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SECOND RICCI CURVATURE OF HOMOGENEOUS BOUNDED DOMAINS

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Abstract. In this paper it is proved that there exists no non-symmetric homogeneous bounded domain whose second Ricci curvature of the Bergman metric is homothetic to the square of the metric. The argument is based on a result of Geatti and the theory of normal j-algebras.

Introduction. There is a countable sequence of conditions H_k which the curvature tensor of harmonic spaces must satisfy. The first condition H_1 is the Einstein condition. Let D be a homogeneous bounded domain in C^n with the Bergman metric g. It is well-known that g always satisfies H_1. Recently Geatti [5] proved that if g satisfies Condition H_2, i.e., the second Ricci curvature of g is homothetic to the square of g, then D is biholomorphic to one of the following:

(1) the unit ball in C^n,
(2) the 6-dimensional classical domain of type IV in Cartan's classification,
(3) the exceptional symmetric domain of dimension 16,
(4) the exceptional symmetric domain of dimension 27,
(5) the domain of dimension 26 of type T(8; 8, 8), where

$$\mathrm{T}(8;8,8):=\left[\begin{array}{cc|c} 1 & 8 & 8 \\ & 1 & 8 \end{array}\right] \tag{0.1}$$

(see (2.4)). It is well-known (Carpenter-Gray-Willmore [3]) that the first four types of domains actually satisfy Condition H_2. In her paper [5] Geatti asked whether the last 26-dimensional domain satisfies Condition H_2 or not. In the present paper, we shall show that the last domain does not satisfy Condition H_2 (Proposition 3.1), so that there exists no non-symmetric homogeneous bounded domain whose Bergman metric satisfies Condition H_2 (Theorem 3.2). Our argument is based on the theory of normal j-algebras which represent the Bergman metric of homogeneous bounded domains.

1. The second Ricci curvature and Condition H_2. Let (M, g) be a Riemannian manifold. Let $\nabla_X Y$ be the covariant derivative with respect to the Levi-Civita connection of g, and set $R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}$, $R(X, Y, Z, W)=g(R(Z, W)Y, X)$. For a tangent vector $x\in T_pM$ at a point $p\in M$, let $S(x)\in \mathrm{End}(T_pM)$ be the mapping defined by

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$$S(x)u=R(u, x)x\,. \tag{1.1}$$

For every positive integer k, the k-th Ricci curvature $\rho^{[k]}$ of g is, by definition, the symmetric covariant tensor field of degree $2k$ satisfying

$$\rho^{[k]}(x, \dots, x)=\operatorname{trace} S(x)^k \qquad \text{for all} \quad x\in T_pM\,. \tag{1.2}$$

For $k=1, 2$, the manifold (M, g) is said to satisfy Condition H_k if there exists a function $\lambda\in C^\infty(M)$ such that

$$\rho^{[k]}(x, \dots, x)=\lambda(p)g(x, x)^k \qquad \text{for all} \quad x\in T_pM,\ p\in M \tag{1.3}$$

(cf. Carpenter-Gray-Willmore [3]). We note that $\rho^{[1]}$ is the usual Ricci curvature and H_1 coincides with the Einstein condition. Let B be a basis of the tangent space T_pM, $g_{ab}=g(a, b)$ for $a, b\in B$, and (g^{ab}) be the inverse matrix of (g_{ab}). For $x\in T_pM$, we have

$$\rho^{[2]}(x, x, x, x)=\sum_{a,b,c,d\in B} g^{ab}g^{cd}R(x, c, x, b)R(x, a, x, d)\,. \tag{1.4}$$

From now on, we assume that M is a complex manifold and g is a Kähler metric on M. Let p be a point in M, T_pM the real tangent space at p, and $j\in\operatorname{End}(T_pM)$ the complex structure on M. The holomorphic tangent space T_p^hM at p is realized as $\{z\in T_pM\otimes C; jz=iz\}=\chi(T_pM)$, where $j\in\operatorname{End}(T_pM\otimes C)$ is the complex linear extension of j and $\chi\colon T_pM\to T_pM\otimes C$ is defined by $\chi(x)=2^{-1}(x-ijx)$. As usual, for $x_a\in T_pM$, set

$$R_{x_1\bar{x}_2x_3\bar{x}_4}=R(\chi(x_1), \overline{\chi(x_2)}, \chi(x_3), \overline{\chi(x_4)})\,, \tag{1.5}$$

$$g_{x_1\bar{x}_2}=g(\chi(x_1), \overline{\chi(x_2)})\,, \tag{1.6}$$

where $g(\cdot, \cdot)$ and $R(\cdot, \cdot, \cdot, \cdot)$ are extended to complex multi-linear mappings. We note that

$$g_{x\bar{x}}=\frac{1}{2}g(x, x)\,. \tag{1.7}$$

We need the following lemma.

LEMMA 1.1. *If B is a subset of T_pM such that $B\cap jB=\varnothing$ and $B\cup jB$ is a basis of T_pM, and if $x\in T_pM$, then*

$$\rho^{[2]}(x, x, x, x)=2\rho_1(x, x, x, x)+2\rho_2(x, x, x, x)\,,$$

where

$$\rho_1(x_1, x_2, x_3, x_4)=\sum_{p,q,r,s\in B} g^{\bar{p}q}g^{\bar{r}s}R_{x_1\bar{x}_2q\bar{r}}R_{x_3\bar{x}_4s\bar{p}}\,, \tag{1.8}$$

$$\rho_2(x_1, x_2, x_3, x_4)=\sum_{p,q,r,s\in B} g^{\bar{p}q}g^{\bar{r}s}R_{x_1\bar{p}x_3\bar{r}}R_{q\bar{x}_2s\bar{x}_4}\,, \tag{1.9}$$

and $(g^{\bar{p}q})_{p,q\in B}$ *is the inverse matrix of* $(g_{p\bar{q}})_{p,q\in B}$.

PROOF. Setting

$$\tilde{R}(a,b)=\sum_{c,d\in B\cup jB} g^{cd}R(x,c,x,b)R(x,a,x,d)\,,$$

we have

$$\rho^{[2]}(x,x,x,x)=\sum_{a,b\in B\cup jB} g^{ab}\tilde{R}(a,b)=\sum_{p,q\in B} g^{\bar{p}q}(\tilde{R}(\overline{\chi(p)},\chi(q))+\tilde{R}(\chi(q),\overline{\chi(p)}))\,. \tag{1.10}$$

For the same reason we have

$$\tilde{R}(\overline{\chi(p)},\chi(q))=\sum_{r,s\in B} g^{\bar{r}s}(R(x,\overline{\chi(r)},x,\chi(q))R(x,\overline{\chi(p)},x,\chi(s)) + R(x,\chi(s),x,\chi(q))R(x,\overline{\chi(p)},x,\overline{\chi(r)})).$$

Since $x=\chi(x)+\overline{\chi(x)}$, it follows that

$$\tilde{R}(\overline{\chi(p)},\chi(q))=\sum_{r,s\in B} g^{\bar{r}s}(R_{x\bar{r}q\bar{x}}R_{x\bar{p}s\bar{x}}+R_{s\bar{x}q\bar{x}}R_{x\bar{p}x\bar{r}})\,.$$

Similarly, we have

$$\tilde{R}(\chi(q),\overline{\chi(p)})=\sum_{r,s\in B} g^{\bar{r}s}(R_{x\bar{r}x\bar{p}}R_{q\bar{x}s\bar{x}}+R_{s\bar{x}x\bar{p}}R_{q\bar{x}x\bar{r}})\,.$$

Substituting these into (1.10) we get the desired formula.

2. The curvature of quasi-symmetric bounded domains. Let D be a homogeneous bounded domain in $\boldsymbol{C}^n$ with the Bergman metric g and p be a point in D. Then the real tangent space T_pD at p possesses the structure $(\mathfrak{g}, j)$ of a normal j-aljebra such that $\mathfrak{g}$ is a Lie algebra which coincides with T_pD as a real vector space, that j is the complex structure of T_pD, and that if ω is the Koszul form of $\mathfrak{g}$, i.e., $\omega\in\mathfrak{g}^*$ is defined by

$$\omega(x)=\frac{1}{2}\,\mathrm{trace}(\mathrm{ad}\,jx-j\circ\mathrm{ad}\,x) \qquad \text{for} \quad x\in\mathfrak{g}\,, \tag{2.1}$$

then it holds that $g(x,y)=\omega[jx,y]$ for $x,y\in T_pD=\mathfrak{g}$. Here, a normal j-algebra $(\mathfrak{g}, j)$ is, by definition (cf. Pyatetskii-Shapiro [6]), a triangular Lie algebra over $\boldsymbol{R}$ with complex structure j satisfying that $[jx,jy]=j[jx,y]+j[x,jy]+[x,y]$ for $x,y\in\mathfrak{g}$, and that if ω is the Koszul form (2.1) of $\mathfrak{g}$, then the bilinear form $\langle\cdot,\cdot\rangle$ given by

$$\langle x,y\rangle=\omega[jx,y] \qquad \text{for} \quad x,y\in\mathfrak{g} \tag{2.2}$$

defines a j-invariant inner product on $\mathfrak{g}$. Two normal j-algebras $(\mathfrak{g}, j)$, $(\tilde{\mathfrak{g}},\tilde{j})$ are said to be isomorphic if there exists an isomorphism $\Phi\colon\mathfrak{g}\to\tilde{\mathfrak{g}}$ of Lie algebras such that $\Phi\circ j=\tilde{j}\circ\Phi$.

Let $\mathscr{H}_n$ be the set of all biholomorphic equivalence classes of homogeneous bounded domains in $\boldsymbol{C}^n$, and let $\mathscr{A}_n$ be the set of all isomorphism classes of normal j-algebras of dimension $2n$. The assertion mentioned above yields the existence of a natural mapping from $\mathscr{H}_n$ to $\mathscr{A}_n$. It is also known that the mapping is bijective (cf. [6]).

Let $(\mathfrak{g}, j)$ be a normal j-algebra. Set $\mathfrak{n}=[\mathfrak{g}, \mathfrak{g}]$. The dimension R of the orthogonal complement $\mathfrak{a}$ of $\mathfrak{n}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ in (2.2) is called the rank of $(\mathfrak{g}, j)$ or the rank of the corresponding homogeneous bounded domain D. For any linear form $\alpha \in \mathfrak{a}^*$ on $\mathfrak{a}$, set $\mathfrak{n}(\alpha)=\{x \in \mathfrak{n}\,;\, [h, x]=\alpha(h)x \text{ for all } h \in \mathfrak{a}\}$. Every element of the set $\Delta := \{\alpha \in \mathfrak{a}^*\,;\, \mathfrak{n}(\alpha) \neq \{0\}\}$ is said to be a root. The structure theorem of Pyatetskii-Shapiro [6] says the following:

(n1) $\mathfrak{a}$ is a non-trivial abelian subalgebra of $\mathfrak{g}$, amd $\mathfrak{n}$ has an orthogonal decomposition $\sum_{\alpha \in \Delta} \mathfrak{n}(\alpha)$.

(n2) There are R roots $\varepsilon_1, \dots, \varepsilon_R$ such that $j\mathfrak{a}$ is the direct sum of the 1-dimensional root spaces $\mathfrak{n}(\varepsilon_a)$, and that any of the other roots is one of the forms $2^{-1}(\varepsilon_a \pm \varepsilon_b)$, $2^{-1}\varepsilon_c$, where $a, b, c \in \{1, \dots, R\}$ with $a<b$.

(n3) $j\mathfrak{n}(2^{-1}(\varepsilon_a+\varepsilon_b))=\mathfrak{n}(2^{-1}(\varepsilon_a-\varepsilon_b))$ when $a<b$, and $j\mathfrak{n}(2^{-1}\varepsilon_a)=\mathfrak{n}(2^{-1}\varepsilon_a)$ for all a.

Let us fix some notation. Set

$$\mathscr{L}=\sum_{a \le b} \mathfrak{n}\left(\frac{1}{2}(\varepsilon_a+\varepsilon_b)\right), \quad \mathscr{U}=\sum_a \mathfrak{n}\left(\frac{1}{2}\varepsilon_a\right),$$

$$n_{ab}=\dim \mathfrak{n}\left(\frac{1}{2}(\varepsilon_a+\varepsilon_b)\right), \quad n_a=\frac{1}{2}\dim \mathfrak{n}\left(\frac{1}{2}\varepsilon_a\right).$$

We then have the decomposition

$$\mathfrak{g}=\mathscr{L}+j\mathscr{L}+\mathscr{U}\,. \tag{2.3}$$

We call the table

$$\left[\begin{array}{ccccc|c} 1 & n_{12} & \cdots & n_{1,R-1} & n_{1R} & n_1 \\ & 1 & \cdots & n_{2,R-1} & n_{2R} & n_2 \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & 1 & n_{R-1,R} & n_{R-1} \\ & & & & 1 & n_R \end{array}\right] \tag{2.4}$$

of multiplicities the type of the normal j-algebra $(\mathfrak{g}, j)$ or the type of the corresponding homogeneous bounded domain D in $\boldsymbol{C}^n$. We note that the sum of all numbers in the type table coincides with the complex dimension n of D, because of the decomposition (2.3). Let $r_a \in \mathfrak{n}(\varepsilon_a) \setminus \{0\}$ be the unique element such that $[jr_a, r_a]=r_a$. It is known (cf. [8]) that if we set $\omega_a := \omega(r_a)=\langle r_a, r_a \rangle$, then

(2.5) $$\omega_a = 1 + \frac{1}{2}\left(\sum_{b>a} n_{ab} + \sum_{b<a} n_{ba}\right) + \frac{1}{2} n_a .$$

Let

(2.6) $$r := r_1 + r_2 + \cdots + r_R \in \mathscr{L} .$$

The following is needed.

LEMMA 2.1 (Pyatetskii-Shapiro [6]). *If* $1 \le a < b < c \le R+1$, $x, x' \in \mathfrak{n}(2^{-1}(\varepsilon_a + \varepsilon_b))$, $y, y' \in \mathfrak{n}(2^{-1}(\varepsilon_b + \varepsilon_c))$ *with the convention* $\varepsilon_{R+1} = 0$, *then*

$$\langle [jx, y], [jx', y'] \rangle + \langle [jx, y'], [jx', y] \rangle = \frac{1}{\omega_b} \langle x, x' \rangle \langle y, y' \rangle .$$

By the identification $\mathfrak{g} = T_p D$ the covariant derivative $\nabla_x y$ with respect to the Levi-Civita connection of the Bergman metric is well-defined as an element of $\mathfrak{g}$ for all $x, y \in \mathfrak{g}$. It is given by

$$\nabla_x y = \frac{1}{2}([x, y] - (\operatorname{ad} x)^t y - (\operatorname{ad} y)^t x) ,$$

where $(\operatorname{ad} x)^t$ is the adjoint operator of $\operatorname{ad} x$ with respect to the inner product $\langle \cdot, \cdot \rangle$ in (2.2). For $x, y \in \mathscr{L}$ and $u, v \in \mathscr{U}$, we define

$$xy = -j \nabla_x y \in \mathscr{L} ,$$

$$\varphi(x) u = -2 j \nabla_x u \in \mathscr{U} ,$$

$$F(u, v) = \frac{1}{4}([ju, v] + i[u, v]) \in \mathscr{L} \otimes \boldsymbol{C} .$$

It is well-known ([4], [2]) that the quasi-symmetry of D in the sense of Satake [7] is equivalent to the property that n_{ab} are constant for all a, b with $a < b$ and so are n_a for all a. We need the following lemma.

LEMMA 2.2. *Let* $x_a \in \mathscr{L}$ *and* $u_b \in \mathscr{U}$. *The following formulas hold*:

(LL|UU) $$R_{x_1 \bar{u}_2 x_3 \bar{u}_4} = 0 ,$$

(LL|LU) $$R_{x_1 \bar{x}_2 x_3 \bar{u}_4} = 0 ,$$

(LU|UU) $$R_{x_1 \bar{u}_2 u_3 \bar{u}_4} = 0 ,$$

(UU|UU) $$R_{u_1 \bar{u}_2 u_3 \bar{u}_4} = 2(\langle F(u_1, u_2), F(u_3, u_4) \rangle + \langle F(u_1, u_4), F(u_3, u_2) \rangle) .$$

Further, if D is quasi-symmetric, then

(LL|LL) $$R_{x_1 \bar{x}_2 x_3 \bar{x}_4} = \frac{1}{4}(\langle x_1 x_2, x_3 x_4 \rangle + \langle x_1 x_4, x_3 x_2 \rangle - \langle x_1 x_3, x_2 x_4 \rangle) ,$$

(LU|LU) $$R_{x_1\bar{x}_2u_3\bar{u}_4}=\frac{1}{2}\langle F(\varphi(x_2)u_3,\ \varphi(x_1)u_4),\ r\rangle\ .$$

PROOF. The first three formulas are given in [1], and formula (UU|UU) in [1], [2]. Formula (LL|LL) follows from Lemma 4.6 and Theorem 2.7 in [2], and (LU|LU) from Lemma 4.7 as well as Theorem 3.7 and Lemma 3.6 in [2].

Since j is a complex structure on each subspace $\mathfrak{n}(2^{-1}\varepsilon_k)$, if we define

$$\sigma(u, v)=\langle u, v\rangle - i\langle ju, v\rangle\ , \tag{2.7}$$

and

$$(\xi+i\eta)u=\xi u+\eta ju \tag{2.8}$$

for $u, v\in\mathscr{U}$ and $\xi, \eta\in\boldsymbol{R}$, then $(\mathfrak{n}(2^{-1}\varepsilon_k), \sigma)$ is a Hermitian linear space. For a subset $B\subset\mathfrak{n}(2^{-1}\varepsilon_k)$ with $B\cap jB=\varnothing$, we note that the following four statements are mutually equivalent:

(b1) B is an orthogonal basis with respect to σ normalized by $\sigma(u, u)=\omega_k$ for $u\in B$.

(b2) $B\cup jB$ is an orthogonal basis with respect to $\langle\cdot,\cdot\rangle$ normalized by $\langle u, u\rangle=\omega_k$ for $u\in B$.

(b3) $[ju,v]=\delta_{uv}r_k$, $[u, v]=0$ for all $u, v\in B$.

(b4) $4F(u, v)=\delta_{uv}r_k$ for all $u, v\in B$.

3. Curvature properties of the 26-dimensional quasi-symmetric domain. In this section we shall show the following.

PROPOSITION 3.1. *The Bergman metric of the homogeneous bounded domain of type* T(8; 8, 8) *does not satisfy Condition* H_2 *(see* (0.1)*). More presisely, if* $\mathfrak{g}=\mathscr{L}+j\mathscr{L}+\mathscr{U}$ *with* $\mathscr{L}=\mathfrak{n}(\varepsilon_1)+\mathfrak{n}(\varepsilon_2)+\mathfrak{n}(2^{-1}(\varepsilon_1+\varepsilon_2))$ *and* $\mathscr{U}=\mathfrak{n}(2^{-1}\varepsilon_1)+\mathfrak{n}(2^{-1}\varepsilon_2)$ *is the corresponding normal j-algebra, then the function* $\rho^{[2]}(x, x, x, x)/\langle x, x\rangle^2$ *is not constant on the space* $\mathfrak{n}(2^{-1}\varepsilon_2)\setminus\{0\}$.

Combining this with a theorem of Carpenter-Gray-Willmore [3] and a theorem of Geatti [5] (see the introduction) we obtain the following.

THEOREM 3.2. *The Bergman metric g of a homogeneous bounded domain D satisfies Condition* H_2 *if and only if D is biholomorphic to one of the following*:

(1) *the unit ball in* $\boldsymbol{C}^n$,

(2) *the* 6*-dimensional classical domain of type* IV *in Cartan's classification*,

(3) *the exceptional symmetric domain of dimension* 16,

(4) *the exceptional symmetric domain of dimension* 27.

Consequently, there exists no non-symmetric homogeneous bounded domain whose Bergman metric satisfies Condition H_2.

To prove Proposition 3.1, we proceed as in the argument in the preceding sections, and assume that D is of type T(8; 8, 8). We first note that D is quasi-symmetric and that

$$\omega_1=\omega_2=9 \tag{3.1}$$

(see (2.5)). Take $m_1, \dots, m_8 \in \mathfrak{n}(2^{-1}(\varepsilon_1+\varepsilon_2))$ so that $\langle m_a, m_b\rangle=\delta_{ab}\omega_1$, or

$$[jm_a, m_b]=\delta_{ab}r_1\,, \tag{3.2}$$

and take $v_1, \dots, v_8 \in \mathfrak{n}(2^{-1}\varepsilon_2)$ so that $\{v_1, \dots, v_8, jv_1, \dots, jv_8\}$ is an orthogonal basis of $\mathfrak{n}(2^{-1}\varepsilon_2)$ with $[jv_a, v_a]=r_2$, or

$$4F(v_a, v_b)=\delta_{ab}r_2\,. \tag{3.3}$$

Consider the homomorphisms

$$\psi_k:=\sqrt{2}\,\mathrm{ad}\,jm_k=\sqrt{2}\,\varphi(m_k): \mathfrak{n}(2^{-1}\varepsilon_2)\to\mathfrak{n}(2^{-1}\varepsilon_1)\,. \tag{3.4}$$

It follows from Lemma 2.1 that ψ_k are isometric isomorphisms commuting with j. Let

$$u_a=\psi_1(v_a)\,, \qquad a=1, \dots, 8\,. \tag{3.5}$$

Then, $\{u_1, \dots, u_8, ju_1, \dots, ju_8\}$ is an orthogonal basis of $\mathfrak{n}(2^{-1}\varepsilon_1)$ with $[ju_a, u_a]=r_1$, or

$$4F(u_a, u_b)=\delta_{ab}r_1\,. \tag{3.6}$$

For $k=1, \dots, 8$, we associate a complex 8×8 matrix $A_k=(\alpha_{kb}^a)_{a,b}$ so that

$$\sqrt{2}\,\varphi(m_k)v_b=\sum_{a=1}^{8}\alpha_{kb}^a u_a \qquad \text{for} \quad b=1, \dots, 8\,, \tag{3.7}$$

where the scalar multiplication $\alpha_{kb}^a u_a$ is given by (2.8). In view of (3.1), Lemma 2.1 implies the following:

$$A_1=I_8\,, \quad A_k^*A_k=I_8\,, \quad \text{and} \quad A_k^*A_l+A_l^*A_k=0 \qquad (k\neq l)\,. \tag{3.8}$$

It follows that $A_k^*=-A_k (k\geq 2)$, $A_kA_l=-A_lA_k (k, l\geq 2, k\neq l)$, $A_k^2=-I_8$ $(k\geq 2)$. Since the eigenvalues of A_2 are i and $-i$, $\boldsymbol{C}^8$ is a direct sum of two spaces S^+ and S^- defined by $S^\pm=\{v\in\boldsymbol{C}^8\,;\, A_2v=\pm iv\}$. Since $A_2A_3=-A_3A_2$, we have $A_3S^+\subset S^-$, $A_3S^-\subset S^+$; therefore, $\dim S^+=\dim S^-=4$. Take a unitary 8×8 matrix U so that

$$U^*A_2U=\begin{bmatrix} iI_4 & 0 \\ 0 & -iI_4 \end{bmatrix}.$$

Let $k\geq 3$. Since $A_2A_k=-A_kA_2$ and $A_k^*=-A_k$, there exists a 4×4 matrix B_k such that

$$U^*A_kU=\begin{bmatrix} 0 & B_k \\ -B_k^* & 0 \end{bmatrix}.$$

It follows from (3.8) that

$$B_k^*B_k=I_4 \ (k\geq 3)\,, \quad B_k^*B_l+B_l^*B_k=0 \quad (k, l\geq 3,\, k\neq l)\,. \tag{3.9$_1$}$$

Take appropriate unitary 4×4 matrices V_1, V_2 so that $V_1^* B_3 V_2 = I_4$. Taking

$$U\begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$$

instead of U in the argument mentioned above, we may assume that

$$(3.9)_2 \qquad B_3 = I_4 .$$

Compare $(3.9)_1$ and $(3.9)_2$ with (3.8). The same argument as in finding B_k from A_k implies the existence of a unitary 4×4 matrix V such that

$$V^*B_4V=\begin{bmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{bmatrix}, \quad V^*B_kV=\begin{bmatrix} 0 & C_k \\ -C_k^* & 0 \end{bmatrix} \quad (k\geq 5),$$

where C_k are 2×2 matrices with the properties

$$C_5=I_2, \quad C_k^*C_k=I_2 \quad (k\geq 5), \quad C_k^*C_l+C_l^*C_k=0 \quad (k, l\geq 5, k\neq l).$$

Similarly, there exists a unitary 2×2 matrix W such that

$$W^*C_6W=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad W^*C_kW=\begin{bmatrix} 0 & d_k \\ -\bar{d}_k & 0 \end{bmatrix} \quad (k\geq 7),$$

where $d_k \in \boldsymbol{C}$ with the properties $d_7=1$, $|d_8|=1$, $\bar{d}_8+d_8=0$. Taking $-m_8$ instead of m_8 if necessary, we may assume that $d_8=i$. Setting

$$T=U\begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} W & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 0 \\ 0 & 0 & 0 & W \end{bmatrix},$$

and taking $(v_1, \dots, v_8)T$ and $(u_1, \dots, u_8)T$ instead of $(v_1, \dots, v_8)$ and $(u_1, \dots, u_8)$, respectively, we may finally assume the following:

$$(3.10)_1 \qquad A_1=I_8, \quad A_2=\begin{bmatrix} iI_4 & 0 \\ 0 & -iI_4 \end{bmatrix}, \quad A_3=\begin{bmatrix} 0 & I_4 \\ -I_4 & 0 \end{bmatrix}, \quad A_k=\begin{bmatrix} 0 & B_k \\ B_k & 0 \end{bmatrix} \quad (k\geq 4),$$

where

$$(3.10)_2 \qquad B_4=\begin{bmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{bmatrix}, \quad B_5=\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}, \quad B_6=\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix},$$

$$B_7=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad B_8=\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

(cf. Geatti [5]). We note that $B:=\{r_1, r_2, m_1, \dots, m_8, u_1, \dots, u_8, v_1, \dots, v_8\}$ is an orthogonal system of $\mathfrak{g}$ normalized by $\langle b, b\rangle=\omega_1$ for $b\in B$ (see (3.1)) and that $B\cup jB$ is a basis of $\mathfrak{g}$. We list the relationships among elements in B (for details, see [1], [2]):

$$r_a r_b=\delta_{ab}r_a\,, \quad r_a m_k=\frac{1}{2}m_k\,, \quad m_k m_l=\frac{1}{2}\delta_{kl}r\,, \tag{3.11}$$

$$\begin{gathered} \varphi(r_a)u_h=\delta_{a1}u_h\,, \quad \varphi(r_a)v_k=\delta_{a2}v_k\,, \\ \varphi(m_k)u_a=\frac{1}{\sqrt{2}}\sum_{b=1}^{8}\bar{\alpha}_{kb}^{a}v_b\,, \quad \varphi(m_k)v_b=\frac{1}{\sqrt{2}}\sum_{a=1}^{8}\alpha_{kb}^{a}u_a\,, \end{gathered} \tag{3.12}$$

$$F(u_a, u_b)=\frac{1}{4}\delta_{ab}r_1\,, \quad F(v_a, v_b)=\frac{1}{4}\delta_{ab}r_2\,, \quad F(v_a, u_c)=\frac{1}{4\sqrt{2}}\sum_{k=1}^{8}\alpha_{ka}^{c}m_k\,. \tag{3.13}$$

We shall show the following assertion:

$$\rho^{[2]}(v_s+v_t, v_s+v_t, v_s+v_t, v_s+v_t)=\begin{cases}12, & \text{if } (s,t)\notin E\\ 14, & \text{if } (s,t)\in E\,,\end{cases} \tag{3.14}$$

where

$$E=\{(1,6), (6,1), (2,5), (5,2), (3,8), (8,3), (4,7), (7,4)\}\,. \tag{3.15}$$

Assertion (3.14) proves Proposition 3.1. To show (3.14) we first note that

$$\begin{aligned} \rho_i(v_s+v_t, v_s+v_t, v_s+v_t, v_s+v_t)&=\rho_i(v_s, v_s, v_s, v_s)+2\rho_i(v_t, v_s, v_s, v_s)\\ &+2\rho_i(v_s, v_t, v_s, v_s)+2\rho_i(v_s, v_s, v_t, v_t)+2\rho_i(v_s, v_t, v_t, v_s)+\rho_i(v_t, v_s, v_t, v_s)\\ &+\rho_i(v_s, v_t, v_s, v_t)+2\rho_i(v_t, v_s, v_t, v_t)+2\rho_i(v_s, v_t, v_t, v_t)+\rho_i(v_t, v_t, v_t, v_t) \end{aligned} \tag{3.16}$$

for $i=1, 2$. Since the basis $B\cup jB$ is orthogonal and normalized by $\langle b, b\rangle=\omega_1$, $b\in B$, it follows from (1.8), (1.9) as well as (1.7), Lemma 2.2 and (3.11)–(3.13) that

$$\begin{aligned} \rho_1(v_a, v_b, v_c, v_d)&=\frac{4}{\omega_1^2}\Bigg(\sum_{p=q=r_2}+\sum_{p=m_k, q=m_l}+\sum_{p=u_k, q=u_l}+\sum_{p=v_k, q=v_l}\Bigg)R_{v_a\bar{v}_b p\bar{q}}\bar{R}_{v_d\bar{v}_c p\bar{q}}\\ &=\frac{1}{16}\delta_{ab}\delta_{cd}+\frac{1}{64}\sum_{j,h}A_{ad}^{jh}\bar{A}_{bc}^{jh}+\frac{1}{64}\sum_{k,l}A_{ab}^{lk}\bar{A}_{dc}^{lk}\\ &+\frac{1}{16}(10\delta_{ab}\delta_{cd}+\delta_{ad}\delta_{bc})\,, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \rho_2(v_a, v_b, v_c, v_d)&=\frac{4}{\omega_1^2}\Bigg(\sum_{p=u_k, q=u_l}+\sum_{p=v_k, q=v_l}\Bigg)R_{v_a\bar{p}v_c\bar{q}}\bar{R}_{v_b\bar{p}v_d\bar{q}}\\ &=\frac{1}{64}\sum_{k,l}(B_{ac}^{kl}+B_{ac}^{lk})(B_{bd}^{kl}+B_{bd}^{lk})^{-}+\frac{1}{8}(\delta_{ab}\delta_{cd}+\delta_{ad}\delta_{bc})\,, \end{aligned} \tag{3.18}$$

where

$$A_{st}^{jh}=\sum_{k=1}^{8}\alpha_{ks}^{j}\bar{\alpha}_{kt}^{h} \quad \text{and} \quad B_{st}^{jh}=\sum_{k=1}^{8}\alpha_{ks}^{j}\alpha_{kt}^{h}.$$

We note that

$$(3.19)_1 \qquad A_{ss}^{jh}=0 \qquad \text{when} \quad j\neq h$$

and that the matrix $(A_{ss}^{jj})_{1\leq j,s\leq 8}$ is given by

$$(3.19)_2 \qquad (A_{ss}^{jj})=2\begin{bmatrix} I & 0 & I & J \\ 0 & I & J & I \\ I & J & I & 0 \\ J & I & 0 & I \end{bmatrix},$$

where $I=I_2$ and J is the 2×2 matrix whose entries are all one, while

$$(3.20) \qquad B_{ss}^{jh}=0 \qquad \text{for all} \quad j, h.$$

Assume $(s, t)\in E$. Then

$$(3.21) \qquad A_{st}^{jh}=0 \qquad \text{for all} \quad j, h,$$

and there exists a permutation τ of $\{1, \dots, 8\}$ such that

$$(3.22) \qquad B_{st}^{jh}=\begin{cases} 2, & (j, h)=(\tau_1, \tau_2) \text{ or } (\tau_3, \tau_4) \\ -2, & (j, h)=(\tau_5, \tau_6) \text{ or } (\tau_7, \tau_8) \\ 0, & \text{otherwise}. \end{cases}$$

Next assume $s\neq t$ and $(s, t)\notin E$. Then, there exists a permutation σ of $\{1, \dots, 8\}$ such that

$$(3.23) \qquad A_{st}^{jh}=\begin{cases} \pm 2, & (j, h)=(\sigma_1, \sigma_2) \text{ or } (\sigma_3, \sigma_4) \\ 0, & \text{otherwise} \end{cases}$$

and

$$(3.24) \qquad B_{st}^{jh}=\begin{cases} 2, & (j, h)=(\sigma_5, \sigma_6) \\ -2, & (j, h)=(\sigma_6, \sigma_5) \\ 0, & \text{otherwise}. \end{cases}$$

LEMMA 3.3. *$\rho_1(v_s, v_s, v_s, v_s)=5/4$ and $\rho_2(v_s, v_s, v_s, v_s)=1/4$.*

PROOF. By (3.17) as well as (3.19) we have

$$\rho_1(v_s, v_s, v_s, v_s)=\frac{1}{16}+\frac{2}{64}\sum_{j,h}|A_{ss}^{jh}|^2+\frac{11}{16}=\frac{5}{4}.$$

By (3.18) as well as (3.20) we have

$$\rho_2(v_s, v_s, v_s, v_s) = \frac{1}{16}\sum_{k,l} |B^{kl}_{ss}|^2 + \frac{1}{4} = \frac{1}{4}.$$

LEMMA 3.4. *If $s \neq t$, then $\rho_i(v_t, v_s, v_s, v_s) = 0$ $(i = 1, 2)$.*

PROOF. By (3.17) we have

$$\rho_1(v_t, v_s, v_s, v_s) = \frac{2}{64}\sum_{j,h} A^{jh}_{ts}\bar{A}^{jh}_{ss} = 0,$$

because $A^{jh}_{ss} = 0$ when $j \neq h$ (by (3.19)), and $A^{jh}_{ts} = 0$ when $j = h$ (by (3.21) and (3.23)). By (3.18) as well as (3.20) we see

$$\rho_2(v_t, v_s, v_s, v_s) = \frac{1}{64}\sum_{k,l} (B^{kl}_{ts} + B^{lk}_{ts})(B^{kl}_{ss} + B^{lk}_{ss})^- = 0.$$

LEMMA 3.5. *If $s \neq t$, then $\rho_1(v_s, v_s, v_t, v_t) = 15/16$ when $(s, t) \notin E$, and $= 11/16$ when $(s, t) \in E$.*

PROOF. By (3.17) we have

$$\rho_1(v_s, v_s, v_t, v_t) = \frac{11}{16} + \frac{1}{64}\sum_{j,h} |A^{jh}_{st}|^2 + \frac{1}{64}\sum_{k,l} A^{lk}_{ss}\bar{A}^{lk}_{tt}.$$

It follows from (3.21) and (3.23) that

$$\sum_{j,h} |A^{jh}_{st}|^2 = \begin{cases} 8, & (s, t) \notin E \\ 0, & (s, t) \in E. \end{cases}$$

It follows from (3.19) that

$$\sum_{k,l} A^{lk}_{ss}\bar{A}^{lk}_{tt} = \sum_{k=1}^{8} A^{kk}_{ss}\bar{A}^{kk}_{tt} = \begin{cases} 8, & (s, t) \notin E \\ 0, & (s, t) \in E. \end{cases}$$

From these we have the formulas.

LEMMA 3.6. *If $s \neq t$, then $\rho_1(v_s, v_t, v_t, v_s) = 5/16$ when $(s, t) \notin E$, and $= 1/16$ when $(s, t) \in E$.*

PROOF. By (3.17) we have

$$\rho_1(v_s, v_t, v_t, v_s) = \frac{1}{64}\sum_{j,h} A^{jh}_{ss}\bar{A}^{jh}_{tt} + \frac{1}{64}\sum_{k,l} |A^{lk}_{st}|^2 + \frac{1}{16}.$$

The assertion follows from the equalities in the proof of Lemma 3.5.

LEMMA 3.7. *If $s \neq t$, then $\rho_i(v_s, v_t, v_s, v_t) = 0$ $(i = 1, 2)$.*

PROOF. By (3.17) as wall as (3.21) and (3.23) we have

$$\rho_1(v_s, v_t, v_s, v_t)=\frac{2}{64}\sum_{j,h} A_{st}^{jh}\bar{A}_{ts}^{jh}=\frac{1}{32}\sum_{j,h} A_{st}^{jh}A_{st}^{hj}=0 .$$

By (3.18) as wall as (3.20) we see

$$\rho_2(v_s, v_t, v_s, v_t)=\frac{4}{64}\sum_{k,l} B_{ss}^{kl}\bar{B}_{tt}^{kl}=0 .$$

LEMMA 3.8. *If $s \neq t$, then $\rho_2(v_s, v_s, v_t, v_t)=\rho_2(v_s, v_t, v_t, v_s)$ and the value is $1/8$ when $(s, t)\notin E$, while is $5/8$ when $(s, t)\in E$.*

PROOF. It follows from (3.18) that

$$\rho_2(v_s, v_s, v_t, v_t)=\rho_2(v_s, v_t, v_t, v_s)=\frac{1}{64}\sum_{k,l}|B_{st}^{kl}+B_{st}^{lk}|^2+\frac{1}{8} .$$

If $(s, t)\in E$ then by (3.22) we have

$$\sum_{k,l}|B_{st}^{kl}+B_{st}^{lk}|^2=2^2\times 8 ;$$

while if $(s, t)\notin E$ then by (3.24) we have

$$\sum_{k,l}|B_{st}^{kl}+B_{st}^{lk}|^2=0 .$$

PROOF OF PROPOSITION 3.1. We shall show (3.14). First assume $(s, t)\notin E$. It follows from (3.16) as well as Lemmas 3.3–3.8 that

$$\rho_1(v_s+v_t, v_s+v_t, v_s+v_t, v_s+v_t)=\frac{5}{4}+0+0+2\times\frac{15}{16}+2\times\frac{5}{16}+0+0+0+0+\frac{5}{4}=5 ,$$

and that

$$\rho_2(v_s+v_t, v_s+v_t, v_s+v_t, v_s+v_t)=\frac{1}{4}+0+0+2\times\frac{1}{8}+2\times\frac{1}{8}+0+0+0+0+\frac{1}{4}=1 .$$

Thus $\rho^{[2]}(v_s+v_t, v_s+v_t, v_s+v_t, v_s+v_t)=12$. Similarly, if $(s, t)\in E$, then $\rho^{[2]}(v_s+v_t, v_s+v_t, v_s+v_t, v_s+v_t)=14$, as desired.

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