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BOUNDARY BEHAVIOR OF FUNCTIONS ON COMPLETE MANIFOLDS OF NEGATIVE CURVATURE

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In 1985, Anderson and Schoen [2] extensively studied positive harmonic functions on a complete, simply connected N-dimensional Riemannian manifold M of bounded negative curvature. In particular, they proved that the sphere at infinity $S(\infty)$ of the manifold is homeomorphic to its Martin boundary, and generalized, to the manifold M, the classical Fatou theorem on the boundary behavior of harmonic functions on the open unit disc.

THEOREM 0.1 (Anderson and Schoen [2]). *Every positive harmonic function on M converges nontangentially at almost every point of $S(\infty)$ with respect to a harmonic measure ω.*

Recently, A. Ancona [1] investigated minimal-fine convergence of functions on M, and proved that the manifold has the so-called "Fatou-Doob" property, which is an extension of Theorem 0.1. (For a detailed statement of the property, see [1, Theorem 4] or Corollary 3.3).

The purpose of this paper is to consider the relationship among nontangential, minimal-fine (fine, for short) and semifine convergence of functions on M. In fact, we shall prove that the ratio u/h of two positive harmonic functions u and h on M converges nontangentially at $Q\in S(\infty)$ to l if and only if l is a semifine limit of u/h at Q (cf. Theorems 3.1 and 3.2). As an immediate consequence of the theorems we obtain the result of Ancona on fine convergence. Furthermore, we introduce the notion of admissible convergence on M, and prove, using this concept, that u/h has a fine limit φ $d\omega$-a.e. on $E\subset S(\infty)$ if and only if u/h converges nontangentially to φ $d\omega$-a.e. on E (cf. Corollary 4.5). Moreover, we prove a local version of the "Fatou-Doob" property of M (cf. Section 5) and as a result, we give a refinement of Theorem 0.1 (cf. Corollary 5.3). This result is a generalization to M of the classical local Fatou theorem on the unit disc. For the local Fatou theorem and the classical theory of fine convergence, we refer the reader to the books of Stein [18] and Doob [8]. Fatou's theorem and its local version play important parts in classical analysis.

Ancona [1] dealt also with a class of elliptic harmonic functions, and all results in

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our paper hold good for certain elliptic harmonic functions (cf. Remark 5.4).

In the classical case, the above mentioned results were obtained earlier by Brelot and Doob [4] for the upper half space in $\boldsymbol{R}^N$, and Korányi and Taylor [17] for symmetric spaces of rank one (see also Hunt and Wheeden [13] and Taylor [19]).

In the following, $C_1, C_2, C_3, \cdots, c_1, c_2, c_3, \cdots$ will denote positive constants depending only on the dimension N and the bounds of the curvature of the manifold M.

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1. Preliminaries. Throughout this paper we denote by (M, g) a complete, simply connected, N-dimensional Riemannian manifold such that the sectional curvatures K_M satisfy $-\infty < -b^2 \le K_M \le -a^2 < 0$ for two positive constants a and b. Let us fix a point ζ in M. We call a curve γ a geodesic if γ is a unit speed geodesic.

We now recall some facts on the sphere at infinity of M: Two geodesics γ_1 and γ_2 are said to be asymptotic if $d(\gamma_1(t), \gamma_2(t))$ is bounded as $t \to +\infty$ where $d(\ ,\ )$ is the distance function with respect to the Riemannian metric of M. The sphere at infinity of M is, by definition, the set $S(\infty)$ of all asymptotic classes of geodesics in M. Then $\bar{M} = M \cup S(\infty)$ is a compactification of M under the following cone topology: Let $C(p, v, \delta)$ be the cone at $v \in T_pM$ of angle δ, that is, $C(p, v, \delta) = \{x \in \bar{M}: \measuredangle_p(v, \dot{\gamma}_{px}(0)) < \delta\}$, where γ_{px} is the geodesic with $\gamma_{px}(0) = p$ and $\gamma_{px}(t) = x$ for some $t \in (0, +\infty]$, and $\measuredangle_p$ denotes an angle in T_pM. Let $T_p(v, \delta, R) = C(p, v, \delta) \setminus B(p, R)$, where $B(p, R) = \{z \in M: d(z, p) < R\}$. Then the sets $T(p, v, \delta)$ $(v \in T_pM, \delta > 0)$ together with the geodesic balls $B(q, r)$ $(q \in M, r > 0)$ form a local basis for a topology of $\bar{M}$. It is called the cone topology and does not depend on the choice of p. Under this topology, $\bar{M}$ and $S(\infty)$ are homeomorphic to the closed unit ball in $\boldsymbol{R}^N$ and its topological boundary, respectively. For further details, see Eberlein and O'Neill [9].

Let $d\omega^p$ be the harmonic measure relative to p and M, that is, $d\omega^p$ is a probability measure on $S(\infty)$ such that

$$Hf(p) = \int_{S(\infty)} f d\omega^p, \qquad f \in C(S(\infty)),$$

where Hf is the unique harmonic function on M with boundary values f. The existence of $d\omega^p$ was proved by Anderson and Sullivan (see [2]). Let $K_Q(z)$ $(Q \in S(\infty), z \in M)$ be the Poisson kernel: K_Q is a positive harmonic function on M such that $K_Q(\zeta) = 1$ and that K_Q extends continuously to the zero function on $S(\infty) \setminus \{Q\}$. By [2], there exists a unique kernel function K_Q at every $Q \in S(\infty)$. For simplicity, set $d\omega = d\omega^\zeta$. From [2] follows that $d\omega^p(Q) = K_Q(p) d\omega(Q)$.

From now on, we denote by $t(x)$ the time such that $\gamma_{\zeta x}(t(x)) = x$ for every $x \in \bar{M} \setminus \{\zeta\}$.

To conclude this preliminary section, we state theorems of Cheng, Yau, Anderson and Schoen to which we will refer often in the main body of our paper.

THEOREM CY (Cheng and Yau [7]; see also [1], [2]). *Let u be a positive harmonic function on $B(x, r)$, $x\in M$, $r>0$. Then, for every $0<s<r$, there exists a constant $C_{1,s}$ depending only on N, a, b and s such that*

$$C_{1,s}^{-1}u(z)\leq u(x)\leq C_{1,s}u(z)\,,$$

for all $z\in B(x, s)$. Moreover, $C_{1,s}$ may be taken arbitrarily close to 1 provided s is small enough.

THEOREM AS (Anderson and Schoen [2]; see also [1], [15]). *Let $p\in M$ and $v\in T_pM$ with $g_p(v, v)=1$. Denote $C=C(p, v, \pi/4)$ and $T=T_p(v, \pi/8, 1)$. Let u and h be positive harmonic functions on $C\cap M$, continuous up to ∂C, and vanishing on $\bar{C}\cap S(\infty)$. Then*

$$C_1\exp\{-C_2d(p, x)\}u(p_0)\leq u(x)\leq C_3\exp\{-C_4d(p, x)\}u(p_0)\,, \tag{1}$$

for $x\in T$, where $p_0=\exp_p(v)$.

$$C_5^{-1}\frac{u(p_0)}{h(p_0)}\leq\frac{u(x)}{h(x)}\leq C_5\frac{u(p_0)}{h(p_0)}\,, \tag{2}$$

for $x\in T$.

2. Fine or semifine convergence and semithin sets. We define fine and semifine convergence adapted to our context. For $Q\in S(\infty)$ and $t\in\boldsymbol{R}$, let $Q(t)=\gamma_{\zeta Q}(t)$, $C(Q, t)=C(Q(t), \dot{Q}(t), \pi/4)$ and $\sigma(Q, t)=C(Q, t)\setminus C(Q, t+1)$. Then a set $E\subset M$ is said to be *thin* (resp. *semithin*) at Q if

$$\lim_{k\to+\infty} R^{E\cap C(Q,k)}K_Q=0 \qquad \left(\text{resp. } \lim_{k\to+\infty} R^{E\cap\sigma(Q,k)}K_Q=0\right),$$

where R^Ff is the reduction of $f\colon M\to\boldsymbol{R}$ relative to $F\subset M$, that is, $R^Ff(z)=\inf\{u(z): u\geq 0$, superharmonic on M, and $u\geq f$ on $F\}$. We denote by $\mathscr{F}(Q)$ (resp. $\mathscr{S}(Q)$) the family of subsets E of M whose complements $M\setminus E$ are thin (resp. semithin) at Q. It is easy to check that $\mathscr{F}(Q)$ and $\mathscr{S}(Q)$ are filters at Q. Obviously, $\mathscr{F}(Q)\subset\mathscr{S}(Q)$.

DEFINITION 2.1. A function $f\colon M\to\boldsymbol{R}$ *converges finely* (resp. *converges semifinely*) to l at $Q\in S(\infty)$ or has a *fine limit* (resp. *semifine limit*) l at $Q\in S(\infty)$ if for every $\varepsilon>0$, there exists a set $E\in\mathscr{F}(Q)$ (resp. $E\in\mathscr{S}(Q)$) such that $|f(z)-l|<\varepsilon$ for all $z\in E$.

REMARK 2.2. Obviously, fine convergence implies semifine convergence. However, the converse is not true in general.

From the argument given in [13, Lemma 5.1] it follows that a set $E\subset M$ is thin at $Q\in S(\infty)$ if and only if $R^EK_Q(p)<K_Q(p)$ for some $p\in M$.

To study fine and semifine convergence, we present examples of non-semithin sets and semithin sets: For $Q\in S(\infty)$, $t>0$ and $d>0$, let

$$T_d^t(Q)=\{z\in M: d(z, \gamma_{\zeta Q})<d \text{ and } d(z, \zeta)>t\}.$$

The set $T_d^t(Q)$ was introduced by Anderson and Schoen [2] and called the nontangential cone.

PROPOSITION 2.3. *Let $\{x_n\}$ be a sequence of points in a nontangential cone $T_d^t(Q)$ such that $t(x_n)$ $(=d(\zeta, x_n))\uparrow +\infty$. Let $r\in(0, 1]$. Then the set*

$$B=\bigcup_{n=1}^{\infty} B(x_n, r)$$

is not semithin at Q, and the set B intersects each element of $\mathscr{S}(Q)$.

PROPOSITION 2.4. *Let $Q\in S(\infty)$, and let $r(n)$ be a positive integer with $r(n)\geq n$, $n=1, 2, \cdots$. Then the set*

$$\mathscr{N}=M\diagdown\left[\bigcup_{n=1}^{\infty} T_n^{r(n)}(Q)\right]$$

is semithin at Q.

REMARK 2.5. The set B in Proposition 2.3 is an analogue of a bubble set which is well-known in classical potential theory (cf. [19]). In the proof of Ancona [1, Theorem 4], it was proved that B is not thin at Q.

To prove Propositions 2.3 and 2.4, we need some lemmas:

LEMMA 2.6. *For $Q\in S(\infty)$ and $t\in \boldsymbol{R}$, let $\Delta(Q, t)=C(Q, t)\cap S(\infty)$. There exists a constant $C_6>0$ such that $\omega^z(\Delta(Q, t))\geq C_6$, for all $z\in C(Q, t+t_0)\cup B(Q(t), t_0)$, where t_0 is a positive constant depending only on the curvature bounds.*

PROOF. By the proof of Anderson and Schoen [2, Lemma 7.4], we have $\omega^{Q(t)}(\Delta(Q, t))\geq C_7>0$. The function $\varphi(z)=1-\omega^z(\Delta(Q, t))$ is a positive harmonic function on M vanishing at $\Delta(Q, t)$. Applying Theorem AS (1), [2, Lemma 6.1(ii)] and Theorem CY to φ, we obtain the lemma. ∎

We will use the following geometric lemma:

LEMMA 2.7. *Let δ be a positive number. Then there exists $\alpha>0$ such that*

$$d(\partial C(Q, t), \partial C(Q, t+\alpha))>\delta$$

for every $Q\in S(\infty)$ and $t>0$.

PROOF. By [2, Lemma 6.1], there exists $\beta>0$ such that $C(Q(t), \dot{Q}(t), \pi/8)\supset C(Q, t+\beta)$. Take any $x\in\partial C(Q, t)\cap M$, $x\neq Q(t)$. Let γ be a geodesic starting at $Q(t)$ with $\sphericalangle_{Q(t)}(\dot{\gamma}_{Q(t)Q}(0), \dot{\gamma}(0))=\pi/8$. Set $\theta=\sphericalangle_{Q(t)}(\dot{\gamma}_{Q(t)x}(0), \dot{\gamma}(0))$, $\tau(x)=d(Q(t), x)$ and

$r(s)=d(x, \gamma(s))$, $s\in \boldsymbol{R}$. Obviously, $\theta\geq\pi/4-\pi/8=\pi/8$. By Rauch's comparison theorem ([16, Corollary 2.7.3]) together with the law of hyperbolic cosine, we have

$$\cosh(ar(s))\geq\cosh(a\tau(x))\cosh(as)-c\cdot\sinh(a\tau(x))\sinh(as)=:f_x(s)$$

for $s>0$, where $c=\cos(\pi/8)$. Since the minimum value of f_x is $\{(1-c^2)(\cosh(a\tau(x)))^2+c^2\}^{1/2}=:\psi(\tau(x))$, we have that

$$d(x, \partial C(Q(t), \dot{Q}(t), \pi/8))\geq a^{-1}\operatorname{arccosh}(a\psi(\tau(x)))\,.$$

Let α_0 be a number such that $a^{-1}\operatorname{arccosh}(a\psi(\alpha_0))>\delta$ and $\alpha_0>\max(\delta, \beta)$. Then $\alpha=2\alpha_0$ is a desired number, because for $x\in\partial C(Q, t)\cap M$, if $d(Q(t), x)\leq\alpha_0$, then $d(x, \partial C(Q, t+\alpha))\geq\alpha_0>\delta$, and if $d(Q(t), x)>\alpha_0$, then

$$d(x, \partial C(Q, t+\alpha))\geq d(x, \partial C(Q(t), \dot{Q}(t), \pi/8))>\delta\,. \qquad \blacksquare$$

Now we are ready to prove the propositions:

PROOF OF PROPOSITION 2.3. For every large $J\in\boldsymbol{N}\equiv\{$positive integers$\}$, there exists $n=n(J)\in\boldsymbol{N}$ such that $n>J$ and $x_n\in C(Q, J)$. Take $k=k(n, J)\in\boldsymbol{N}$ with $k\geq J$ and $x_n\in\sigma(Q, k)$. By Lemma 2.7, there exists $\alpha>0$ satisfying

$$d(\partial C(Q, t), \partial C(Q, t+\alpha))>2\,.$$

Since $x_n\in\sigma(Q, k)$, we have

$$B(x_n, r)\subset\sigma(Q, k-\alpha, \alpha)\cup\sigma(Q, k, \alpha)\cup\sigma(Q, k+\alpha, \alpha)\,,$$

where $\sigma(Q, j, \alpha)=C(Q, j)\setminus C(Q, j+\alpha)$.

For simplicity, set $B(n)=B(x_n, r)$ and $\sigma(j)=\sigma(Q, j, \alpha)$, $j\in\boldsymbol{R}$. Then we get that

$$R^{B\cap\{\sigma(k-\alpha)\cup\sigma(k)\cup\sigma(k+\alpha)\}}K_Q(\zeta)\geq R^{B(n)}K_Q(\zeta)\,.$$

It is simple to check that $R^{B(n)}K_Q$ is harmonic on the outside of the closure of $B(n)$ (cf. [12]) and vanishes continuously on $S(\infty)$. Hence, by Theorems AS and CY, we have that for every $z\in\partial B(x_n, 2r)$,

$$R^{B(n)}K_Q(\zeta)\geq C_8\frac{R^{B(n)}K_Q(z)}{K_Q(z)}\geq C_9R^{B(n)}1(z)\,,$$

where C_8 and C_9 depend only on N, a, b, d and r.

To estimate $R^{B(n)}1(z)$, we use the "bounded geometry" property of M (cf. [1, p. 497]), that is, there is a constant $r_0>0$ independent of x_n such that the normal coordinate φ at x_n satisfies

$$|\varphi(p)-\varphi(q)|\leq d(p, q)\leq c_1|\varphi(p)-\varphi(q)|\,,$$

for every $p, q\in B(x_n, r_0)$. We may assume that $r<r_0/100$. Let y_n be a point in $\partial B(x_n, 3r)$, and set $\tilde{R}^{B(n)}1=\inf\{f: f\geq0$, superharmonic on $\varphi^{-1}(\{X\in\boldsymbol{R}^N: |X-\varphi(y_n)|<5r\})$ and $f\geq1$

on $B(n)\}$. Then $R^{B(n)}1(y_n)\geq \tilde{R}^{B(n)}1(y_n)$. From the "bounded geometry" property and Bishop's comparison theorem (cf. [6, p. 66]), it follows that the Laplace-Beltrami operator Δ_M on M is uniformly elliptic on $\varphi^{-1}(\{X\in \boldsymbol{R}^N : |X-\varphi(y_n)|<5r\})$. Denote by $\tilde{\omega}$ the Δ_M-harmonic measure relative to $\varphi(y_n)$ and $\varphi(B(y_n, 3r))$. Then by the superharmonicity of $\tilde{R}^{B(n)}1$ and [5, Lemma 3.5] we have that

$$\tilde{R}^{B(n)}1(y_n)\geq \int \tilde{R}^{B(n)}1 d\tilde{\omega}\geq \tilde{\omega}\{\varphi[\partial B(y_n, 3r)\cap B(n)]\}\geq C_{10}>0 .$$

This implies that $R^{B(n)}K_Q(\zeta)\geq C_{1,d,r}>0$, where $C_{1,d,r}$ is a constant depending only on N, a, b, d and r. Hence, there is $j(k)\in\{k-\alpha, k, k+\alpha\}$ such that

$$R^{B(n)\cap\sigma(j(k))}K_Q(\zeta)\geq 3^{-1}\cdot C_{1,d,r} .$$

Further, $\sigma(j(k))\subset\sigma(Q, j(k))\cup\cdots\cup\sigma(Q, j(k)+[\alpha]+1)$, where $[\alpha]$ denotes the integral part of α. Therefore, we can take a number $i(k)\in\{j(k), \cdots, j(k)+[\alpha]+1\}$ such that

$$R^{B(n)\cap\sigma(Q, i(k))}K_Q(\zeta)\geq 3^{-1}\{[\alpha]+2\}^{-1}C_{1,d,r}=: C_{2,d,r} .$$

Note that $i(k)\geq J-\alpha$ by the choice of k. Consequently, we have a sequence $\{\imath(n)\}\subset \boldsymbol{N}$ such that $\imath(n)\uparrow +\infty$ as $n\to+\infty$ and $R^{B\cap\sigma(Q, \imath(n))}K_Q(\zeta)\geq C_{2,d,r}$ for every n. This guarantees that the set B is not semithin at Q.

We now prove the remaining part of Proposition 2.3. If there is a set $E\in\mathscr{S}(Q)$ with $E\cap B=\varnothing$, then $B\subset M\setminus E$, and $M\setminus E$ is semithin at Q. Thus the set B must be semithin at Q. This contradicts the first part of the proposition. ∎

PROOF OF PROPOSITION 2.4. By Theorems AS and CY, we have

$$K_Q(x)\leq C_{11}\exp\{-C_7 d(x, Q(k))\}K_Q(Q(k)) ,$$

for every $x\in\sigma(Q, k)$ and $k\in \boldsymbol{N}$. Hence for every $\varepsilon>0$ there is $\delta(\varepsilon)>0$ independent of k such that $K_Q\leq\varepsilon K_Q(Q(k))$ on $\sigma(Q, k)\cap[M\setminus B(Q(k), \delta(\varepsilon))]$, $k\in \boldsymbol{N}$.

Let n be an arbitrary integer with $n>\delta(\varepsilon)$. For every $m>r(n)$

$$[M\setminus T_n^{r(n)}(Q)]\cap\sigma(Q, m)\subset[M\setminus B(Q(m), \delta(\varepsilon))]\cap\sigma(Q, m) ,$$

because if x is an element of the left-hand side, then $d(\zeta, x)\geq m>r(n)$ and $d(Q(m), x)\geq d(\gamma_{\zeta Q}, x)\geq n>\delta(\varepsilon)$. Hence $K_Q\leq\varepsilon K_Q(Q(m))$ on $\mathscr{N}\cap\sigma(Q, m)$. Consequently

$$R^{\mathscr{N}\cap\sigma(Q,m)}K_Q(\zeta)\leq\varepsilon K_Q(Q(m))R^{\mathscr{N}\cap\sigma(Q,m)}1(\zeta)\leq\varepsilon K_Q(Q(m))R^{\sigma(Q,m)}1(\zeta) .$$

We now show that $K_Q(Q(m))R^{\sigma(Q,m)}1(\zeta)$ is bounded by a constant independent of m. Let $f(z)=\omega^z(\Delta(Q, m-t_0))$. By Lemma 2.6, $R^{\sigma(Q,m)}1\leq C_6^{-1}f$. Hence

$$R^{\sigma(Q,m)}1(\zeta)\leq C_6^{-1}\omega(\Delta(Q, m-t_0))\leq C_{12}\omega(\Delta(Q, m)) \quad \text{(cf. [2, Lemma 7.4])} .$$

On the other hand, from Theorems AS and CY, we have

$$\frac{1}{\omega(\Delta(Q,m))} \geq C_{13}^{-1} \frac{K_Q(Q(m))}{\omega^{Q(m)}(\Delta(Q,m))} \geq C_{13}^{-1} K_Q(Q(m)).$$

Consequently, $K_Q(Q(m))R^{\sigma(Q,m)}1(\zeta) \leq C_{12}C_{13}$ which implies the proposition. ∎

3. Semifine and nontangential convergence. As is well-known, the study of nontangential convergence was begun in 1906 by P. Fatou, and his result has been developed by A. P. Calderón, E. M. Stein, L. Carleson and others. In particular, Brelot and Doob [4] found relationship among nontangential limits, fine limits and semifine limits of functions on upper half-spaces, and improved the Fatou theorem and its local version, the so-called local Fatou theorem, by proving them for quotients of positive harmonic functions. Somewhat later, Hunt and Wheeden [13] generalized the results of Brelot and Doob to Lipschitz domains in $\boldsymbol{R}^N$.

In the following sections, we extend these results to the manifold M. To state our results we recall the definition of nontangential convergence on M.

DEFINITION (Anderson and Schoen [2]). A function f: $M \to \boldsymbol{R}$ *converges nontangentially* to l at $Q \in S(\infty)$ or has a *nontangential limit* l at Q if for every $d>0$ and $\varepsilon>0$, there exists a $t>0$ such that $|f(z)-l|<\varepsilon$ for all $z \in T_d^t(Q)$.

Our results in this section are the following:

THEOREM 3.1. *Let f: $M \to \boldsymbol{R}$ be a function which has a nontangential limit l at $Q \in S(\infty)$. Then f converges semifinely to l at Q.*

PROOF. For every $\varepsilon>0$ and $n \in \boldsymbol{N}$, there exists $r(n)=r(n,\varepsilon) \in \boldsymbol{N}$ such that $r(n) \geq n$ and $|f-l|<\varepsilon$ on $T_n^{r(n)}(Q)$. Let $E(\varepsilon)=\bigcup_{n=1}^{\infty} T_n^{r(n)}(Q)$. Then $|f-l|<\varepsilon$ on $E(\varepsilon)$ and $E(\varepsilon) \in \mathscr{S}(Q)$ by Proposition 2.4. Thus l is the semifine limit of f at Q. ∎

The converse of Theorem 3.1 can be proved when f is the ratio of two positive harmonic functions:

THEOREM 3.2. *Let u and h be positive harmonic functions on M. Then u/h has a nontangential limit l at $Q \in S(\infty)$ if and only if u/h has the semifine limit l at Q.*

PROOF. The "only if" part is a special case of Theorem 3.1. We prove the "if" part. Suppose that u/h does not converge nontangentially to l at Q. Then there exist $\varepsilon>0$, $d>0$ and $\{x_n\} \subset T_d^0(Q)$ such that $t(x_n) \uparrow +\infty$ and $|(u/h)(x_n)-l| \geq \varepsilon$ for all n. By Theorem CY there is $r>0$ such that $|(u/h)(z)-l| \geq \varepsilon/2$ for every $z \in \bigcup_{n=1}^{\infty} B(x_n, r)=: B$. From Proposition 2.3 it follows that for every $E \in \mathscr{S}(Q)$ the set B intersects E. Therefore l is not a semifine limit of u/h at Q. ∎

By the Anderson and Schoen theorem [2, Theorem 6.5] for every nonnegative harmonic function f on M there is a unique, finite, positive Borel measure μ on $S(\infty)$ such that

$$f(z)=\int_{S(\infty)} K_Q(z)d\mu(Q), \qquad z\in M.$$

We denote by μ_f the measure μ.

As a direct consequence of Theorem 3.2, we have a result of Ancona in the case of harmonic functions:

COROLLARY 3.3 (cf. Ancona [1, Theorem 4]). *Let u and h be positive harmonic functions on M. Then we have the following*:

(1) *If u/h has a fine limit l at $Q\in S(\infty)$, then u/h converges nontangentially to l at Q.*

(2) *The ratio u/h converges nontangentially to the Radon-Nikodym derivative $d\mu_u/d\mu_h$ μ_h-almost everywhere in $S(\infty)$.*

PROOF. Since a fine limit of a function is also its semifine limit, Theorem 3.2 implies (1). The statement (2) is an immediate consequence of (1), Remark 2.2 and the generalization of Fatou-Naim-Doob theorem obtained by Gowrisankaran [11]. ∎

4. Fine and admissible convergence. In the case of upper half-spaces, nontangential convergence does not imply pointwise fine convergence, but does almost everywhere (cf. Brelot and Doob [4]; [13], [17], [19]). In this section we will consider this fact in our setting. For technical reasons we introduce the notion of admissible domains and admissible convergence:

DEFINITION 4.1. (1) For $z\in M$ let $z(t)=\gamma_{\zeta z}(t+t(z))$, $t\in\boldsymbol{R}$. If $z\in M$ and $\alpha\in\boldsymbol{R}$, then $C(z,\alpha)$ denotes the set $\{x\in\bar{M};\ \sphericalangle_{z(\alpha)}(\dot{z}(\alpha),\gamma_{z(\alpha)x}(0))<\pi/4\}$. For $Q\in S(\infty)$ and $\alpha\in\boldsymbol{R}$ we call the set $\{z\in M: Q\in C(z,\alpha)\}$ *an admissible domain* at Q and denote it by $\Gamma(Q,\alpha)$.

(2) For $t>0$ let $\Gamma^t(Q,\alpha)=\Gamma(Q,\alpha)\cap\{z\in M:\ d(\zeta,z)>t\}$. A function f: $M\to\boldsymbol{R}$ *converges admissibly* to l at $Q\in S(\infty)$ or has an *admissible limit* l at Q if for every $\alpha\in\boldsymbol{R}$ and $\varepsilon>0$ there exists $t>0$ such that $|f(z)-l|<\varepsilon$ for all $z\in\Gamma^t(Q,\alpha)$.

Before stating our results, we mention a motivation for the definition of admissible domains. Consider the upper half-plane $\boldsymbol{R}^2_+=\boldsymbol{R}\times(0,\infty)$. The classical nontangential cone $\Gamma_\alpha(x)$ is defined as $\{(y,t)\in\boldsymbol{R}^2_+: |y-x|<\alpha t\}$, $x\in\boldsymbol{R}$, $\alpha>0$. Here $|y-x|$ and t can be regarded as the distances between x and y, and between (y,t) and $\partial\boldsymbol{R}^2_+(=\boldsymbol{R})$, respectively. Since the distance between any point in M and $S(\infty)$ is $+\infty$, we cannot translate the definition of nontangential domains $\Gamma_\alpha(Q)$ to our case. On the other hand, nontangential cones are also characterized by tangent vectors at $\partial\boldsymbol{R}^2_+$ but, in general, $S(\infty)$ is a $C^{(a/b)}$-manifold (cf. Anderson and Schoen [2]).

Now, we recall a characterization of $\Gamma_\alpha(x)$ in $\boldsymbol{R}^2_+$ which does not use distance and tangent vectors at the boundary. Our definition of admissible domains is motivated by the following observation. For $z=(z_0,t)\in\boldsymbol{R}^2_+$, let $\mathscr{C}(z,\alpha)=\{y\in\boldsymbol{R}:\ \sphericalangle_z(\overline{zy},\overline{zz_0})<\alpha\}$, where $\overline{zy}$ is the line segment $\overline{z(y,0)}$. Then $z\in\Gamma_\alpha(x)$ if and only if $x\in\mathscr{C}(z,\alpha)$.

Admissible domains are related to nontangential cones defined by Anderson and

Schoen as follows:

PROPOSITION 4.2. *Let* $Q \in S(\infty)$, $\alpha \in \boldsymbol{R}$ *and* $T>0$. *Then*:

(1) *The set* $\Gamma^T(Q, \alpha)$ *contains the line* $\{\gamma_{\zeta Q}(t) : t>T\}$.

(2) *The intersections of* $S(\infty)$ *with the closure of* $\Gamma^T(Q, \alpha)$ *in* $\bar{M}$ *consists of one point* $\{Q\}$.

(3) *There exists a nontangential cone* $T_d^T(Q)$ *such that* $\Gamma^T(Q, \alpha) \subset T_d^T(Q)$. *Hence* $\Gamma^T(Q, \alpha)$ *is a nontangential domain in the sense of Anderson and Schoen* [2, *Definition* 7].

(4) *If a function* f *on* M *has a nontangential limit* l *at* $Q \in S(\infty)$, *then* f *converges admissibly to* l *at* Q.

PROOF. (1) is obvious. Since (2) and (4) are consequences of (3), we prove only (3). We use the ideas of Anderson and Schoen [2, Lemma 6.1 and Lemma 7.2]. Let $x \in \Gamma^T(Q, \alpha)$. Denote by $R(t)$ the surface consisting of geodesic segments joining ζ with points in $\{\gamma_{xQ}(s) : t>s>0\}$, $t>0$. Then $R(t)$ is smooth away from ζ and has the Gaussian curvature $K_{R(t)} \leq -a^2$ (see [2, p. 453]). Applying the Gauss-Bonnet theorem to $R(t) \setminus B(\zeta, \varepsilon)$, $\varepsilon>0$, we have $\operatorname{area}(R(+\infty)) \leq a^{-2}(\pi+\theta)$, where $\theta = \sphericalangle_x(\dot{\gamma}_{xQ}(0), \dot{\gamma}_{\zeta x}(t(x)))$. Let $d_0 = d(x, \gamma_{\zeta Q})$. By Bishop's comparison theorem [6, p. 68] $2^{-1}(\pi-\theta)d_0^2 \leq$ area $(R(+\infty))$. Hence

$$d_0 \leq \{2(\pi+\theta)/a^2(\pi-\theta)\}^{1/2} =: d(\theta) .$$

If $\alpha=0$, then by the definition of x, the angle θ is less than $\pi/4$. Hence we have $\Gamma^T(Q, \alpha) \subset \Gamma^T(Q, 0) \subset T^T_{d(\pi/4)}(Q)$ for every $\alpha \geq 0$.

Assume that $\alpha<0$. Let $y = \gamma_{\zeta x}(t(x)+\alpha)$ and $\psi = \sphericalangle_y(\dot{\gamma}_{yQ}(0), \dot{\gamma}_{yx}(0))$. Obviously, $\psi \leq \pi/4$. For $t>0$ let $\theta(t)$ be the angle at x between $\gamma_{\zeta x}$ and the geodesic joining x with $\gamma_{yQ}(t)$. By the same argument as in [2, Lemma 6.1], we have that

$$\begin{aligned}\lim_{t \to +\infty} \sup \cos(\pi-\theta(t)) &\leq \coth(b\,|\alpha|) - \{\sinh(b\,|\alpha|)[\cosh(b\,|\alpha|) + 2^{-1/2}\sinh(b\,|\alpha|)]\}^{-1} \\ &=: F(|\alpha|) .\end{aligned}$$

Hence $\theta \leq \pi - \arccos(F(|\alpha|))$. Thus

$$d_0^2 \leq 2[2\pi - \arccos(F(|\alpha|))]/a^2 \arccos(F(|\alpha|)) ,$$

which completes the proof of (3). ∎

Now we state our result.

THEOREM 4.3. *Let* E *be a set of* $S(\infty)$ *and* f: $M \to \boldsymbol{R}$ *be a function which has an admissible limit* f^* *on* E. *Then* f *converges finely to* f^* *at* $d\omega$-*almost every point of* E, *that is, there is a set* $E_0 \subset S(\infty)$ *such that* $\omega(E_0)=0$ *and that* f *admits a fine limit* f^* *on each point* Q *in* $E \cap (S(\infty) \setminus E_0)$.

PROOF. The following proof is an application of the proof of [4, Theorem 9] and

[17, Theorem 4.1] to the context of the manifold M.

Let $\alpha \in \boldsymbol{R}$ and denote $V(f, t)(Q)=\sup\{|f(x)-f(y)|: x, y \in \Gamma^t(Q, \alpha)\}$ for $Q \in S(\infty)$. It is easy to check that $V(f, t)$ is lower semicontinuous and that $V^*(f)=\lim_{t \to +\infty} V(f, t)$ is Borel measurable. Hence the set $F=\{V^*(f)=0\}$ is a Borel set which contains E. From this fact, we may assume that E is Borel measurable. By Egorov's theorem we have that for every $n \in \boldsymbol{N}$ there is a compact set $D_n \subset F$, with $\omega(F \setminus D_n)<1/n$, such that $V(f, t)$ converges uniformly to 0 on $D_n \cap E$.

Now we show that f^* is continuous on $D_n \cap E$. Indeed, let $Q_0 \in D_n \cap E$. For every $\varepsilon>0$ there is $T=T(\varepsilon)>0$ such that $V(f, T)<\varepsilon$ on $D_n \cap E$ and $|f-f^*(Q_0)|<\varepsilon$ on $\Gamma^T(Q_0, \alpha)$. Take a point x in $\Gamma^T(Q_0, \alpha)$. Then $Q_0 \in C(x, \alpha) \cap S(\infty)$. Note that $C(x, \alpha) \cap S(\infty)$ is an open set in $S(\infty)$. Let Q_1 be an arbitrary point in $C(x, \alpha) \cap D_n \cap E$. There is τ ($>T$) so that $|f^*-f^*(Q_1)|<\varepsilon$ on $\Gamma^\tau(Q_1, \alpha)$. Hence for a point $y \in \Gamma^\tau(Q_1, \alpha)$ we obtain that

$$|f^*(Q_1)-f^*(Q_0)| \leq |f^*(Q_1)-f(y)|+|f(y)-f(x)|+|f(x)-f^*(Q_0)|<3\varepsilon ,$$

which implies that f^* is continuous at Q_0.

Now we continue the proof of Theorem 4.1. For $\varepsilon>0$ let $T=T(\varepsilon, n)$ be a positive number such that $V(f, T)<\varepsilon$ on $D_n \cap E$. Then, by the continuity of f^* for every $Q_0 \in D_n \cap E$ there is a neighborhood W of Q_0 in the cone topology such that

$$\bigcup \{\Gamma^T(Q, \alpha): Q \in D_n \cap E\} \cap W \subset \{y \in M: |f(y)-f^*(Q_0)|<2\varepsilon\} .$$

Therefore the proof is reduced to the following lemma:

LEMMA 4.4. *For every Borel measurable set $A \subset S(\infty)$, $T>0$ and $\alpha \in \boldsymbol{R}$, put $S(A, \alpha, T)=\bigcup \{\Gamma^T(Q, \alpha): Q \in A\}$. Then $M \setminus S(A, \alpha, T)$ is thin $d\omega$-a.e. on A.*

PROOF. Let $w(z)=\omega^z(A)$, $z \in M$. By the Fatou-Naim-Doob theorem [11, Theorem 8], w converges finely to the characteristic function χ_A $d\omega$-a.e. on $S(\infty)$. Hence for every $\beta \in (0, 1)$ the set $\{w \leq \beta\}$ is thin $d\omega$-a.e. on A. Denote by A_0 the exceptional set. We will prove that

$$\{w>\beta\} \cap T_\zeta(\dot{\gamma}_{\zeta Q}(0), \pi/4, T) \subset S(A, \alpha, T) \tag{4.1}$$

for every $Q \in A \setminus A_0$ and for a constant β which is sufficiently close to 1. Since the complement of every neighborhood of Q in the cone topology is thin at Q, (4.1) implies the desired result.

To prove (4.1) we suppose that there is a point z in $M \setminus S(A, T, \alpha)$ which is contained in the left-hand side of (4.1). Then $C(z, \alpha) \cap S(\infty) \subset S(\infty) \setminus A$. Hence by Lemma 2.6 we have

$$1-w(z) \geq \omega^z(C(z, \alpha) \cap S(\infty)) \geq C_{14}>0 ,$$

where C_{14} is a constant depending only on N, a, b and α. This contradicts the fact that

$w(z) > \beta$ if $\beta > 1 - C_{14}$. ∎

Using the results obtained in Sections 3 and 4, we will show the following:

COROLLARY 4.5. *Let u and h be positive harmonic functions on M. For every set* $E \subset S(\infty)$, *the following conditions are mutually equivalent*:

(1) *u/h converges nontangentially $d\omega$-a.e. on E.*

(2) *u/h converges admissibly $d\omega$-a.e. on E.*

(3) *u/h converges finely $d\omega$-a.e. on E.*

(4) *u/h converges semifinely $d\omega$-a.e. on E.*

PROOF. The implications $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$ and $(4) \Rightarrow (1)$ are immediate consequences of Proposition 4.2, Theorems 4.3 and 3.2, respectively. By Remark 2.2, the implication $(3) \Rightarrow (4)$ is obvious. ∎

5. Local Fatou theorem. In this section we consider the local Fatou theorem on M.

THEOREM 5.1. (1) *Let h be a positive harmonic function on M. Let E be a Borel measurable subset of $S(\infty)$, and u be a harmonic function on $\bigcup \{T_d^t(Q) : Q \in E\}$ for some $t, d > 0$, such that $T_{d/2}^t(Q) \supset \Gamma^t(Q, \alpha)$ for some $\alpha \in \boldsymbol{R}$. Assume that u/h is bounded below on each $T_d^t(Q)$, $Q \in E$. Then there are sets F_1 and F_2 such that $\omega(F_1) = \mu_h(F_2) = 0$ and that u/h converges admissibly at every point in $E \setminus (F_1 \cup F_2)$.*

(2) *Moreover, suppose that u is harmonic on $\bigcup_{Q \in F} \bigcup_{k=1}^{\infty} T_k^{t(k)}(Q)$ $(=: H)$ for some $t(k) > 0$ and $F \subset E$ with $\omega(E \setminus F) = 0$. If u/h is bounded below on H, then u/h converges nontangentially on $F \setminus (F_1 \cup F_2)$.*

REMRK. By the proof of Proposition 4.2, $T_e^t(Q) \supset \Gamma^t(Q, 0)$ when $e > d(\pi/4)$.

A theorem of this type was obtained by L. Carleson, M. Brelot and J. L. Doob for the upper half-space (cf. [18], [4]) and by Korányi and Taylor [17] for symmetric spaces of rank one. We refer also to Jerison and Kenig [14] and Taylor [19] for nontangentially accessible domains.

Using Lemma 4.4 and the argument in the proof of Theorem 3.2, we can apply a method of the proof of Korányi and Taylor [17, Theorem 3.5] to our context. Thus to prove Theorem 5.1 it is enough to show the following lemma.

LEMMA 5.2. *For every $\beta < \alpha$, $t > 0$ and $\varepsilon > 0$, there are $k > 0$ and a compact subset F of E such that $\omega(E \setminus F) < \varepsilon$ and $S(F, \beta, k) \subset S(E, \alpha, t)$.*

PROOF. As in the proof of Jerison and Kenig [14, (6.2)], we use Calderón's density argument. By Anderson and Schoen [2, p. 456], there is $\tau > 0$ such that if $\Delta(Q, t) \cap \Delta(Q', t) \neq \varnothing$, then $\Delta(Q', t) \subset \Delta(Q, t - \tau)$, $Q, Q' \in S(\infty)$. By the doubling condition we have

$$\omega(\Delta(Q, t+\alpha+\beta-2\tau)) \leq C_{15}\omega(\Delta(Q, t))$$

for all $Q \in S(\infty)$ and $t>0$, where C_{15} is a constant independent of Q and t. Now, let

$$A_j = \left\{Q \in E : \frac{\omega(\Delta(Q, r) \cap E)}{\omega(\Delta(Q, r))} \geq 1-(2C_{15})^{-1}\right\},$$

$j=1, 2, \cdots$. By the weak (1, 1)-estimate for the maximal function defined in [2, p. 454], we easily have $\omega(E \setminus \bigcup_{j=1}^{\infty} A_j)=0$. Hence for every $\varepsilon>0$ there is $k>0$ with $\omega(E \setminus A_k)<\varepsilon$.

Let $Q \in A_k$ and $x \in \Gamma^{k+\tau-\beta}(Q, \beta)$. Set $S=\Delta(Q, t(x)-\tau+\beta)$ and $S'=C(x, t(x)+\alpha) \cap S(\infty)$. Since $Q \in C(x, t(x)-\tau+\beta) \cap \Delta(Q, t(x)-\tau+\beta)$, we have

$$S' \subset C(x, t(x)+\beta) \cap S(\infty) \subset S \subset C(x, t(x)+\beta-2\tau) \cap S(\infty).$$

Hence, $\omega(S) \leq C_{15}\omega(S')$. From this it follows that

$$\omega(S' \cap E) \geq \omega(S \cap E)-\omega(S \setminus S') \geq (2C_{15})^{-1}>0.$$

Consequently, $S' \cap E \neq \varnothing$. Thus we have $x \in S(E, \alpha, k)$. ∎

As a simple consequence of Theorem 5.1 we have the following:

COROLLARY 5.3. *Let E, h and u be as in Theorem* 5.1(1) (*resp.* (2)). *If $h=1$, then u converges admissibly* (*resp. nontangentially*) $d\omega$-a.e. *on E.*

PROOF. Theorem 5.1 and Theorem 4.3 imply that u converges finely $d\omega$-a.e. on E. Thus, by a slight modification of the proof of Theorem 3.2, we obtain the corollary. ∎

REMARK 5.4. By the theory of Ancona [1], all results of our paper can be extended to L-harmonic functions. The operator L is defined by

$$L(u)=\operatorname{div}(\mathscr{A}(\nabla u))+B\cdot\nabla u+\operatorname{div}(uC)+\gamma u,$$

where B and C are vector fields on M, $\mathscr{A}$ is a section of $\operatorname{End}(T(M))$ and γ is a real function on M, and they are assumed to satisfy the following conditions:

(1) $\mathscr{A}$, B, C and γ are measurable.

(2) For every $(x, \xi) \in T(M)$,

$$\lambda^{-1}\|v\|^2 \leq \langle \mathscr{A}_x(v), v\rangle \leq \|\mathscr{A}_x\| \, \|v\|^2 \leq \lambda\|v\|^2,$$

$$\|B\|_{p,x}+\|C\|_{p,x}+\|\gamma\|_{q,x} \leq \lambda,$$

where $\| \ \|_{r,x}$ is the $L^r(B(x, r_0), dv_g)$-norm (r_0 is given in the proof of Proposition 2.3), $r \geq 1$ and λ, p, q are constants such that $1<\lambda<+\infty$, $N<p\leq+\infty$ and $N/2<q\leq+\infty$.

(3) $L(1)=0$, L is weakly coersive and the L-Green function G_x satisfies $G_x=o(\rho_x^{\alpha})$ near the boundary $S(\infty)$, where $\rho_x(y)=d(x, y)$ and α is a positive constant.

(4) For every $Q \in S(\infty)$ and $t \in \boldsymbol{R}$, $\omega^{Q(t)}(\Delta(Q, t)) \geq c$, $d\omega^{Q(t)}$ is the L-harmonic

measure relative to $Q(t)$ where c is a positive constant independent of Q and t. The existence and uniqueness of $d\omega^{Q(t)}$ are guaranteed by the conditions (1)–(3) and [1, Theorem 9].

REMARK 5.5. In order to study harmonic functions on M it is useful to consider the Hardy spaces H^p on M, for which we refer the reader to [3], where the relationship among H^p, BMO and their probabilistic approach are discussed.

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