

THE BEHAVIOUR ON RADIAL FUNCTIONS OF MAXIMAL OPERATORS ALONG ARBITRARY DIRECTIONS AND THE KAKEYA MAXIMAL OPERATOR

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1. Introduction. For locally integrable functions f defined on \mathbf{R}^n and a positive integer N we define the Kakeya maximal operator K_N as

$$K_N f(x) = \sup_{x \in R \in \mathcal{B}_N} \frac{1}{|R|} \int_R |f(y)| dy,$$

where \mathcal{B}_N denotes the class of all rectangles in \mathbf{R}^n of eccentricity N ; that is, congruent with any dilate of the rectangle $[0, 1]^{n-1} \times [0, N]$, and where $|A|$ denotes Lebesgue measure of the set A .

Given $R \in \mathcal{B}_N$, we can always find a cube in \mathbf{R}^n , Q , with $R \subset Q$ and $|Q| = N^{n-1} |R|$. Hence, we see that K_N is an operator of weak type $(1, 1)$ with a constant which does not exceed $C_n N^{n-1}$. (Here and throughout the paper, C_n denotes always a constant depending only on dimension, although its value may vary from one place to another.)

Using this estimate together with the trivial inequality

$$\|K_N f\|_\infty \leq \|f\|_\infty,$$

we obtain via the Marcinkiewicz interpolation theorem that K_N is bounded on all $L^p(\mathbf{R}^n)$, $1 < p < \infty$, and moreover

$$\|K_N f\|_p \leq C_n \frac{p}{p-1} N^{(n-1)/p} \|f\|_p.$$

It is conjectured however that for $p=n$ a better estimate holds, namely,

$$(1) \quad \|K_N f\|_{L^n(\mathbf{R}^n)} \leq C_n (1 + \log N)^{\alpha(n)} \|f\|_{L^n(\mathbf{R}^n)}$$

for some constant $\alpha(n)$ depending only on dimension.

This conjecture is closely related to a longstanding classical conjecture about the boundedness of Bochner-Riesz means in \mathbf{R}^n . Estimate (1) was obtained for the particular case $n=2$ by Córdoba [2] (see also Fefferman [4], Strömberg [9] and Wainger [10]) who used it to give a new proof of the celebrated Carleson-Sjölin theorem ([1], [4]).

In a recent paper, Igari [5] has shown that estimate (1) holds in \mathbf{R}^n for some constant $\alpha(n)$ when we restrict ourselves to the class of radial functions. His result is the following:

Let \mathcal{B}'_N be the class of all rectangles in \mathbf{R}^n congruent with $[0, 1]^{n-1} \times [0, N]$ and let K'_N be the corresponding maximal operator. Then for every radial function f we have

$$\|K'_N f\|_{L^n(\mathbf{R}^n)} \leq C_n(1 + \log N)^{1+1/n} \|f\|_{L^n(\mathbf{R}^n)}.$$

His proof, although elementary, does not give a clear insight on the role played by radial functions in this particular problem; moreover, the exponent $\alpha(n) = 1 + 1/n$ is not best possible even for the larger operator K_N .

The purpose of this note is to present a simple proof of the following:

THEOREM 1. *There exists a constant C_n such that for every radial function f one has*

- (i) $t |\{x \in \mathbf{R}^n: K_N f(x) > t\}|^{1/n} \leq C_n(1 + \log N)^{(n-1)/n} \|f\|_{L^n(\mathbf{R}^n)}$
- (ii) $\|K_N f\|_{L^n(\mathbf{R}^n)} \leq C_n(1 + \log N) \|f\|_{L^n(\mathbf{R}^n)}$.

Given a finite set of unit vectors $\Omega = \{\omega_i\}_{i=1}^N$, we also define M_Ω as the maximal operator over rectangles in \mathbf{R}^n having one side parallel to one of the given directions ω_i . When these N directions are uniformly distributed on S^{n-1} , we will simply write M_N instead of M_Ω .

It is easy to see that in \mathbf{R}^n , K_N is majorized by $M_{N^{n-1}}$ and therefore Theorem 1 follows from the more general:

THEOREM 2. *Let Ω denote a collection of N unit vectors in \mathbf{R}^n . Then there exists a constant C_n such that*

- (i') $t |\{x \in \mathbf{R}^n: M_\Omega f(x) > t\}|^{1/n} \leq C_n(1 + \log N)^{(n-1)/n} \|f\|_{L^n(\mathbf{R}^n)}$
- (ii') $\|M_\Omega f\|_{L^n(\mathbf{R}^n)} \leq C_n(1 + \log N) \|f\|_{L^n(\mathbf{R}^n)}$

for every radial function f .

Our proof relies entirely on the study of the behaviour of the “universal” maximal operator \mathcal{M} (supremum over all rectangles in \mathbf{R}^n) on radial functions and this reduces matters to computing the constant of boundedness of the one-dimensional Hardy-Littlewood maximal operator on $L^p(\mathbf{R})$ with respect to the weight $w(t) = |t|^{n-1}$, $p > n$.

In addition to that, we will show with an easy example that the dependence on N of the constants in Theorem 1 (and hence in Theorem 2 too) cannot be improved. In particular, with the notation in (1), $\alpha(n) = 1$ is sharp for the class of radial functions.

The estimates in Theorem 2 for general functions are known to be true in dimension 2 for the operator M_N (see [9], [3], [10]). The problem for N arbitrary directions is still open, even with possibly bigger constants $(1 + \log N)^\beta$, $\beta \geq 1$.

In the last section we will give a more precise form of Theorem 2 when $n = 2$ and, as a consequence, we will prove the boundedness on radial functions of the maximal operator defined along a Cantor set of directions.

Finally, we would like to point out that we began thinking on these problems in conversations with J. L. Rubio de Francia whose ideas continue to influence our mathematics.

2. The universal maximal operator on radial functions. As we mentioned above, the main ingredient in this paper is to show that for radial functions in \mathbf{R}^n the operator \mathcal{M} , defined as the supremum of averages over all rectangles, behaves well on $L^p(\mathbf{R}^n)$ for $p > n$.

PROPOSITION 3. *There exists C_n such that for every $p > n$ and every radial function we have*

$$(2) \quad t |\{x \in \mathbf{R}^n: \mathcal{M} f(x) > t\}|^{1/p} \leq C_n \left(\frac{p}{p-n}\right)^{1/p'} \|f\|_{L^p(\mathbf{R}^n)}$$

where $1/p + 1/p' = 1$.

REMARK. Observe that for $n \geq 2$, \mathcal{M} is not of weak type (n, n) . In fact, if f is the characteristic function of the unit ball in \mathbf{R}^n , $\mathcal{M} f \notin L^{n,\infty}(\mathbf{R}^n)$.

Also since K_N and M_N are majorized obviously by \mathcal{M} , estimate (2) holds for these two operators on radial functions. For general functions, the constant of boundedness cannot be independent of N as the construction of the Keakeya set shows.

PROOF OF PROPOSITION 3. A simple geometric argument shows that

$$\mathcal{M} f(x) \leq C_n \mathcal{M}_0 f(x)$$

where

$$\mathcal{M}_0 f(x) = \sup_{|\omega|=1} \sup_{r>0} \frac{1}{r} \int_0^r |f(x + t\omega)| dt .$$

Now, given f radial we denote by f_0 its radial projection on $[0, \infty)$. Thus, we have $f(x) = f_0(|x|)$ for all $x \in \mathbf{R}^n$.

Next we show

$$(3) \quad \mathcal{M}_0 f(x) \leq 2 f_0^\dagger(|x|) ,$$

where f_0^\dagger is the Hardy-Littlewood maximal function of f_0 . In order to prove (3) we will need the following:

LEMMA 4. *Let $\phi: [a, b] \rightarrow \mathbf{R}_+$ be monotonic and convex. Then, for every function h ,*

$$I = \frac{1}{b-a} \int_a^b |h(\phi(s))| ds \leq h^\dagger(\phi(a)) .$$

PROOF. For simplicity we will assume ϕ is C^1 on (a, b) . Then a change of variables gives

$$I = \frac{1}{b-a} \int_{\phi(a)}^{\phi(b)} |h(u)| w(u) du ,$$

with $w(u) = 1/\phi'(\phi^{-1}(u))$, and the result follows from the fact that w is monotone decreasing with constant sign and

$$\frac{1}{b-a} \int_{\phi(a)}^{\phi(b)} w(u) du = 1. \quad \blacksquare$$

Since f is radial, so is $\mathcal{M}_0 f$. Hence, in order to reduce matters to a one-dimensional problem, it suffices to show

$$(4) \quad \mathcal{M}_0 f(v, 0, \dots, 0) \leq 2f_0^\dagger(v), \quad v \geq 0.$$

Fix $|w|=1$, $w=(w_1, \dots, w_n)$, $r>0$ and $v>0$. Set $\phi(t) = |(v, 0, \dots, 0) + tw| = (v^2 + 2vtw_1 + t^2)^{1/2}$. Observe that

$$I = \frac{1}{r} \int_0^r |f((v, 0, \dots, 0) + tw)| dt = \frac{1}{r} \int_0^r |f_0(\phi(t))| dt.$$

Clearly ϕ is convex on $(0, \infty)$ and monotone decreasing for $t \leq -vw_1$ and increasing for $t \geq -vw_1$. Thus for $w_1 \geq 0$ or for $w_1 < 0$, but $r < -vw_1$, (4) follows directly from Lemma 4. Now, if $w_1 < 0$ and $-vw_1 < r < -2vw_1$ then

$$I \leq \frac{1}{-vw_1} \int_0^{-2vw_1} |f_0(\phi(t))| dt = 2 \frac{1}{-vw_1} \int_0^{-vw_1} |f_0(\phi(t))| dt \leq 2f_0^\dagger(v),$$

with the equality above due to the symmetry properties of ϕ , whereas if $w_1 < 0$ and $-2vw_1 < r$, then

$$I \leq \max\left(\frac{1}{-2vw_1} \int_0^{-2vw_1} |f_0(\phi(t))| dt, \frac{1}{r+2vw_1} \int_{-2vw_1}^r |f_0(\phi(t))| dt\right) \leq 2f_0^\dagger(v).$$

To conclude the proof of Proposition 3, we just need to use polar coordinates, estimate (4) and the fact that $w(t) = |t|^{n-1}$ is in the Muckenhoupt class A_p for every $p > n$ with A_p -constant

$$C_{w,p} = \sup \left\{ \left(\frac{1}{|I|} \int_I w \right)^{1/p} \left(\frac{1}{|I|} \int_I w^{-1/(p-1)} \right)^{1/p'} : I \text{ interval} \right\} \leq C \left(\frac{p}{p-n} \right)^{1/p'}$$

(see [6]).

REMARK. A straightforward computation, with the help perhaps of Hardy's inequality (see Stein-Weiss [8]) gives the strong type inequality

$$\left(\int_0^\infty (h^\dagger(t))^p t^{n-1} dt \right)^{1/p} \leq C \frac{p}{p-n} \left(\int_0^\infty |h(t)|^p t^{n-1} dt \right)^{1/p}, \quad p > n$$

for h supported in $(0, \infty)$.

A consequence of this is the strong type estimate for the universal maximal func-

tion \mathcal{M}

$$(5) \quad \|\mathcal{M}f\|_{L^p(\mathbf{R}^n)} \leq C_n \frac{p}{p-n} \|f\|_{L^p(\mathbf{R}^n)}$$

for every f radial.

Both estimates (5) and (2) are sharp as the example $f(x) = |x|^{-n/p} \chi_{\{1 \leq |x| \leq L\}}$ for large L shows.

3. Proof of Theorem 2. Theorem 2 follows by interpolating estimate (2) and trivial estimates for M_Ω on, say, $L^{3/2}(\mathbf{R}^n)$. The only thing we have to be sure of is that the interpolation theorem we use is also valid for operators defined on radial functions. This is clearly the case for the Riesz convexity theorem as a quick look at its proof shows (see e.g. Stein-Weiss [8]).

For the weak type we will use:

PROPOSITION 5. *Let T be a linear and positive (i.e., $|Tf(x)| \leq (T|f|)(x)$) operator such that T is bounded $L^{p_0}(\mathbf{R}^n) \rightarrow L^{p_1, \infty}(\mathbf{R}^n)$ with constant M_i , $i=0, 1$, $0 < p_1 \leq p_2 \leq \infty$. Then, for $0 \leq \theta \leq 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$, we have*

$$(6) \quad t |\{x \in \mathbf{R}^n : |Tf(x)| > t\}|^{1/p} \leq e^{1/p} M_0^{1-\theta} M_1^\theta \|f\|_p.$$

If T is defined only on radial functions, (6) still holds for such functions.

To obtain (i') in Theorem 2 we apply the usual "linearisation" techniques to M_Ω , together with Proposition 5 with $p_0 = 3/2$, $M_0 = C_n N$ (any $C_n N^\beta$, $\beta > 0$, will work), $p_1 = n + 1/\log N$ and $p = n$. In order to compute M_1 we use estimate (2). To obtain (ii') we use the ordinary Riesz convexity theorem with the same parameters as before and with M_1 computed now from estimate (5).

The proof of Proposition 5, with perhaps a larger constant, is essentially given in Sagher [7]. Here we simply keep track of the constants appearing and check our claim on radial functions.

We start with some notation: Given a function g and $r > 0$, we define

$$\begin{aligned} \lambda_g(t) &= |\{x \in \mathbf{R}^n : |g(x)| > t\}| \\ g^*(t) &= \inf \{s : \lambda_g(s) \leq t\} \\ g_r^{**}(t) &= \sup_{|E|=t} \left(\frac{1}{t} \int_E |g(x)|^r dx \right)^{1/r} = \left(\frac{1}{t} \int_0^t (g^*(s))^r ds \right)^{1/r}. \end{aligned}$$

As is well known,

$$\sup_{t>0} t \lambda_g^{1/p}(t) = \sup_{t>0} t^{1/p} g^*(t) = \|g\|_{p, \infty}^*.$$

Also, $g^*(t) \leq g_r^{**}(t)$ and, if $r < p$,

$$\sup_{t>0} t^{1/p} g_r^{**}(t) \leq \left(\frac{p}{p-r}\right)^{1/r} \|g\|_{p,\infty}^* .$$

Define $\alpha(z) = p((1-z)/p_0 + z/p_1)$, $0 \leq \text{Re } z \leq 1$. Thus $\alpha(\theta) = 1$. Given a function f , which without loss of generality we assume non-negative and simple, we define $F_x(z) = Tf^{\alpha(z)}(x)$.

Observe that if f is radial, so is $f^{\alpha(z)}$. Also, F_x is analytic and satisfies the estimates for $y \in \mathbf{R}$,

$$|F_x(iy)| \leq T |f^{\alpha(iy)}|(x) = Tf^{p/p_0}(x) = g(x)$$

and

$$|F_x(1+iy)| \leq T |f^{\alpha(1+iy)}|(x) = Tf^{p/p_1}(x) = \bar{g}(x) .$$

From the three line theorem we have

$$|Tf(x)| = |F_x(\theta)| \leq (g(x))^{1-\theta} (\bar{g}(x))^\theta .$$

Now, Hölder's inequality gives

$$(Tf)_r^{**}(t) \leq (g_r^{**}(t))^{1-\theta} (\bar{g}_r^{**}(t))^\theta .$$

Hence,

$$\begin{aligned} \|Tf\|_{p,\infty}^* &\leq \left(\frac{p_0}{p_0-r}\right)^{(1-\theta)/r} \|Tf^{p/p_0}\|_{p_0,\infty}^{*(1-\theta)} \left(\frac{p_1}{p_1-r}\right)^{\theta/r} \|Tf^{p/p_1}\|_{p_1,\infty}^{*\theta} \\ &\leq \left(\frac{p_0}{p_0-r}\right)^{(1-\theta)/r} \left(\frac{p_1}{p_1-r}\right)^{\theta/r} M_0^{1-\theta} M_1^\theta \|f\|_p \end{aligned}$$

and the proposition follows by letting $r \rightarrow 0$. ■

To show that the constants $(1 + \log N)^{(n-1)/n}$ and $(1 + \log N)$ in (i) and (ii), respectively, of Theorem 1 are sharp we simply consider the function $f(x) = (1 + |x|)^{-1} \chi_{\{|x| \leq N\}}$. By taking rectangles of dimensions $(j/N) \times (j/N) \times \cdots \times (j/N) \times j$, it is easy to see that for $j-1 < |x| < j$ we have

$$K_N f(x) \geq c_n \frac{\log j}{j}, \quad j = 1, 2, \dots, N .$$

Hence,

$$\|K_N f\|_{L^n(\mathbf{R}^n)} \geq c_n (1 + \log N)^{(n+1)/n}$$

and

$$\left| \left\{ x \in \mathbf{R}^n : K_N f(x) > c_n \frac{\log N}{N} \right\} \right|^{1/n} \geq c'_n N$$

whereas

$$\|f\|_{L^n(\mathbb{R}^n)} \sim (\log N)^{1/n}.$$

4. The maximal function along a Cantor set of directions. Given a real number θ , we define the maximal function in \mathbb{R}^2 along the direction θ as

$$M_\theta f(x) = \sup_{a < 0 < b} \frac{1}{b-a} \int_a^b |f(x + t(\cos \theta, \sin \theta))| dt.$$

Thus, $M_\theta f = M_{\theta+\pi} f$. If A is a subset of $[0, \pi)$ we also define

$$M_A f(x) = \sup\{M_\theta f(x) : \theta \in A\}.$$

For a finite set A we can apply Theorem 2 to obtain

$$\|M_A f\|_{L^2(\mathbb{R}^2)} \leq C(1 + \log \#A) \|f\|_{L^2(\mathbb{R}^2)},$$

where $\#A$ denotes the cardinality of A and f is a radial function.

In this section we will prove the following sharper result:

THEOREM 6. *Let $0 = \theta_1 < \theta_2 < \dots < \theta_N < \pi$ be given and let $A = \{\theta_k\}_{k=1}^N$. Define $\alpha_k = \theta_{k+1} - \theta_k$, $k = 1, 2, \dots, N$ with $\theta_{N+1} = \pi$. Then, for $1 < p \leq 2$, there exists a constant C_p such that for every radial function f*

$$(7) \quad \|M_A f\|_{L^p} \leq C_p E_p(A) \|f\|_{L^p}$$

where

$$E_p(A) = \left(\sum_1^N \alpha_k^{p-1} \left(1 + \log^+ \frac{1}{\alpha_k} \right)^{2(p-1)} \right)^{1/p}.$$

REMARKS. (a) Observe that $\theta_k = (k-1)\pi/N$, $k = 1, 2, \dots, N$, the uniformly distributed case, is a particular, already known, case of Theorem 6. It turns out that this is also the key case for the proof of (7).

(b) It is worth noting that

$$\sup \left\{ \left(\sum_1^N \alpha_k \left(1 + \log^+ \frac{1}{\alpha_k} \right)^2 \right)^{1/2} : \alpha_k \geq 0, \sum_1^N \alpha_k = \pi \right\} \leq C(1 + \log N)$$

and hence (7) is a better estimate than (ii') in Theorem 2.

COROLLARY 7. *Let A denote the ordinary 1/3-Cantor set. Then for $p > 1 + \log 2 / \log 3$, there exists a constant C_p such that for every radial function f*

$$(8) \quad \|M_A f\|_p \leq C_p \|f\|_p.$$

PROOF. Given $J \in \mathbb{N}$, we let $A_J = \{x = \sum_{j=0}^J a_j 3^{-j} : a_j \in \{0, 2\}\}$. Since $A_J \uparrow A$, it suffices to prove (8) for M_{A_J} with a constant C_p independent of J .

But

$$E_p(A_J) \sim \left(1 + \sum_{j=0}^J 2^j \frac{1}{3^{j(p-1)}} (1 + \log 3^j)^{2(p-1)} \right)^{1/p} \leq C \left(\sum_{j=0}^{\infty} \frac{2^j 3^{2j(p-1)}}{3^{j(p-1)}} \right)^{1/p} = C_p$$

and $C_p < \infty$ provided $2 < 3^{p-1}$. ■

In order to prove Theorem 6 we will need the following lemmas:

LEMMA 8. *Let $x \in \mathbf{R}^2$, $x = (x_1, x_2)$ and set $\theta_x = \arctan(x_2/x_1)$. Then for every f radial, we have:*

- (a) $\sup_{\theta_x - \pi/2 < \theta \leq \theta_0} M_\theta f(x) \leq 2M_{\theta_0} f(x)$ if $\theta_x - \pi/2 < \theta_0 < \theta_x$.
- (b) $\sup_{\theta_1 \leq \theta < \theta_x + \pi/2} M_\theta f(x) \leq 2M_{\theta_1} f(x)$ if $\theta_x < \theta_1 < \theta_x + \pi/2$.

The proof of Lemma 8 is very similar to the way we proved estimate (3) and in fact requires the same convexity result, Lemma 4.

Let us assume with no loss of generality $\theta_x = 0$. Fix $\theta_x - \pi/2 = -\pi/2 < \theta_0 < \theta_x = 0$ and $-\pi/2 < \theta \leq \theta_0$. Consider for instance the case

$$0 < r < |x| \cos \theta.$$

For $0 < t < r$, let $\phi(t)$ be defined by the condition

$$|x + te^{i(\theta + \pi)}| = |x + \phi(t)e^{i(\theta_0 + \pi)}|.$$

Then $\phi(t)$ is an increasing convex function and if f is radial

$$I = \frac{1}{r} \int_0^r |f(x + te^{i(\theta + \pi)})| dt = \frac{1}{r} \int_0^r |f_{\theta_0, x}(\phi(t))| dt$$

where $f_{\theta_0, x}(u) = f(x + ue^{i(\theta_0 + \pi)})$.

From Lemma 4, $I \leq f_{\theta_0, x}^\dagger(0) = M_{\theta_0} f(x)$. The remaining cases, as well as part (b) of the lemma, can be proven in the same way. ■

LEMMA 9. *With the same notation as above, if $x \in \mathbf{R}^2$, $0 < \theta_1 - \theta_0 < \pi/2$ and $\theta_0 \leq \theta_x \leq \theta_1$, then for every radial function f*

$$\sup_{\theta_1 \leq \theta \leq \theta_0 + \pi} M_\theta f(x) \leq 2 \max(M_{\theta_0} f(x), M_{\theta_1} f(x)).$$

PROOF. Apply Lemma 8 to the cases $\theta_1 \leq \theta \leq \theta_x + \pi/2$ and $\theta_x + \pi/2 \leq \theta \leq \theta_0 + \pi$.

LEMMA 10. *For all $\theta \in [0, \pi)$, for all $0 < \alpha < \pi$, and for all f radial,*

$$(9) \quad \iint_{D_{\theta, \alpha}} (M_\theta f)^p(x) dx \leq C_p \alpha^{p-1} \left(1 + \log^+ \frac{1}{\alpha} \right)^{2(p-1)} \int |f|^p dx$$

where $D_{\theta,\alpha} = \{x \in \mathbf{R}^2 \mid |\theta_x - \theta| < \alpha \text{ or } |\theta_x + \pi - \theta| < \alpha\}$.

PROOF. With no loss of generality we may assume $\theta = 0$. Take $N \in \mathbf{N}$ such that $\pi/2N < \alpha \leq \pi/N$. It suffices to prove (9) for $\alpha = \pi/N$. Consider the operator M_N corresponding to directions $\theta_k = (k-1)\pi/N$, $k = 1, 2, \dots, N$. Now for every f radial, $M_N f$ is a periodic function in the angular variable of period π/N . Moreover

$$\iint_{\mathbf{R}^2} (M_N f(x))^p dx \leq C_p N^{2-p} (1 + \log N)^{2(p-1)} \int |f|^p dx.$$

Hence

$$\iint_{D_{0,\pi/N}} (M_0 f)^p dx \leq \iint_{D_{0,\pi/N}} (M_N f)^p dx \leq \frac{C_p}{N} N^{2-p} (1 + \log N)^{2(p-1)} \int |f|^p dx.$$

PROOF OF THEOREM 6. Clearly we have

$$\iint_{\mathbf{R}^2} (M_A f)^p(x) dx \leq \sum_{k=1}^N \iint_{D_{\theta_k, \max(\alpha_{k+1}, \alpha_k)}} (M_A f)^p(x) dx$$

and the theorem follows from Lemmas 9 and 10.

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