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MANIFOLDS OVER FUNCTION ALGEBRAS AND MAPPING SPACES

Dedicated to Professor Ichiro Satake on his sixtieth birthday

SHOSHICHI KOBAYASHI*

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1. Introduction. The space of smooth mappings from one manifold into another is an infinite dimensional manifold modelled on a Fréchet space. However, the concept of a Fréchet manifold is so general that its general theory is not very effective. In studying a mapping space, it is customary to complete it to a Banach manifold by an appropriate Sobolev norm. In this paper we shall show that for certain problems such as computing the curvature of a mapping space it is unnecessary to complete the space to a Banach manifold. For this purpose we introduce a very restricted category of infinite dimensional manifolds that comprises mapping spaces.

An n-dimensional manifold is modelled on $\boldsymbol{R}^n$ with the usual structure sheaf $\mathcal{O}_{\boldsymbol{R}^n}$ of germs of C^∞ real valued functions. While an ordinary manifold is defined over the field of real numbers $\boldsymbol{R}$, a manifold in this new category is defined over a function algebra A and is "finite dimensional" over A. Let S be a fixed compact manifold (with or without boundary), and let A be the algebra of C^∞ real valued functions on S. Let V be a real vector bundle (of finite rank) over S, and let $E=\Gamma(S, V)$ denote the space of C^∞ sections of V over S. In the next section we define a very small structure sheaf $\mathcal{O}_E$ on E; it is not the sheaf of germs of ordinary C^∞ real valued functions on E but is a certain sheaf of germs of A-valued functions on E. With $(E, \mathcal{O}_E)$ as a model space, we can define a manifold, called A-manifold, in the same way as we define an ordinary manifold modelled on $(\boldsymbol{R}^n, \mathcal{O}_{\boldsymbol{R}^n})$.

A justification for introducing the concept of A-manifold is that the space M^S of C^∞ mappings from S to an ordinary (finite dimensional) manifold M is an A-manifold and that some differential geometric properties of such a mapping space can be studied more easily as properties of an A-manifold. If M is a Riemannian manifold and S is a compact manifold with a volume element, the mapping space M^S has an induced Riemannian metric. In [1] Freed calculated its curvature by suitably completing M^S to a Banach manifold. By introducing the concept of a Riemannian A-metric and interpreting the induced Riemannian metric of M^S as the integrated form of a Riemannian A-metric, we define the Levi-Civita connection of M^S and compute its curvature with-

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out completing M^S.

A large part of this paper is devoted to the basic theory of A-manifolds. For the sole purpose of computing the curvature of a mapping space in an elementary manner, the general theory developed here is probably excessive. But we are hoping that these basic facts on A-manifolds presented here may have other applications.

For a number of interesting problems related to mapping spaces, A-manifolds are often inadequate. When a problem involves derivatives of mappings, our structure sheaf $\mathcal{O}_E$ is simply too small. For this reason we introduce in the last section a sequence of larger sheaves, i.e., sheaves of $A^{(r)}$-functions for $r=0, 1, 2, \cdots$. For problems involving r-th partial derivatives of mappings, $A^{(r)}$-functions are needed. It is definitely more interesting and non-trivial to develop the theory of $A^{(r)}$-manifolds for $r \geq 1$. The first step in such a task would be to extend classical theorems of analysis such as the implicit function theorem to $A^{(r)}$-functions. However, we confine ourselves in this paper to explaining $A^{(1)}$-functions and $A^{(1)}$-vector fields with examples.

2. Model spaces for A-manifolds. We fix a compact manifold S, and let A be the algebra of real C^∞ functions over S. Let V be a real vector bundle (of finite rank) over S and, $E=\Gamma(S, V)$ the space of C^∞ sections of V over S. Then E is a finitely generated reflexive projective A-module, its dual E^* being isomorphic to the space $\Gamma(S, V^*)$ of sections of the dual vector bundle V^*.

We have the usual C^∞-topology on E, which makes E into a Fréchet space. But we shall not be concerned with this topology at this moment since the structure sheaf can be defined in terms of a more coarse topology, namely the C^0-topology defined as follows. We define it by specifying its neighborhood system. For $\alpha \in E$, let $N \subset V$ be a neighborhood of the section $\alpha(S) \subset V$. Let $\tilde{N}$ be the neighborhood of α in E given by

$$\tilde{N} = \{\xi \in E;\ \xi(s) \in N \text{ for } s \in S\}.$$

Thus $\tilde{N}$ consists of all sections that lie in N.

We construct a sheaf $\mathcal{O}_E$ on E by specifying what an A-function is. Let f be a real C^∞ function on N. Then f induces a mapping $\tilde{f}$: $\tilde{N} \to A$ by

$$(\tilde{f}(\xi))(s) = f(\xi(s)) \qquad \text{for} \quad \xi \in E, \quad s \in S. \tag{2.1}$$

Let $\mathcal{O}_E(\tilde{N})$ be the set of A-valued functions $\tilde{f}$ on $\tilde{N}$ obtained in this way. We call $\tilde{f}$ the *A-function* on $\tilde{N}$ corresponding to the function f on N.

A typical neighborhood of $\alpha \in E$ in the C^∞-topology consists of points ξ of $\tilde{N}$ satisfying additional conditions on derivatives of ξ. For such a neighborhood, say $\tilde{N}'$, we set $\mathcal{O}_E(\tilde{N}') = \mathcal{O}_E(\tilde{N})$.

With $\mathcal{O}_E$ as its structure sheaf, E is a ringed space. The function algebra A and the ringed space $(E, \mathcal{O}_E)$ will play the roles analogous to those of $\boldsymbol{R}$ and $\boldsymbol{R}^n$, respectively.

Let V' be another vector bundle over the same base manifold S and let $E' = \Gamma(S, V')$. We define a (local) morphism from $(E, \mathcal{O}_E)$ to $(E', \mathcal{O}_{E'})$. If $\varphi: N \to V'$ is a fiber-preserving C^∞ map which induces the identity transformation on the base manifold S, it induces a map $\tilde{\varphi}: \tilde{N} \to E'$ by

$$(\tilde{\varphi}(\xi))(s) = \varphi(\xi(s)) \qquad \text{for} \quad \xi \in \tilde{N}. \tag{2.2}$$

If g is a C^∞ function on E', then

$$\widetilde{g \circ \varphi} = \tilde{g} \circ \tilde{\varphi}. \tag{2.3}$$

This shows that, for every smooth $\tilde{g} \in \mathcal{O}_{E'}(E')$, $\tilde{g} \circ \tilde{\varphi}$ is in $\mathcal{O}_E(\tilde{N})$, i.e., $\tilde{\varphi}$ is a morphism of the ringed space $(\tilde{N}, \mathcal{O}_E|_{\tilde{N}})$ into the ringed space $(E', \mathcal{O}_{E'})$. We call $\tilde{\varphi}$ the *A-map induced by* φ. We can show that, conversely, all morphisms $F: \tilde{N} \to E'$ arise in this way.

We shall show that, conversely, if $F: \tilde{N} \to E'$ is a morphism between these ringed spaces, then $F = \tilde{\varphi}$ for some fiber-preserving C^∞ map $\varphi: N \to V'$ which induces the identity transformation on S. Let $\xi \in \tilde{N}$ and $s \in S$. We are forced to define $\varphi(\xi(s))$ by

$$\varphi(\xi(s)) = (F\xi)(s).$$

Then we have to verify that $\varphi(\xi(s))$ depends only on $\xi(s)$, not on ξ. In other words, we have to show that $(F\xi)(s) = (F\xi')(s)$ for any $\xi' \in N$ such that $\xi'(s) = \xi(s)$. Let g be an arbitrary C^∞ function on V'. Since F is a morphism of $(\tilde{N}, \mathcal{O}_E|_{\tilde{N}})$ into $(E', \mathcal{O}_{E'})$, there is a C^∞ function f on N such that

$$\tilde{g} \circ F = \tilde{f} \qquad \text{on} \quad \tilde{N}.$$

Then

$$\tilde{g}(F\xi) = \tilde{f}(\xi), \qquad \tilde{g}(F\xi') = \tilde{f}(\xi').$$

Hence,

$$g((F\xi)(s)) = f(\xi(s)), \qquad g((F\xi')(s)) = f(\xi'(s)).$$

Since $\xi(s) = \xi'(s)$, we have $f(\xi(s)) = f(\xi'(s))$. Hence,

$$g((F\xi)(s)) = g((F\xi')(s)).$$

Since this holds for all g, we have

$$(F\xi)(s) = (F\xi')(s),$$

which proves our assertion.

Let $\alpha \in E$ (resp. $\beta \in E'$) and $N \subset V$ (resp. $N' \subset V'$) a neighborhood of $\alpha(S) \subset V$ (resp. $\beta(S) \subset V'$). A fiber-preserving diffeomorphism $\varphi: N \to N'$ (which covers the identity transformation of S)induces an invertible morphism $\tilde{\varphi}: (\tilde{N}, \mathcal{O}_E|_{\tilde{N}}) \to (\tilde{N}', \mathcal{O}_{E'}|_{\tilde{N}'})$, and the inverse $\tilde{\varphi}^{-1}$ is induced by φ^{-1}. We call $\tilde{\varphi}$ the *A-diffeomorphism induced by* φ.

We have shown that if $F: (\tilde{N}, \mathcal{O}_E|_{\tilde{N}}) \to (\tilde{N}', \mathcal{O}_{E'}|_{\tilde{N}'})$ is an isomorphism of the ringed spaces (i.e., an invertible morphism such that F^{-1} is also a morphism), then F is the A-diffeomorphism induced by some diffeomorphism $\varphi: N \to N'$.

For brevity, by a *local diffeomorphism* of V to V' we mean a fiber-preserving diffeomorphism of a neighborhood N of a section $\alpha(S) \subset V$ onto a neighborhood N' of a section $\beta(S) \subset V'$ which induces the identity transformation on S. Specializing what we have shown above to the case $V' = V$, we see that the pseudogroup of local A-diffeomorphisms of E is isomorphic to the pseudogroup of fiber-preserving local diffeomorphisms of V.

Having constructed the pseudogroup of local A-diffeomorphisms of E, we are in a position to define a manifold modelled on E. An *A-manifold modelled on E* is a Hausdorff topological space $\mathscr{M}$ covered with coordinate charts $(U_i, \varphi_i)_{i \in I}$, where $\{U_i\}$ is an open cover of $\mathscr{M}$ and each φ_i is a homeomorphism of U_i onto an open set $\tilde{N}_i \subset E$ such that

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is an A-diffeomorphism. (Here, $\tilde{N}_i$ is the open set obtained from an open set $N_i \subset V$ which projects onto S, i.e., which is a neighborhood of a section of V.) If we need not specify the model space E, then we call $\mathscr{M}$ simply an *A-manifold.*

An A-function on $\mathscr{M}$ is a mapping $f: \mathscr{M} \to A$ such that $f \circ \varphi_i^{-1}$ is an A-function on $\varphi_i(U_i) \subset E$ for every coordinate chart (U_i, φ_i).

In order to construct a global object on a manifold by patching local objects together, we often use a partition of unity. However, our structure sheaf is too small to construct a partition of unity by A-functions on an A-manifold. To see this, let N_1 and N_2 be neighborhoods of a section $\alpha(S)$ in V such that $\bar{N}_1 \subset N_2$. Then by *Tietze's theorem*, there is a C^∞ real valued function f on V such that (i) $f = 1$ on N_1, (ii) $f = 0$ outside of N_2, and (iii) $0 \le f \le 1$ on V. Then the induced A-function $\tilde{f}$ has the property that $\tilde{f} = 1$ on $\tilde{N}_1$, but it does not have the property that $\tilde{f} = 0$ outside of $\tilde{N}_2$ since a section $\xi \in E$ which lies partially in N_2 and partially outside N_2 is not a point of $\tilde{N}_2$ but $\tilde{f}(\xi) \neq 0$.

Although we may not have a partition of unity by A-functions, in applications we will be dealing mostly with naturally induced Riemannian metrics and connections rather than constructing them from local data.

(2.4) REMARK. In the definition of A-manifold, we could have used the algebra A of continuous functions on any compact topological space S and the model space $E = \Gamma(S, V)$ consisting of all continuous sections of a topological vector bundle V. Then, in defining an A-function, one should consider a continuous function on $N \subset V$ which is C^∞ on each fibre $V_s \cap N$. Accordingly, a *local diffeomorphism* of V to V' should be a fibre-preserving homeomorphism of N onto N' which induces the identity transformation on S and a diffeomorphism on each fibre $V_s \cap N$.

3. Mapping spaces. We fix a compact manifold S. Given an n-dimensional manifold M, let M^S be the space of smooth maps from S into M with the C^∞-topology. This space may not be connected. Each connected component consists of maps which are homotopic to each other. Obviously, the most important component is the one containing the constant maps.

In particular, $\boldsymbol{R}^S = A$ and $(\boldsymbol{R}^n)^S = (\boldsymbol{R}^S)^n = A^n$.

Let $x \in M^S$, i.e., $x\colon S \to M$. Let $x^{-1}TM$ be the pull-back of the tangent bundle TM by x; it is a vector bundle of rank n over S. If x and y are in the same connected component of M^S, i.e., if they are homotopic to each other, then the two vector bundles $x^{-1}TM$ and $y^{-1}TM$ are isomorphic to each other. If x is homotopic to a constant map, then $x^{-1}TM$ is isomorphic to the product bundle $S \times \boldsymbol{R}^n$.

We define the *tangent space* $T_x(M^S)$ of M^S at x by

$$T_x(M^S) = \Gamma(S, x^{-1}TM)\,. \tag{3.1}$$

It is a finitely generated reflexive projective A-module.

We shall define an A-manifold structure in M^S by identifying a neighborhood of x in M^S with a neighborhood of the origin in $T_x(M^S)$. More precisely,

(3.2) THEOREM. *For each $x \in M^S$, the connected component of M^S containing x is an A-manifold modelled on $E = \Gamma(S, x^{-1}TM)$.*

PROOF. Choose a Riemannian metric on M, and let $\exp_{x(s)}\colon T_{x(s)}M \to M$ be the ordinary exponential map for the Riemannian manifold M. We define the induced exponential map

$$\exp_x\colon T_x(M^S) \to M^S$$

by

$$(\exp_x \xi)(s) = \exp_{x(s)} \xi(s) \qquad \text{for} \quad \xi \in T_x(M^S)\,, \quad s \in S\,. \tag{3.3}$$

Then it is not hard to see that $\exp_x$ gives a homeomorphism from a neighborhood $\tilde{N}_x$ of the origin in $T_x(M^S)$ onto a neighborhood U_x of x in M^S. We have to verify that coordinate changes are A-diffeomorphisms. Let $y \in M^S$ be another point in the same connected component as x. We have a homeomorphism $\exp_y\colon \tilde{N}_y \to U_y$. Then we have to verify that

$$\exp_y^{-1} \circ \exp_x\colon \exp_x^{-1}(U_x \cap U_y) \to \exp_y^{-1}(U_x \cap U_y)$$

is an A-diffeomorphism (in the sense of Section 2). Set

$$\tilde{N}_{x,y} = \exp_x^{-1}(U_x \cap U_y) \subset \tilde{N}_x\,, \qquad \tilde{N}_{y,x} = \exp_y^{-1}(U_x \cap U_y) \subset \tilde{N}_y\,.$$

Let $N_{x,y} \subset x^{-1}TM$ (resp. $N_{y,x} \subset y^{-1}TM$) be the neighborhood of the zero section corresponding to $\tilde{N}_{x,y} \subset T_x(M^S)$ (resp. $\tilde{N}_{y,x} \subset T_y(M^S)$). For each $s \in S$, $\exp_{y(s)}^{-1} \circ \exp_{x(s)}$ is a diffeomorphism of $N_{x,y} \cap \pi^{-1}(s)$ onto $N_{y,x} \cap \pi^{-1}(s)$. Varying $s \in S$, we obtain a diffeomorphism of $N_{x,y}$ onto $N_{y,x}$, which induces the map $\exp_y^{-1} \circ \exp_x\colon \tilde{N}_{x,y} \to \tilde{N}_{y,x}$. Thus the connected component of M^S containing x is an A-manifold modelled on $E = \Gamma(S, x^{-1}TM)$, in the sense of Section 1. q.e.d.

We denote the tangent bundle of M^S by $T(M^S)=\bigcup T_x(M^S)$. Now we consider $(TM)^S$. Then there is a natural isomorphism

$$(TM)^S \cong T(M^S)\,. \tag{3.4}$$

In fact, $\xi\in(TM)^S$ can be considered as a tangent vector of M^S at the point $x=\pi\circ\xi\in M^S$ since $\xi(s)\in T_{x(s)}M$ for all $s\in S$. So without ambiguity we may write TM^S for the tangent bundle of M^S, i.e.,

$$TM^S=T(M^S)=(TM)^S\,. \tag{3.5}$$

Similarly, we define the cotangent space $T^*_x(M^S)$ by

$$T^*_x(M^S)=\Gamma(S, x^{-1}T^*M)\,. \tag{3.6}$$

Under the natural pointwise pairing

$$\Gamma(S, x^{-1}TM)\times\Gamma(S, x^{-1}T^*M)\to A\,,$$

$T^*_x(M^S)$ is the dual A-module of $T_x(M^S)$. We denote the cotangent bundle of M^S by $T^*(M^S)=\bigcup T^*_x(M^S)$. Then as in the case of the tangent bundle, we have a natural isomorphism $T^*(M^S)\cong(T^*M)^S$. So we can write

$$T^*M^S=T^*(M^S)=(T^*M)^S\,. \tag{3.7}$$

Let N be another manifold. Each smooth map $f\colon M\to N$ induces a map

$$f^S\colon M^S\to N^S\,,$$

called the *extension* of f, by

$$f^S(x)=f\circ x\,. \tag{3.8}$$

The extension $\pi^S\colon(TM)^S\to M^S$ of $\pi\colon TM\to M$ is consistent with the projection $T(M^S)\to M^S$ and the isomorphism (3.4).

If $p\colon M\times N\to M$ and $q\colon M\times N\to N$ are the projections, $p^S\times q^S$ defines an isomorphism

$$(M\times N)^S\cong M^S\times N^S\,. \tag{3.9}$$

An ordinary function on M^S is a mapping $M^S\to\boldsymbol{R}$. An *A-function* is a mapping $f\colon M^S\to A$ such that $f\circ\exp_x$ is an A-function for every x in the sense of Section 2.

Fix a point x_o of M^S and consider the graph $\Gamma_{x_o}=\{(s, x_o(s));\, s\in S\}\subset S\times M$. Let f be a real valued smooth function defined in a neighborhood of this graph in $S\times M$. Then f defines an A-function $\tilde f$ on M^S by

$$(\tilde f(x))(s)=f(s, x(s))\,. \tag{3.10}$$

We shall show that every A-function defined in a neighborhood of x_o arises in this way. Let $V=x_o^{-1}TM$. Using a Riemannian metric of M and its exponential map exp, we define a mapping Exp: $V \to S \times M$ by

$$\operatorname{Exp}(\xi) = (s, \exp_{x_o} \xi) \qquad \text{for} \quad s \in V_s .$$

Then Exp defines a diffeomorphism from a neighborhood of the zero section in V onto a neighborhood of the graph Γ_{x_o} in $S \times M$. Now our assertion follows from the definition of an A-function.

So the ring of germs of A-functions as $x_o \in M^S$ is naturally isomorphic to the ring of germs of real valued smooth functions at $\Gamma_{x_o} \subset S \times M$.

But we proved a little more:

(3.11) LEMMA. *Given $x_0 \in M^S$, there is a neighborhood N of Γ_{x_0} in $S \times M$ such that the ring of real C^∞ functions on N is naturally isomorphic to the ring of A-functions on the corresponding neighborhood $\tilde{N}$ of x_0 in M^S.*

A global version of (3.11) reads as follows:

(3.12) PROPOSITION. *Every smooth function f: $M \to \mathbf{R}$ defines on A-function $\tilde{f}$ on M^S by* (3.10). *Conversely, for any connected component $\mathcal{M}$ of M^S, every A-function F on $\mathcal{M}$ comes from a real C^∞ function f on $S \times M$.*

Thus the ring of real C^∞ functions on $S \times M$ is naturally isomorphic to the ring of A-functions on $\mathcal{M}$.

PROOF. It suffices to construct f from F. Fix a point $(o, p) \in S \times M$. Then given $x, y \in \mathcal{M} \subset M^S$ such that $p = x(o) = y(o)$, we have only to show that $(F(x))(o) = (F(y))(o)$. (Then we set $f(o, p) = (F(x))(o) = (F(y))(o)$.) Since x and y are homotopic to each other, we have a finite sequence of mappings $x = x_0, \cdots, x_k = y \in \mathcal{M}$ such that $p = x_i(o)$ and x_i is close enough to x_{i-1} in the C^0-topology of M^S (i.e., x_i lies in the neighborhood $\tilde{N}$ of x_{i-1} as described in (3.11)). Then the assertion follows from (3.11). q.e.d.

For applications it is useful to have the following generalization of (3.2). The proof is almost identical to that of (3.2).

(3.13) THEOREM. *Let B be a fibre bundle over a compact manifold S with projection p. Let $T'B$ be the subbundle of the tangent bundle TB consisting of vertical vectors. Let $\Gamma(S, B)$ be the space of C^∞ sections of B. For each section x: $S \to B$, the connected component of $\Gamma(S, B)$ containing x is an A-manifold modelled on $E = \Gamma(S, x^{-1}T'B)$.*

(3.14) REMARK. Every real C^∞ function f on B defines an A-function $\tilde{f}$ on $\Gamma(S, B)$ by

$$(\tilde{f}(x))(s) = f(x(s)) \qquad \text{for} \quad x \in \Gamma(S, B)\,, \quad s \in S\,.$$

Then both (3.11) and (3.12) extend to this case if $S \times M$ is replaced by B.

The space of sections $\Gamma(S, B)$ may be obtained also as an A-submanifold of the mapping space B^S; locally it can be described as the zeros of a finite number of A-functions.

(3.15) REMARK. As we remarked in (2.5), for any compact topological space S and for the space M^S of continuous mappings from S to M the results in this section are still valid.

4. Vector fields. In order to discuss local properties of vector fields on an A-manifold it suffices to consider vector fields on its model space. As in Section 2, let $E=\Gamma(S, V)$ be the space of smooth sections of V. Let $\alpha\in E$ and let X be a vertical vector field defined only along the section $\alpha(S)$, i.e., for each $s\in S$, X_s is a tangent vector to the fibre V_s at $\alpha(s)$. Such a vector field X may be identified with a tangent vector $\tilde{X}\in T_\alpha E$.

We may also regard a tangent vector $\tilde{X}\in T_\alpha E$ as a derivation on the algebra of A-functions defined in a neighborhood of α. Let f be a real C^∞ function defined in a neighborhood of the section $\alpha(S)\subset V$, and $\tilde{f}$ the corresponding A-function defined in a neighborhood of $\alpha\in E$. Then we define $\tilde{X}\tilde{f}\in A$ by

$$(\tilde{X}\tilde{f})(s)=(Xf)(\alpha(s))\in \boldsymbol{R} \qquad \text{for} \quad s\in S\,. \tag{4.1}$$

It is easy to verify

$$\tilde{X}(\tilde{f}+\tilde{g})=\tilde{X}\tilde{f}+\tilde{X}\tilde{g}\,, \tag{4.2}$$

$$\tilde{X}(\tilde{f}\tilde{g})=\tilde{X}\tilde{f}\cdot\tilde{g}(\alpha)+\tilde{f}(\alpha)\cdot\tilde{X}\tilde{g}\,. \tag{4.3}$$

Conversely, if $\tilde{X}$ is an A-linear mapping of the algebra of (germs of) smoooth A-functions at $\alpha\in E$ into A satisfying (4.2) and (4.3) and if we define X by (4.1), then X satisfies similar formulas on the algebra of (germs of) C^∞ functions around $\alpha(S)$ and is seen to be a vertical vector field on V defined along $\alpha(S)$.

Since a tangent vector at α is defined as a vertical vector field of V defined along the section $\alpha(S)$, a cotangent vector at α may be defined as a cotangent vector field (i.e., a 1-form) on V defined along $\alpha(S)$ modulo the horizontal cotangent vector fields along $\alpha(S)$. More precisely, let $\Gamma(S, \alpha^{-1}T^*V)=\Gamma(\alpha(S), T^*V)$ denote the space of sections of the cotangent bundle T^*V over $\alpha(S)$. The space $\Gamma(S, T^*S)$ of 1-forms of S, pulled back by the projection $\pi\colon V\to S$, may be regarded as a subspace of $\Gamma(\alpha(S), T^*V)$. Then the cotangent space $T^*_\alpha E$ of E at α is defined by

$$T^*_\alpha E=\Gamma(S, \alpha^{-1}T^*V)/\Gamma(S, T^*S)\,. \tag{4.4}$$

Now we shall introduce the concept of A-vector field. Let N be a neighborhood of $\alpha(S)\subset V$ and $\tilde{N}$ the corresponding neighborhood of $\alpha\in E$. Let $\xi\in\tilde{N}$. Then every vertical vector field X on $N\subset V$, restricted to $\xi(S)\subset N$, defines a tangent vector $\tilde{X}_\xi\subset T_\xi E$. The vector field $\tilde{X}$ on $\tilde{N}$ thus obtained is called the *A-vector field* induced by X. The correspondence $X \longmapsto \tilde{X}$ gives an isomorphism of the Lie algebra of vertical vector fields on N onto the Lie algebra of A-vector fields on $\tilde{N}$.

Since the vector bundle V is locally a product bundle, i.e., $\pi^{-1}U \cong U \times \boldsymbol{R}^n$ for a small open set $U \subset S$, using a natural coordinate system $x^1, \cdots, x^n$ in $\boldsymbol{R}^n$ we can express a vertical vector field X on V in the form:

$$X = \sum_{i=1}^{n} u^i \frac{\partial}{\partial x^i}, \tag{4.5}$$

where each $u^i = u^i(x^1, \cdots, x^n, s)$ is a function of $x^1, \cdots, x^n, s$. We may consider (4.5) as a local expression for $\tilde{X}$ also.

Given a finite dimensional manifold M, we shall now describe A-vector fields on M^S. Let $S \times TM$ denote the pull-back of the tangent bundle TM by the projections $S \times M \to M$. We should regard $S \times M$ as a (product) bundle over S and then $S \times TM$ as the space of vertical vectors of this bundle. For any $x_o \in M^S$, the pull-back bundle $x_o^{-1}TM$ is naturally isomorphic to the restriction of the vector bundle $S \times TM$ to the graph $\Gamma_{x_o} = \{(s, x(s)); s \in S\}$ of x. So a tangent vector $X \in T_x M^S$ is a section of the vector bundle $S \times TM$ defined along the graph Γ_{x_o}. Every local section X of the bundle $S \times TM$, defined in a neighborhood N of the graph $\Gamma_{x_o} \subset S \times M$, gives rise to an A-vector field $\tilde{X}$ in the corresponding neighborhood $\tilde{N}$ of $x \in M^S$. Explicitly,

$$\tilde{X}_x(s) = X(s, x(s)) \qquad \text{for} \quad x \in \tilde{N}, \quad s \in S. \tag{4.6}$$

If f is a real C^∞ function on N and $\tilde{f}$ is the corresponding A-function on $\tilde{N}$, then $\tilde{X}\tilde{f}$ is by definition the A-function on $\tilde{N}$ corresponding to Xf, i.e.,

$$\widetilde{Xf} = \tilde{X}\tilde{f}. \tag{4.7}$$

In this way, $\tilde{X}$ acts as a derivation on the ring of germs of A-functions at $x_o \in M^S$.

In terms of a local coordinate system $x^1, \cdots, x^n$ of M, a local A-vector field $\tilde{X}$ at $x_o \in M^S$ is therefore represented by

$$X = \sum \xi^i \frac{\partial}{\partial x^i}, \tag{4.8}$$

where each $\xi^i = \xi^i(s, x)$ is a function of $s^1, \cdots, s^m$ and $x^1, \cdots, x^n$.

The proofs of (4.9) and (4.10) are similar to those of (3.11) and (3.12).

(4.9) LEMMA. *Given $x_0 \in M^S$, there is a neighborhood N of Γ_{x_0} in $S \times M$ such that the Lie algebra of real C^∞ vertical vector fields on N is naturally isomorphic to the Lie algebra of A-vector fields on the corresponding neighborhood $\tilde{N}$ of x_0 in M^S.*

(4.10) PROPOSITION. *For any connected component $\mathscr{M}$ of M^S, the Lie algebra of real C^∞ vertical vector fields on $S \times M$ is isomorphic to the Lie algebra of A-vector fields on $\mathscr{M}$ under the correspondence given by* (4.6).

(4.11) REMARK. As in the case of (3.11) and (3.12) (see Remark (3.14)), both (4.9) and (4.10) extend to the space of sections $\Gamma(S, B)$ of a bundle B over S.

5. Differential forms. As in Section 2, let V be a vector bundle of rank n over a compact manifold S with projection π. Let $\alpha \in E = \Gamma(S, V)$ and $N \subset V$ a neighborhood of the section $\alpha(S) \subset V$. Let

$$\mathscr{A}(N) = \sum \mathscr{A}^p(N)$$

be the algebra of differential forms on N. Similarly, let $\mathscr{A}(S)$ be the algebra of differential forms on S. Let $\mathscr{I}(N)$ be the ideal of $\mathscr{A}(N)$ generated by $\pi^*(\sum_{p>0} \mathscr{A}^p(S))$. The algebra of *vertical differential forms* on N is defined to be

$$\mathscr{A}(N/S) = \mathscr{A}(N)/\mathscr{I}(N) . \tag{5.1}$$

We write

$$\mathscr{A}(N/S) = \sum_{p=0}^{n} \mathscr{A}^p(N/S) .$$

We note that, for $p>n$, $\mathscr{A}^p(N)$ is contained in the ideal $\mathscr{I}(N)$ so that $\mathscr{A}^p(N/S) = 0$.

Since the ideal $\mathscr{I}(N)$ is closed under d, exterior differentiation

$$d\colon \mathscr{A}^p(N/S) \to \mathscr{A}^{p+1}(N/S)$$

is well defined. In particular, every element f of A behaves like a constant under d, i.e., $d(\pi^* f) \in \mathscr{I}(N)$ so that $df = 0$ in $\mathscr{A}(N/S)$.

We consider an element of $\mathscr{A}^p(N/S)$ as a p-form on $\tilde{N} \subset E$, where $\tilde{N}$ is the neighborhood of $\alpha \in E$ corresponding to N. Thus we set

$$\mathscr{A}^p(\tilde{N}) = \mathscr{A}^p(N/S) . \tag{5.2}$$

In the preceding section, we defined an A-vector field on $\tilde{N}$ as a vertical vector field on N. It is a straightforward matter (by going back to N) to verify the usual relations between vector fields and differential forms on $\tilde{N}$. We note that the definition of $\mathscr{A}^p(\tilde{N})$ is consistent with the definition of the cotangent space given by (4.4). We call an element of $\mathscr{A}^p(\tilde{N})$ an *A-form* of degree p.

Assume that $\tilde{N}$ is contractible to α in the sense that there is a map

$$h\colon \tilde{N} \times [0, 1] \to \tilde{N}$$

such that

$$h(v, 1) = v , \qquad h(v, 0) = \alpha(\pi(v)) , \qquad \pi(h(v, t)) = \pi(v) \qquad \text{for} \quad v \in N , \quad t \in [0, 1] .$$

The last condition says that the homotopy h leaves each fibre $N_s = \pi^{-1}(s) \cap N$ invariant. Then the Poincaré lemma holds for $\mathscr{A}(\tilde{N})$. To prove this assertion, we apply the usual proof of the Poincaré lemma to forms on N. First we define an operator

$$K\colon \mathscr{A}^{p+1}(N\times[0,1])\to\mathscr{A}^p(N)$$

by setting

$$(5.3)\qquad K\theta=(-1)^p\int_0^1\theta_2 dt \qquad \text{for}\quad \theta=\theta_1+\theta_2\wedge dt\in\mathscr{A}^{p+1}(N\times[0,1]),$$

where θ_1 and θ_2 are forms of degree $p+1$ and p, respectively, not involving dt. The integral on the right hand side of (5.1) means the integral of the coefficients of θ_2 with respect to dt. Then we obtain

$$(5.4)\qquad Kd(h^*\omega)+dK(h^*\omega)=\omega-h_0^*\omega \quad \text{for}\quad \omega\in\mathscr{A}^p(N),$$

where $h_0\colon N\to N$ is defined by $h_0(v)=h(v,0)=\alpha(\pi v)$. Since $h^*(\mathscr{I}(N))\subset\mathscr{I}(N\times[0,1])$ and $h_0^*\omega\in\mathscr{I}(N)$ and since $K(\mathscr{I}(N\times[0,1]))\subset\mathscr{I}(N)$, we see from (5.4) that if $d\omega\in\mathscr{I}(N)$, then

$$\omega=dK(h^*\omega)\qquad \text{modulo}\quad \mathscr{I}(N).$$

Interpreted on $\tilde N$, this means that if $\omega\in\mathscr{A}^p(\tilde N)$ is closed, then $\omega=d\omega'$ for some $\omega'\in\mathscr{A}^{p-1}(\tilde N)$.

If $\omega\in\mathscr{A}^n(\tilde N)$, the integral $\int_{\tilde N}\omega\in A$ is defined by

$$(5.5)\qquad \left(\int_{\tilde N}\omega\right)(s)=\int_{N_s}\omega_N \qquad \text{for}\quad s\in S,$$

where $N_s=N\cap\pi^{-1}s$ and ω_N is an n-form on N representing ω. We note that the integral $\int_{\tilde N}\omega$ is not a real number but is a function on S.

(5.6) REMARK. In applications, S is often equipped with a measure μ_S, or even with a Riemannian metric. Then integrating $\int_{\tilde N}\omega$ over S with respect to μ_S, we obtain a real numbers $\int_S(\int_{\tilde N}\omega)\mu_S$.

Similarly, if $\theta\in\mathscr{A}^{n-1}(\tilde N)$, then the boundary integral $\int_{\partial\tilde N}\theta\in A$ is defined by

$$(5.7)\qquad \left(\int_{\partial\tilde N}\theta\right)(s)=\int_{\partial N_s}\theta_N.$$

The Stokes formula

$$(5.8)\qquad \int_{\tilde N}d\theta=\int_{\partial\tilde N}\theta$$

follows from the usual Stokes formula for N_s and θ_N.

We consider now differential forms on the mapping space M^S. Let $S\times T^*M$denote the pull-back of the cotangent bundle T^*M by the projection $S\times M\to M$. Then the germs of 1-forms at $x_o\in M^S$ are identified with the germs of sections of the bundle $S\times T^*M$ along the graph $\Gamma_{x_o}\subset S\times M$. More generally, the germs of p-forms at $x_o\in M^S$

are identified with the germs of sections of $S\times\wedge^p T^*M$ along the graph Γ_{x_0}. We call a section of $S\times\wedge^p T^*M$ a *vertical* p-form.

In terms of local coordinates $s^1, \cdots, s^m$ and $x^1, \cdots, x^n$ of S and M, a vertical p-form on M^S can be therefore expressed locally as

$$\omega=\sum f_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}\,, \tag{5.8}$$

where each coefficient $f_{i_1\cdots i_p}$ is a function of $s^1, \cdots, s^m$ and $x^1, \cdots, x^n$.

More precisely, we have the following (cf. (3.11) and (4.9)).

(5.9) LEMMA. *Given $x_0\in M^S$, there is a neighborhood N of Γ_{x_0} in $S\times M$ such that the algebra of real C^∞ vertical differential forms on N is naturally isomorphic to the algebra of A-forms on the corresponding neighborhood $\tilde{N}$ of x_0 in M^S.*

Correspondingly to (3.12) and (4.10) we have the following global version of (5.9).

(5.10) PROPOSITION. *For any connected component $\mathscr{M}$ of M^S, the algebra of real C^∞ vertical differential forms on $S\times M$ is naturally isomorphic to the algebra of A-forms on $\mathscr{M}$.*

Thus, if ω is a vertical p-form and $X_1, \cdots, X_p$ are vertical vector fields on $S\times M$ and if $\tilde{\omega}$ and $\tilde{X}_1, \cdots, \tilde{X}_p$ are the corresponding A-form and A-vector fields on $\mathscr{M}\subset M^S$, then $\tilde{\omega}(\tilde{X}_1, \cdots, \tilde{X}_p)$ is the A-function that corresponds to the real valued function $\omega(X_1, \cdots, X_p)$ on $S\times M$.

From (5.10) it follows that the de Rham cohomology of A-forms on each connected component $\mathscr{M}$ of M^S is isomorphic to $H^*(M, \boldsymbol{R})\otimes A$.

6. Affine connections. Let $\mathscr{M}$ be an A-manifold modelled on $E=\Gamma(S, V)$. An affine connection on $\mathscr{M}$ can be described in terms of covariant differentiation ∇. Given A-vector fields X and Y on $\mathscr{M}$, an A-vector field $\nabla_X Y$ is assigned in such a way that

(i) $(X, Y)\longmapsto\nabla_X Y$ is bilinear over A,

(ii) $\nabla_{fX}Y=f\nabla_X Y$ and $\nabla_X(fY)=Xf\cdot Y+f\cdot\nabla_X Y$ for any A-function f on $\mathscr{M}$.

A curve $x=x(t)$ in $\mathscr{M}$ is a geodesic if its velocity vector $\dot{x}=dx/dt$ satisfies the equation

$$\nabla_{\dot{x}}\dot{x}=0\,. \tag{6.1}$$

We shall express $\nabla_X Y$ in terms of local coordinate systems of S and V. Since the question is local, we may assume that $\mathscr{M}=E=\Gamma(S, V)$.

Let $s^1, \cdots, s^m$ be a local coordinate system in S. As in (4.5), using a local fibre coordinate system $x^1, \cdots, x^n$ for V, we express X and Y in the form

$$X=\sum u^i\frac{\partial}{\partial x^i}\qquad Y=\sum v^i\frac{\partial}{\partial x^i}\,, \tag{6.3}$$

where the components u^i and v^i are functions of $s^1, \cdots, s^m$ and $x^1, \cdots, x^n$. Define

Christoffel's symbols Γ^i_{jk} as functions of $s^1, \cdots, s^m, x^1, \cdots, x^n$ by

(6.4) $$\nabla_{\frac{\partial}{\partial x^j}}\left(\frac{\partial}{\partial x^k}\right)=\sum \Gamma^i_{jk}\frac{\partial}{\partial x^i}\cdot$$

Then

(6.5) $$\nabla_X Y=\sum u^j \nabla_j v^i \frac{\partial}{\partial x^i},$$

where

(6.6) $$\nabla_j v^i=\frac{\partial v^i}{\partial x^j}+\sum \Gamma^i_{jk}v^k .$$

The torsion T and curvature R of ∇ are given by the usual formula:

(6.7) $$T(X, Y)=\nabla_X Y-\nabla_Y X-[X, Y], \qquad R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]} .$$

Since a point of E is a section of V, a curve $x(t)$ in E with parameter t is locally given by

(6.8) $$x(t)=(x^i(t, s)) .$$

Its velocity vector is given by

(6.9) $$\dot{x}=\sum \frac{dx^i}{dt}\frac{\partial}{\partial x^i} .$$

Hence the equation of a geodesic is expressed locally as

(6.10) $$\frac{d^2x^i}{dt^2}+\sum \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=0 .$$

So the equation looks the same as in the classical case except for the fact that both x^i and Christoffel's symbols Γ^i_{jk} depend on $s \in S$. Since S is compact, it follows that for any initial point $x(0)$ and any initial velocity $\dot{x}(0)$ there is a unique geodesic $x=x(t)$ at least for sufficiently small t, $|t|<\varepsilon$. (We note that ε can be chosen independently of s since S is compact.)

Given a finite dimensional manifold M and an affine connection ∇, we consider its extension to M^S. More generally, let $\nabla=\nabla_s$ be a family of affine connections (i.e., covariant differentiation) on M parametrized by $s \in S$. It induces an affine connection $\tilde{\nabla}$ on M^S:

(6.11) $$\tilde{\nabla}_X \tilde{Y}=\widetilde{\nabla_X Y} .$$

Since every A-vector field on M^S is of the form $\tilde{X}$ for some vector field X on M which depends on the parameter $s \in S$, (6.11) defines a connection in M^S.

In particular, given a connection ∇ on M (independent of $s \in S$), there is an induced connection $\tilde{\nabla}$ on M^S.

Fix a point $x \in M^S$, and let $V = x^{-1}TM$. Let $(x^1, \cdots, x^n)$ be a local coordinate system in M. For the connection ∇, Christoffel's symbols Γ^i_{jk} are functions of $x^1, \cdots, x^n$, $s^1 \cdots, s^m$ since ∇ depends on the parameter $s \in S$. By an obvious identification, we use $\partial/\partial x^1, \cdots, \partial/\partial x^n$ as a local basis for V. By abuse of notation, we shall denote the corresponding local fibre coordinate system of V also by $x^1, \cdots, x^n$. For the connection $\tilde{\nabla}$, Christoffel's symbols are given by (6.4). It is not hard to see that they coincide with Christoffel's symbols for ∇.

The curvature $\tilde{R}$ of $\tilde{\nabla}$ is given as a ntural extension of the curvature R of ∇. Thus, if $X, Y, Z \in T_x(M^S) = \Gamma(S, x^{-1}TM)$, then $(\tilde{R}(X, Y)Z \in T_x(M^S)$ is given by

(6.12) $$(\tilde{R}(X, Y)Z)(s) = R(X(s), Y(s))Z(s)\,.$$

The same holds with the torsion tensor.

(6.13) REMARK. We shall explain a slight technical problem we encounter when we try to treat an affine connection as a special case of a connection in a principal bundle. Let P be a principal bundle over M with structure group G. The right action of G on P extends to a right action of G^S on P^S. However, it is not quite correct to say that P^S is a principal A-bundle over M^S with structure group G^S. Since some mappings of S into M may not lift to a mapping of S into P, P^S may not project *onto* M^S. So we consider

$$M^S_o = \{x \in M^S;\ x^{-1}P \cong S \times G \text{ (product bundle)}\}\,.$$

In general, M^S_o is a union of some connected components of M^S. Then P^S is a principal A-bundle over M^S_o with structure group G^S, and every connection in P induces a connection in P^S. So with this approach we fail to treat some components of M^S.

7. Riemannian structures. An *inner product b* in a vector bundle V over S defines, at each point $s \in S$, an inner product b_s in the fibre V_s of V in such a way that b_s varies smoothly with s. Any two inner product b and b' in V are equivalent in the sense that there is an isomorphism $f\colon V \to V$ such that $b'(v, w) = b(fv, fw)$ for all $v, w \in V_s$, $s \in S$. Each inner product b in V gives rise to an *inner product* in $E = \Gamma(S, V)$:

$$\tilde{b}\colon E \times E \to A$$

by

(7.1) $$(\tilde{b}(\xi, \eta))(s) = b(\xi(s), \eta(s)) \qquad \text{for} \quad \xi, \eta \in E\,, \quad s \in S\,.$$

Then $\tilde{b}$ satisfies the following conditions:

(i) $\tilde{b}\colon E \times E \to A$ is symmetric and bilinear over A;

(ii) $\tilde{b}$ is positive definite in the sense that, for each $\xi \in E$, the function $\tilde{b}(\xi, \xi) \in A$ is nonnegative everywhere on S and vanishes at $s \in S$ if and only if ξ vanishes at s. (We note that (ii) is stronger than the condition that $\tilde{b}(\xi, \xi) \geq 0$ on S for every $\xi \in E$, and $\equiv 0$ on S if and only if $\xi = 0$).

(iii) the inner product b defines an A-module isomorphism $E \approx E^*$ sending $\xi \in E$ to $\tilde{b}(\xi, \cdot) \in E^*$.

Conversely, an inner product $\tilde{b}$ on E satisfying (i) and (ii) above comes from an inner product b in V. (We note that (iii) is a consequence of (i) and (ii).)

An inner product in E is unique up to an equivalence in the sense explained above.

Let $\mathcal{M}$ be an A-manifold modelled on $E = \Gamma(S, V)$. A *Riemannian A-metric g* on $\mathcal{M}$ defines at each point $x \in \mathcal{M}$ an inner product g_x on $T_x\mathcal{M}$ satisfying (i), (ii) (and hence (iii)) above and is "smooth" in x in the sense that

(iv) if X and Y are A-vector fields on $\mathcal{M}$, then $g(X, Y)$ is an A-function on $\mathcal{M}$.

Then there is a unique torsionfree affine connection preserving the A-metric g; this is called the *Levi-Civita connection* of $(\mathcal{M}, g)$. The proof of this assertion is exactly the same as in the classical case. Namely, given A-vector fields X and Y on $\mathcal{M}$, we define an A-vector field $\nabla_X Y$ by the following equation:

$$\begin{aligned}2g(\nabla_X Y, Z) = X\cdot g(Y, Z) + Y\cdot g(X, Z) - Z\cdot g(X, Y)\\ + g([X, Y], Z) + g([Z, X], Y) + g(X, [Z, Y])\,,\end{aligned}$$

which should hold for all A-vector fields Z on $\mathcal{M}$. By (iv), the equation above determines a vector field $\nabla_X Y$, and the usual proof of the proposition in the finite dimensional case is valid, (see the second proof of Theorem 2.2 of Chapter IV in [2]).

With the notation of (6.3) the A-metric can be expressed locally in the same form as in the classical case:

$$g(X, Y) = \sum g_{ij} u^i v^j\,, \qquad \text{where} \quad g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right). \tag{7.2}$$

We note that the coefficients g_{ij} are functions of $s^1, \cdots, s^m, x^1, \cdots, x^n$.

Christoffel's symbols Γ^i_{jk} can be expressed in terms of g_{ij} by the well known formula, the proof being identical to the classical case.

Given two points p and q in $\mathcal{M}$, let $\Omega_{p,q}$ be the space of curves $c\colon [0, 1] \to M$ with $c(0) = p$ and $c(1) = q$. The A-energy $E(c)$ of c is defined by

$$E(c) = \int_0^1 g(\dot{c}(t), \dot{c}(t))dt \in A\,, \qquad \text{where} \quad \dot{c} = dc/dt\,. \tag{7.3}$$

We note that $E(c)$ is not a real number but is a non-negative function on S.

We can calculate the first variation of $E(c)$ as in the classical case. If c_τ, $-\varepsilon < \tau < \varepsilon$, is a variation of c, i.e., a 1-parameter family of curves belonging to $\Omega_{p,q}$ such that $c = c_0$ and if X is the infinitesimal variation of c induced by c_τ, i.e., $X = (\partial c_\tau / \partial \tau)_{\tau=0}$, then

$$\left(\frac{dE(c_\tau)}{d\tau}\right)_{\tau=0} = -2\int_0^1 g(X, \ddot{c})dt\,, \tag{7.4}$$

where $\ddot{c}=\nabla_{\dot{c}}\dot{c}$ is the second covariant derivative of c. It follows that c is a critical point of the energy functional E (i.e., $(dE(c_\tau)/d\tau)_{\tau=0}=0$ for all variations c_τ of c if and only if c is a geodesic.

Suppose that S is given a measure μ_S. Then an Riemannian A-metric g on $\mathscr{M}$ gives rise to a *real* Riemannian metric $\bar{g}$:

$$\bar{g}(X, Y)=\int_S g(X, Y)\mu_S . \tag{7.5}$$

We have also the *real* energy

$$\bar{E}(c)=\int_0^1 \bar{g}(\dot{c}(t), \dot{c}(t))dt=\int_S E(c)\mu_S . \tag{7.6}$$

Since differentiation d/dt commutes with integration with respect to μ_S, it follows that a critical point of $E(c)$, i.e., a geodesic c is a critical point of $\bar{E}(c)$.

Let $X, Y\in T_x\mathscr{M}$. Unless X and Y are linearly independent at each point $s\in S$, we cannot speak of the A-sectional curvature of the A-plane spanned by X, Y. If they are, then the A-sectional curvature is the element of A given by

$$\frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y)-g(X, Y)^2} . \tag{7.7}$$

Integrating (7.7) over S we obtain the *real* sectional curvature.

The Ricci tensor Ric is defined as an A-bilinear form on $T_x\mathscr{M}$, $x\in\mathscr{M}$; the trace of $Z\longmapsto R(Z, Y)X$ makes sense and is denoted $\mathrm{Ric}(X, Y)$. Again, integrating $\mathrm{Ric}(X, Y)$ over S, we obtain the *real* Ricci tensor. Similarly, for the scalar curvature.

Let M be an ordinary finite dimensional manifold and g a Riemannian metric on M parametrized by $s\in S$. (Thus, g may be considered as a Riemannian structure in the vector bundle $S\times TM$ over $S\times M$, where $S\times TM$ denotes the subbundle of $T(S\times M)$ consisting of vertical tangent vectors with respect to the projection $S\times M\to S$). Then g induces a Riemannian A-metric $\tilde{g}$ on M^S in a natural manner. At $x\in M^S$, the inner product $\tilde{g}_x$ in T_xM^S is given by

$$(\tilde{g}_x(X, Y))(s)=g_{(s,x(s))}(X(s), Y(s)) \qquad \text{for} \quad X, Y\in T_xM^S , \quad s\in S . \tag{7.8}$$

The proof of the following proposition is straightforward.

(7.9) PROPOSITION. *The Levi-Civita connection ∇ of a Riemannian metric g on M parametrized by S induces, via* (6.11), *the Levi-Civita connection $\tilde{\nabla}$ of the Riemannian A-metric $\tilde{g}$ on M^S.*

Hence the curvature $\tilde{R}$ of $\tilde{g}$ can be expressed in terms of the curvature R of g by (6.12).

If S is equipped with a volume element μ_S, then we define as in (7.5) a real Riemannian metric $\bar{g}$ by integrating $\tilde{g}$ over S.

(7.10) PROPOSITION. *The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ is also the Levi-Civita connection for $\bar{g}$.*

PROOF. Since $\tilde{\nabla}$ has no torsion, it suffices to verify that it preserves the metric $\bar{g}$. Let X and Y be A-vector fields on M^S and let $Z \in T_x M^S$. We integrate the equation

$$Z(\tilde{g}(X, Y)) = \tilde{g}(\nabla_Z X, Y) + \tilde{g}(X, \nabla_Z Y)$$

over S. The right hand side yields $\bar{g}(\nabla_Z X, Y) + \bar{g}(X, \nabla_Z Y)$ by the very definition of $\bar{g}$. In order to see that the left hand side yields $Z(\bar{g}(X, Y))$, we consider in general an A-function f on an A-manifold M as a real valued function F on $S \times \mathcal{M}$ by setting

$$F(s, x) = (f(x))(s) \qquad \text{for} \quad (s, x) \in S \times \mathcal{M} .$$

Then the differentiation by Z and the integration by μ_S commute:

$$Z\left(\int F(s, x)\mu_S\right) = \int Z(F(s, x))\mu_S .$$

Apply this formula to $f = \tilde{g}(X, Y)$. Then we obtain

$$Z(\bar{g}(X, Y)) = \int Z(\tilde{g}(X, Y))\mu_S . \qquad \text{q.e.d.}$$

This is consistent with Freed's curvature computation for mapping spaces in [1].

(7.11) REMARK. In applications, we have to consider sometimes mapping spaces with base point. Fix a point $(s_o, p_o) \in S \times M$, and let $M_o^S = (M, p_o)^{(S, s_o)}$ be the set of mappings $x \in M^S$ such that $x(s_o) = p_o$. Its tangent space $T_x M_o^S$ at x is the subspace of $T_x M^S$ given by

(7.12) $$T_x M_o^S = \{\xi \in \Gamma(S, x^{-1}TM);\ \xi(s_o) = 0\} .$$

Hence, the A-vector field $\tilde{X}$ of M^S corresponding to a vertical vector field X of $S \times M$ is tangent to M_o^S if and only if X vanishes at (s_o, p_o). If Y is another vertical vector field on $S \times M$ vanishing at (s_o, p_o), then $\nabla_X Y$ is again a vertical vector field of $S \times M$ vanishing at (s_o, p_o). Hence, M_o^S is a totally geodesic submanifold of M^S and its curvature is obtained by "restricting" the curvature of M^S.

Since $T_x M_o^S$ is not a projective A-module, M_o^S is not an A-manifold in the sense defined in Section 2.

An important example of $(M, p_o)^{(S, s_o)}$ is provided by the group of based loops in a Lie group G.

8. Enlarged structure sheaves. We have noted already that the class of A-functions on an A-manifold is too small to include some of the most important functions. In this section we shall show how to enlarge our structure sheaf $\mathcal{O}_E$ to a desired size.

Let V be a real vector bundle of rank n over a compact manifold S, and $E=\Gamma(S, V)$ as in Section 2. We define first the r-th jet bundle $V^{(r)}$ of V. We say that two local sections ξ and η of V defined in a neighborhood of a point $s\in S$ define the same *r-jet* at o if their partial derivatives of order $\leq r$ coincide at s, and we write $j_s^r(\xi)=j_s^r(\eta)$. Thus, $j_s^r(\xi)$ is the equivalence class consisting of local sections η with the same r-jet as ξ at s. Let $V_s^{(r)}$ be the set of all r-jets of local sections at s; it is a vector space of dimension $n(m+r)!/m!r!$. Let $V^{(r)}=\bigcup_s V_s^{(r)}$; it is a vector bundle over S.

Every $\xi\in E=\Gamma(S, V)$ gives rise to a section $j^r\xi$ of $V^{(r)}$ in a natural manner. Let $\alpha\in E$. Given a neighborhood N of $(j^r\alpha)(S)$ in $V^{(r)}$, the subset $\tilde{N}\subset E$ defined by

$$\tilde{N}=\{\xi\in E;\ (j^r\xi)(S)\subset N\} \tag{8.1}$$

is a typical neighborhood of α in E the C^r-topology.

We construct a sheaf $\mathcal{O}_E^{(r)}$ on E, called the *sheaf of germs* of *$A^{(r)}$-functions* on E. Every real C^∞ function f on N induces a mapping $\tilde{f}$: $\tilde{N}\to A$ by

$$(\tilde{f}(\xi))(s)=f(j_s^r\xi) \qquad \text{for} \quad \xi\in E\,, \quad s\in S\,. \tag{8.2}$$

Let $\mathcal{O}_E^{(r)}(\tilde{N})$ be the set of A-valued functions $\tilde{f}$ on $\tilde{N}$ obtained in this way. We call $\tilde{f}$ the *$A^{(r)}$-function* on $\tilde{N}$ corresponding to f.

Clearly, we have

$$\mathcal{O}_E^{(r)}\subset\mathcal{O}_E^{(r+1)}\,. \tag{8.3}$$

As soon as we have the sheaf $\mathcal{O}_E^{(r)}$ of germs of $A^{(r)}$-functions on the model space E, we have the concept of *$A^{(r)}$-manifold* and that of *$A^{(r)}$-map* as in Section 2.

In order to define an $A^{(r)}$-vector field, consider a vertical vector field X defined on $N\subset V^{(r)}$. For each $\xi\in\tilde{N}$, the vector field X gives a vertical vector field $X_{|j^r\xi}$ at the section $j^r\xi(S)\subset\tilde{N}$. Apply the natural projection $V^{(r)}\to V$ to $X_{|j^r\xi}$ to obtain a vertical vector field X_ξ at the section $\xi(S)\subset V$. Let $\tilde{X}_\xi\in T_\xi E$ be the vector corresponding to X_ξ. The vector field $\tilde{X}$ on $\tilde{N}$ thus obtained is called the *$A^{(r)}$-vector field* induced by X.

As in Section 3, let M^S be the mapping space of S into M; its connected components are all A-manifolds (see (3.2)). We consider M^S as the space of sections of the product bundle $S\times M$ over S, and we denote the r-th jet bundle of B by $B^{(r)}$.

We shall exhibit an $A^{(1)}$-function which is not an A-function on M^S. Assuming that S is oriented, let μ_S be a volume form of S; it is an everywhere positive m-form on S, (where $m=\dim S$). Let ω be any m-form on M. We define a mapping $\tilde{f}$: $M^S\to A$ by

$$x^*\omega=\tilde{f}(x)\mu_S \qquad \text{for} \quad x\in M^S\,. \tag{8.4}$$

Then $\tilde{f}$ is not an A-function but is an $A^{(1)}$-function. In terms of local coordinates $(s^1, \cdots, s^m)$ and $(x^1, \cdots, x^n)$ of S and M, write

$$\mu_S=a\cdot ds^1\wedge\cdots\wedge ds^m\,,$$

$$\omega=\frac{1}{m!}\sum b_{j_1\cdots j_m}dx^{j_1}\wedge\cdots\wedge dx^{j_m}$$

so that

$$\frac{x^*\omega}{\mu_S}=\frac{1}{a}\sum b_{j_1\cdots j_m}\frac{\partial x^{j_1}}{\partial s^1}\cdots\frac{\partial x^{j_m}}{\partial s^m}.$$

Let (s^i, x^j, x^j_i) be the local coordinate system in $B^{(1)}$ induced naturally from $(s^1, \cdots, s^m)$ and $(x^1, \cdots, x^n)$; thus $x^j_i=\partial x^j/\partial s^i$. Then $\tilde{f}$ corresponds to the real C^∞ function f on $B^{(1)}$ given by

$$f=\frac{1}{a}\sum b_{j_1\cdots j_m}x^{j_1}_1\cdots x^{j_m}_m. \tag{8.5}$$

For example, consider the group $\mathscr{D}(S)$ of diffeomorphisms of S. Since it is open in the mapping space S^S, it is an A-manifold. Given a volume element μ_S on S, let $\mathscr{D}(S, \mu_S)$ be the group of volume-preserving diffeomorphisms of S. For each diffeomorphism x of S, let $x^*\mu_S=f(x)\mu_S$, where $f(x)$ is a C^∞ real valued function on S. Then the mapping f: $\mathscr{D}(S)\to A$ is an $A^{(1)}$-function but is not an A-function. The group $\mathscr{D}(S, \mu_S)$ defined by $f(x)=1$ is not an A-submanifold of $\mathscr{D}(S)$.

Every vertical vector field X on $B^{(r)}$ induces an $A^{(r)}$-vector field $\tilde{X}$ on M^S in a natural way; (the construction is similar to that of the $A^{(r)}$-vector field $\tilde{X}$ on E explained above).

We shall now give an example of $A^{(1)}$-vector field on M^S. Let Y be a vector field on S. At each $x\in M^S$, define $\bar{Y}_x\in T_xM^S=\Gamma(S, x^{-1}TM)$ by

$$\bar{Y}_x(s)=x_*(Y_s). \tag{8.6}$$

Then $\bar{Y}$ is an $A^{(1)}$-vector field on M^S although it is not an A-vector field. In fact, in terms of the local coordinate system (s^i, x^j, x^j_i) introduced above, for

$$Y=\sum\eta^i(s)\frac{\partial}{\partial s^i},$$

$\bar{Y}$ is induced by the following vertical vector field on $B^{(1)}$.

$$\sum\eta^i(s)x^j_i\frac{\partial}{\partial x^j}. \tag{8.7}$$

Vector fields of type $\bar{Y}$ are necessary in discussing stability of mappings. (A mapping $x: S\to M$ is said to be stable if every element of T_xM^S is written as a sum of two vectors $\tilde{X}_x$ and $\bar{Y}_x$, where X is a vector field on M and Y is a vector field on S.)

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Department of Mathematics
University of California
Berkeley, California, 94720
U.S.A.