(x_0)$ lies in the τ-cycle $X = \{x_0, x_1, \cdots, x_p\}$. Thus, $\Gamma_+(x_0) \cap B_+ \subseteqq X$. Let V_0 be the neighborhood of x_0 in Z_0 given by (6.9) and let $\gamma_0 \in \Gamma_+$ be the simple loop at x_0. By the definition of B_1, the set $(V_0 \setminus \{x_0\}) \cap B_+ \subseteqq B_0$ clusters at x_0. Elementary symbolic dynamics implies that γ_0 contracts this set to x_0, provided V_0 is small enough. If $z \in (V_0 \setminus \{x_0\}) \cap B_+$, then (6.15) and (6.4) imply that $\Gamma_+(z) \cap B_+$ is contained in a τ-cycle and (6.9) implies that $\Gamma_+(z) \cap B_+$ is not contained in a τ-cycle. □

(6.18) COROLLARY. *There is a finite, possibly empty set F such that $Z_0 \cap B_+ = F$ or $F \cup C$, where C is a Cantor set.*

PROOF. By the structure theory of compact, totally disconnected subsets of $\boldsymbol{R}$ [Pi], this is an easy consequence of (6.17). □

(6.19) LEMMA. *The set $\mathscr{J}_+$ is finite.*

PROOF. It will be enough to prove that, in (6.18), the possibility that $Z_0 \cap B_+ = F \cup C$ does not, in fact, occur. Indeed, if $Z_0 \cap B_+ = F$ and if $\mathscr{J}_+$ were infinite, then some point of F would belong to $B_1 = \varnothing$.

Suppose that $Z_0 \cap B_+ = F \cup C$ and obtain a contradiction as follows.

(1) Let $A = [x_0, \bar{x}_0]$ be a gap of C. If A contains infinitely many elements of $\mathscr{J}_+$, they must cluster in A, necessarily only at x_0 and/or $\bar{x}_0$. But this cluster point would belong to $B_1 = \varnothing$.

(2) By step (1), either A is an element of $\mathscr{J}_+$, or some point of ∂A, say x_0, is an element of B_0.

If $x_0 \in B_0$, then x_0 is τ-cyclic (6.15) and, by (6.4), $\Gamma_+(x_0) \cap B_+ \subset B_0$, hence $\Gamma_+(x_0) \cap B_+$ lies in the τ-cycle.

If $A \in \mathscr{J}_+$, then, with finitely many possible exceptions, A is a τ-cyclic gap of Z_0 (6.8) and the τ-cycle of x_0 also contains $\Gamma_+(x_0) \cap B_+$, again with the at most finitely many exceptions allowed by (6.8).

(3) By step (2), choose A and $x_0 \in \partial A$ so that $\{x_0, x_1, \cdots, x_p\} = X$ is a τ-cycle and $\Gamma_+(x_0) \cap B_+ \subseteqq X$.

Similarly, choose a sequence $\{A_k\}_{k=1}^{\infty}$ of gaps of C clustering at x_0 and having τ-cyclic endpoint $z_k \in \partial A_k$ with exactly the same property as x_0.

(4) Let $\gamma_0 \in \Gamma_+$ be the simple loop at x_0. By symbolic dynamics, $\gamma_0^n(A_k) \to x_0$ as $n \to \infty$, for k sufficiently large. An application of (6.9) to $z = z_k \in A_k$, k large enough, then contradicts the fact that $\Gamma_+(z) \cap B_+$ lies in the τ-cycle of z. □

PROOF OF (6.1). If infinitely many τ-cyclic points are semi-isolated in Z_0, then (6.8) implies that $\mathscr{J}_+$ is an infinite set. □

As already observed, this completes the proof of Theorem 2.

7. Concluding remarks. It is not hard to produce exceptional minimal sets (or, more generally, LMS) that are not Markovian. One way is to start with a Markov example in which one of the generators of Γ, say $h = h_1$, is a contraction to a point $x \in C$ that is not semi-isolated in C, having the property that h is C^∞-tangent to the identity at x. One can replace the generator h with $\tilde{h}$, an element that agrees with h on one side of x and with h^{-1} on the other side. The new pseudogroup, call it $\tilde{\Gamma}$, has the same orbits as Γ, but the generator of $H_x(\tilde{\Gamma}(x), C)$ is not represented by a contraction. By Theorem 1, the exceptional, $\tilde{\Gamma}$-minimal set C cannot be Markovian. Again, one could replace h with two generators, $\tilde{h}$ and $\bar{h}$, one agreeing with h on one side of x and with the identity on the other side, the other being similar, but with the sides of x reversed. In this case, $H_x(\tilde{\Gamma}, C) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$, again contrary to the conclusion of Theorem 1.

Another naturally occuring class of non-Markov examples has been suggested by Inaba (private communication). They are finitely generated, nonelementary Fuchsian groups of the second kind which have *parabolic* elements. The limit set of such a group is an exceptional minimal set, but it is not Markovian because a parabolic element is neither a contraction nor an expansion at its unique fixed point.

In an earlier version of this paper we proposed some conjectures that now seem to us to have been overly optimistic. One of these was that the general case might be orbit equivalent to the Markov case. In the special case of a transversely projective foliation that is transverse to a fibration by circles, Inaba has verified this conjecture (Ibid.). It would be interesting to remove the hypothesis of transversality to an S^1-fibration.

It is not hard to show that the general exceptional LMS gives rise to a multitude of Markov sub-pseudogroups that, in some sense, form a "dense" subsystem. It should be possible to exploit this fact.

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