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ON THE GENERALIZATION OF FROSTMAN'S THEOREM DUE TO S. KOBAYASHI

To Professor Tadashi Kuroda on the occasion of his sixtieth birthday

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1. For a single-valued meromorphic function $f(z)$ in a domain D of the z-plane and a boundary point ζ of D, the range of values $R_D(f, \zeta)$ of f at ζ is defined by $R_D(f, \zeta) = \bigcap_{r>0} f(D \cap U(\zeta, r))$, where $U(\zeta, r)$ denotes the open disc $|z-\zeta| < r$. We denote by $H_{|f|}(z)$ and $H_{|f|^2}(z)$ the least harmonic majorants of $|f(z)|$ and $|f(z)|^2$ in D, respectively.

In the case where D is the unit disc, it is known as Frostman's theorem [1] that if $|f(z)| < 1$ in $|z| < 1$ and Fatou's boundary function f^* of f satisfies $|f^*(\eta)| = 1$ almost everywhere on $|\eta| = 1$ and if f is not analytic at ζ, $|\zeta| = 1$, then $R_{|z|<1}(f, \zeta)$ covers the unit disc $|w| < 1$ except possibly for a set of capacity zero, where capacity means logarithmic capacity. In this case $H_{|f|}(z) = H_{|f|^2}(z) \equiv 1$ in $|z| < 1$ and the assumption that f is not analytic at ζ is equivalent to the existence of a sequence $\{z_n\}$ of points in $|z| < 1$ converging to ζ with $\lim_{n\to\infty} f(z_n) = 0$.

Recently, as a generalization of the above theorem to the case of general domains, Kobayashi [2] has given the following theorem.

THEOREM. *Suppose that $|f(z)| < 1$ in D and that $\zeta \in \partial D$ is a regular boundary point with respect to the Dirichlet problem. If there exists a sequence $\{z_n\}$ of points in D converging to ζ for which $H_{|f|^2}(z_n) \to 1$ and $f(z_n) \to a$ with $|a| < 1$ as $n \to \infty$, then $R_D(f, \zeta)$ covers the unit disc except possibly for a set of capacity zero.*

Our aim of the present note is to show that the standard argument in the theory of cluster sets gives a much simpler proof of Kobayashi's theorem and includes the case where ζ is an irregular boundary point. We shall prove:

THEOREM. *Suppose that $|f(z)| < 1$ in D and that there exists a sequence $\{z_n\}$ of points in D converging to $\zeta \in \partial D$ for which $H_{|f|}(z_n) \to 1$ and*

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$f(z_n) \to a$ with $|a| < 1$ as $n \to \infty$. Then we have the following alternatives:

(1) *The range of values $R_D(f, \zeta)$ covers the unit disc except possibly for a set of capacity zero; this is always the case if ζ is a regular boundary point.*

(2) *$H_{|f|}(z) \equiv 1$ in D and there is $r_0 > 0$ such that $\partial D \cap \overline{U(\zeta, r_0)}$ is of capacity zero, so that f is analytic throughout in $U(\zeta, r_0)$.*

Since $H_{|f|}(z) \geqq H_{|f|^2}(z)$, our assumption is a little weaker than that of Kobayashi.

2. PROOF. For $\rho > 0$, we denote by Δ_ρ the open disc $|w| < \rho$. Now suppose that (1) is not the case. Then there is $r_1 > 0$ such that the capacity of $\Delta_1 - f(D \cap \overline{U(\zeta, r_1)})$ is positive. Hence for some ρ_0, $0 < \rho_0 < 1$, $\bar{\Delta}_{\rho_0} - f(D \cap \overline{U(\zeta, r_1)})$ contains a closed set E of positive capacity. We consider the function $H(w)$ in the open set $\Delta_1 - E$ which coincides with the least harmonic majorant of $|w|$ in each connected component. Obviously, $H(f(z)) \geqq |f(z)|$ in $D \cap U(\zeta, r_1)$. Since E is of positive capacity, $H(w) < 1$ there and for any ρ, $0 < \rho < 1$, there exists λ_ρ, $0 < \lambda_\rho < 1$, such that $H(w) \leqq \lambda_\rho$ in $\Delta_\rho - E$.

Let $h(z)$ be the function in the open set $D \cap U(\zeta, r_1)$ which coincides with the solution of the Dirichlet problem with boundary values $1 - H(f(z))$ on $\partial U(\zeta, r_1) \cap D$ and 0 otherwise in each connected component. Then the function

$$\tilde{H}(z) = \begin{cases} H(f(z)) + h(z) & \text{in} \quad D \cap U(\zeta, r_1) \\ 1 & \text{in} \quad D - U(\zeta, r_1) \end{cases}$$

is continuous and superharmonic in D and satisfies $\tilde{H}(z) \geqq |f(z)|$ there. Hence $\tilde{H}(z) \geqq H_{|f|}(z)$ in D.

(a) Let ζ be a regular boundary point. Then $h(z) \to 0$ $(z \to \zeta)$ Therefore

$$\begin{aligned} 1 = \lim_{n\to\infty} H_{|f|}(z_n) \leqq \limsup_{n\to\infty} \tilde{H}(z_n) &= \limsup_{n\to\infty} H(f(z_n)) + \lim_{n\to\infty} h(z_n) \\ &= \limsup_{n\to\infty} H(f(z_n)) \leqq \lambda_\rho < 1 \quad (|a| < \rho < 1)\,. \end{aligned}$$

This is absurd, and (1) of the theorem is proved.

3. We shall proceed with our proof.

(b) Let ζ be an irregular boundary point. Then

$$\begin{aligned} 1 = \lim_{n\to\infty} H_{|f|}(z_n) \leqq \liminf_{n\to\infty} \tilde{H}(z_n) &\leqq \limsup_{n\to\infty} H(f(z_n)) + \liminf_{n\to\infty} h(z_n) \\ &\leqq \lambda_\rho + \liminf_{n\to\infty} h(z_n) \quad (|a| < \rho < 1)\,. \end{aligned}$$

Threfore $\liminf_{n\to\infty} h(z_n) \geqq 1 - \lambda_\rho > 0$.

Let $\{J_k\}$ be unions of a finite number of closed arcs on $\partial U(\zeta, r_1)$ such that $J_k \subset J_{k+1}$ and $\cup_k J_k = \partial U(\zeta, r_1) \cap D$. Let $\{h_k(z)\}$ be the solutions of the Dirichlet problem with boundary values $1 - H(f(z)) = h(z)$ on J_k and 0 otherwise in $D \cap U(\zeta, r_1)$ and let $\{w_k(z)\}$ be the harmonic measures of $\partial U(\zeta, r_1) \cap D - J_k$ with respect to the disc $U(\zeta, r_1)$. Then $\{w_k(z)\}$ converges to zero uniformly on any compact set in $U(\zeta, r_1)$ and

$$h_k(z) \leqq h(z) \leqq h_k(z) + w_k(z)$$

in $D \cap U(\zeta, r_1)$, so that there exists some k_0 for which $\liminf_{n\to\infty} h_{k_0}(z_n) = \mu > 0$.

(b.1) Suppose that $H_{|f|}(z) \not\equiv 1$ and set $\min_{z \in J_{k_0}}(1 - H_{|f|}(z)) = m > 0$.

Choosing $\alpha > 0$ to satisfy $\alpha h_{k_0}(z) \leqq m$ on J_{k_0}, we have $\alpha h_{k_0}(z) \leqq 1 - H_{|f|}(z)$, that is, $H_{|f|}(z) \leqq 1 - \alpha h_{k_0}(z)$ in $D \cap U(\zeta, r_1)$. Hence we have

$$1 = \lim_{n\to\infty} H_{|f|}(z_n) \leqq 1 - \alpha \liminf_{n\to\infty} h_{k_0}(z_n) = 1 - \alpha\mu < 1\ ,$$

which is absurd. Thus we have $H_{|f|}(z) \equiv 1$ in D.

(b.2) We have just seen that $H_{|f|}(z) \equiv 1$ in D. We note that the totality I of irregular boundary points of D is of capacity zero.

For $\eta \in \partial D$, the cluster set $C_D(f, \eta)$ of f at η is defined by $C_D(f, \eta) = \bigcap_{r>0} \overline{f(D \cap U(\eta, r))}$, that is, $\alpha \in C_D(f, \eta)$ if and only if there exists a sequence $\{y_n\}$ of points in D converging to η with $\lim_{n\to\infty} f(y_n) = \alpha$. We see from (a) that $C_D(f, \eta)$ is a closed subset of the unit circle $\partial\Delta_1$ for $\eta \in \partial D \cap U(\zeta, r_1) - I$, because $H_{|f|}(z) \equiv 1$ and the condition $\lim_{n\to\infty} H_{|f|}(y_n) = 1$ is satisfied always.

For ρ with $\max\{\rho_0, |a|\} < \rho < 1$, we consider the inverse image D_ρ of Δ_ρ. The component containing z_n is denoted by D_n (which may coincide with other $D_{n'}$).

Since the capacity of I is zero, we can take r_2, $0 < r_2 < r_1$ such that the circle $\partial U(\zeta, r_2)$ passes through the gap of I. Then we see from the fact just mentioned above that the intersection Z_n of $\bar{D}_n$ with $\partial D \cap \overline{U(\zeta, r_2)}$ is a closed subset of I so that its capacity is zero. Suppose that $D_n \subset U(\zeta, r_2)$. Since the boundary ∂D_n of D_n consists of the level curves $|f(z)| = \rho$ and Z_n of capacity zero, $f(D_n)$ covers Δ_ρ with possible exception of capacity zero, which contradicts our assumption that $\bar{\Delta}_{\rho_0} - f(D \cap \overline{U(\zeta, r_1)})$ contains E of positive capacity. Therefore D_n has a boundary point ζ_n on the circle $\partial U(\zeta, r_2)$. Now suppose that there is an infinite number of distinct components $\{D_{n_k}\}$. Let ζ_∞ be an accumulation point of the sequence $\{\zeta_{n_k}\}$. If $\zeta_\infty \in D$, we are led to a contradiction that infinitely many level curves $|f(z)| = \rho$ meet a small neighbourhood of ζ_∞. If $\zeta_\infty \in \partial D$, $C_D(f, \zeta_\infty) \cap \partial\Delta_\rho \neq \emptyset$

so that $\zeta_\infty \in I$. On the other hand, ζ_∞ is a point of $\partial U(\zeta, r_2)$ which does not pass over I. Thus $\zeta_\infty \in \partial D$ is also impossible and we can conclude that there is only a finite number of distinct components. In this case, there is at least one component, say D_1, containing a subsequence $\{z_{n_k}\}$ of $\{z_n\}$.

The following is a well-known theorem on cluster sets (cf. Noshiro [3]).

THEOREM. *Let Z be a closed subset of capacity zero of ∂D. If ζ is a point of Z such that $(\partial D - Z) \cap U(\zeta, r) \neq \emptyset$ for any $r>0$, then the set*

$$\Omega = C_D(f, \zeta) - C_{\partial D - Z}(f, \zeta)$$

is empty or open, and when $\Omega \neq \emptyset$, $R_D(f, \zeta)$ covers Ω except possibly for a set of capacity zero. Here the boundary cluster set $C_{\partial D-Z}(f, \zeta)$ is defined by

$$C_{\partial D-Z}(f, \zeta) = \bigcap_{r>0} \overline{\Big(\bigcup_{\eta \in (\partial D - Z) \cap U(\zeta, r)} C_D(f, \eta) \Big)} ,$$

that is, $\alpha \in C_{\partial D-Z}(f, \zeta)$ if and only if there is a sequence $\{\eta_n\}$ of points of $\partial D - Z$ converging to ζ such that we can take $w_n \in C_D(f, \eta_n)$ with $\lim_{n\to\infty} w_n = \alpha$.

Now suppose that $(\partial D - I) \cap U(\zeta, r) \neq \emptyset$ for any $r > 0$. Then obviously $(\partial D_1 - Z_1) \cap U(\zeta, r) \neq \emptyset$ for any $r > 0$. We apply the above theorem taking D_1 and Z_1 as D and Z there, respectively. Since $C_{\partial D_1 - Z_1}(f, \zeta) \subset \partial \Delta_\rho$ and $a = \lim_{k\to\infty}(z_{n_k}) \in \Delta_\rho$ is a cluster value of f at ζ so that $a \in C_{D_1}(f, \zeta)$, we see that $\Omega = \Delta_\rho$ and $R_{D_1}(f, \zeta)$, consequently $R_D(f, \zeta)$, covers Δ_ρ with possible exception of capacity zero, which contradicts our assumption $R_D(f, \zeta) \cap E = \emptyset$. Thus there exists r_0, $0 < r_0 \leqq r_2$, such that $(\partial D - I) \cap \overline{U(\zeta, r_0)} = \emptyset$. This means that $\partial D \cap \overline{U(\zeta, r_0)}$ is a closed set of capacity zero and $f(z)$ is analytic throughout in $U(\zeta, r_0)$. Our proof is now complete.

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